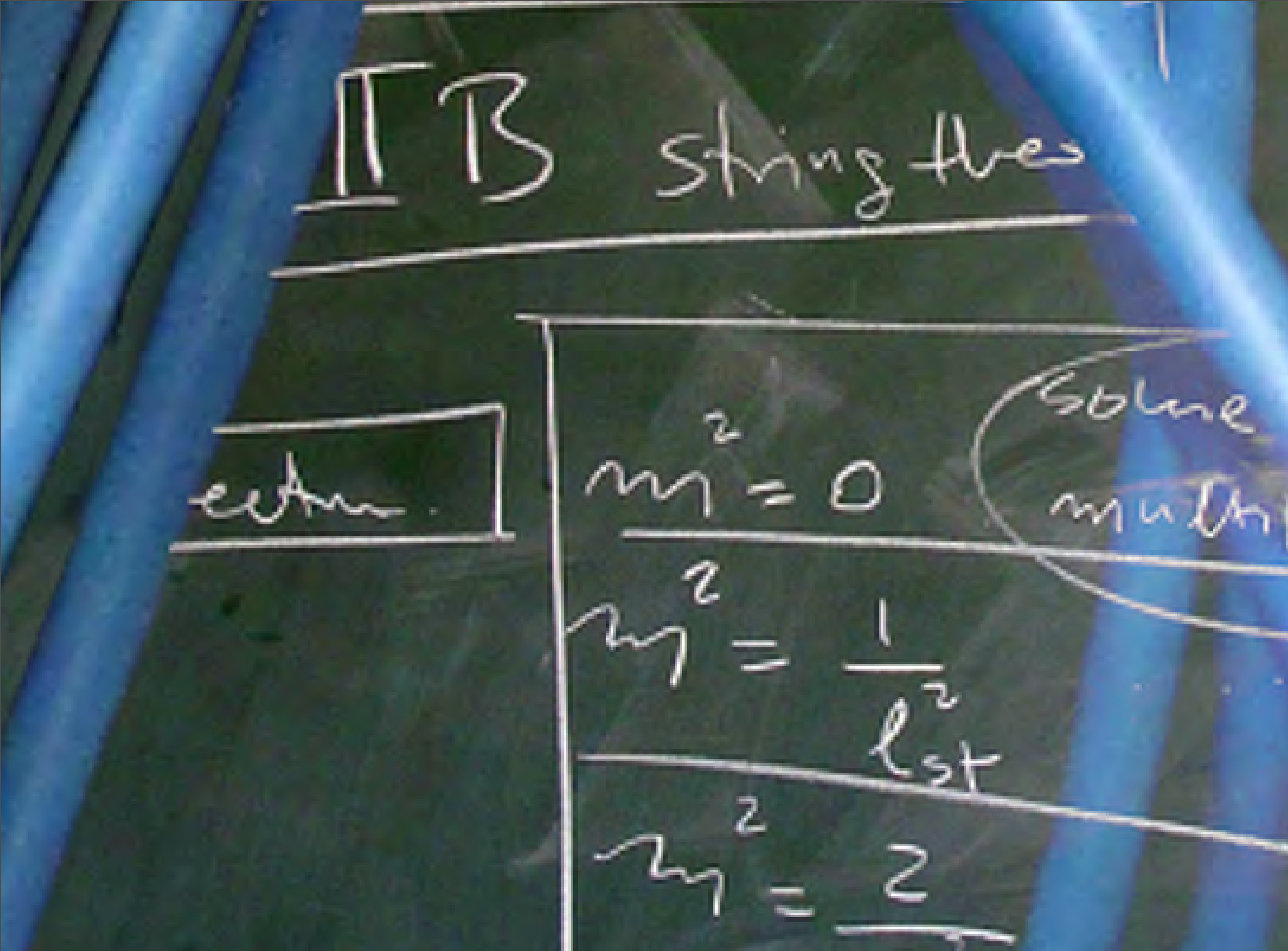

Three
String
Generations
at
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Deformations of special geometry: in search of the topological string

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SPECIAL GEOMETRY

N=2, d=4 vector supermultiplets: *(rigid or local supersymmetry)*

E/M duality (symplectic equivalence) (Wilsonian effective action)

Lagrangian encoded in a holomorphic homogeneous function $F(X)$

period vector $X^I, \; F_I = \partial_I F$

$I = 0, 1, \ldots, n$

dW, Van Proeyen,1984

Cecotti, Ferrara, Girardello, 1989

Calabi-Yau moduli spaces

homology cycles (symplectic equivalence)

holomorphic three-form Ω with periods

$$X^I = \int_{A^I} \Omega \qquad F_I = \int_{B_I} \Omega$$

$I = 0, 1, \ldots, b_{21}$

Strominger,1990

Deformations and all that

Supergravity effective action

Higher-derivative couplings: mainly chiral (‘*F*-terms’)

Non-holomorphic corrections: from integrating out massless modes

Dixon, Kaplunovsky, Louis, 1991

Topological string

Genus-*g* partition functions of a twisted non-linear sigma model with a *CY* target space

Holomorphic anomaly: pinched cycles of the Riemann surface

Bershadsky, Cecotti, Ooguri, Vafa, 1994

The two are related through string theory:

The partition functions capture certain string amplitudes, which are also describable by an effective (supergravity) action.

Antoniadis, Gava, Narain, Taylor, 1993

How to explore/understand this relation ?

Lessons learned from BPS black holes

vector multiplets contain scalars X^I

projectively defined: $X^I \longrightarrow Y^I$

Lagrangian encoded in a holomorphic homogeneous function

$$\boxed{F(\lambda Y) = \lambda^2 \, F(Y)}$$

No need to work in terms of the (complicated) effective actions!

Attractor equations *(horizon behaviour)*

$$Y^I - \bar{Y}^I = \mathrm{i} p^I \qquad \text{magnetic charges}$$

$$F_I - \bar{F}_I = \mathrm{i} q_I \qquad \text{electric charges}$$

Ferrara, Kallosh, Strominger, 1996

homogeneity: entropy and area are proportional to Q^2

$$\frac{R_{\mathrm{hor}}}{l_{\mathrm{s}}} \sim g_{\mathrm{s}}\, Q$$

large $Q \longrightarrow$ macroscopic black hole

“large black hole”

Satisfy Bekenstein-Hawking area law

Possible (subleading) corrections?

➢ other invariant (higher-derivative) couplings

➢ non-holomorphic corrections (non-Wilsonian)

Special class of subleading corrections: *('F-terms')*

extend with one 'extra' (composite) complex field, originating from pure supergravity

$\boxed{\Upsilon}$

$$F(Y) \longrightarrow F(Y, \Upsilon)$$

Weyl multiplet background

Cardoso, dW, Mohaupt, 1998

homogeneity: $\boxed{F(\lambda Y, \lambda^2 \Upsilon) = \lambda^2\, F(Y, \Upsilon)}$

Υ-dependence leads to terms $\propto (R_{\mu\nu\rho}{}^{\sigma})^2$ in effective action

attractor equations remain valid with $\boxed{\Upsilon = -64}$

subleading corrections!

$Y^I - \bar{Y}^I = \mathrm{i} p^I$ magnetic charges

$F_I - \bar{F}_I = \mathrm{i} q_I$ electric charges

Cardoso, dW, Käppeli, Mohaupt, 2000

covariant under dualities!

contains Υ

This will modify the Bekenstein-Hawking area law !

Conveniently described by a variational principle (extending previous work). *Behrndt, Cardoso, dW, Kallosh, Lüst, Mohaupt, 1996*

BPS entropy function:

$$\Sigma(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - q_I(Y^I+\bar{Y}^I) + p^I(F_I+\bar{F}_I)$$

$Y^I+\bar{Y}^I$ and $F_I+\bar{F}_I$ play the role of electro- and magnetostatic potentials at the horizon

$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$ *'free energy'*

$\delta\Sigma = 0 \;\Leftrightarrow$ **attractor equations** (and $\Upsilon = -64$)

$$\pi\Sigma\big|_* = \mathcal{S}_{\rm macro}(p,q)$$

similar to, but different from, Sen's entropy function *Sen, 2005*

- *invariant under the relevant dualities!*
- *the free energy follows from the requirement that the variation of the entropy function yields the attractor equations*
- *non-holomorphic corrections (non-Wilsonian) can be incorporated*

Early example : N=4 supersymmetry with S-duality

Cardoso, dW, Mohaupt, 1999
Cardoso, dW, Käppeli, Mohaupt, 2004

$$F = -\frac{Y^1}{Y^0} Y^a \eta_{ab} Y^b + \frac{\mathrm{i}}{256\,\pi}\Big[\Upsilon \log\eta^{12}(S) + \boxed{\bar\Upsilon \log\eta^{12}(\bar S)} + \tfrac{1}{2}(\Upsilon+\bar\Upsilon)\boxed{\log(S+\bar S)^6}\Big]$$

harmonic

non-holomorphic

required by S-duality

related to threshold correction

Harvey, Moore, 1996

consistent with 1/4 BPS degeneracy

Dijkgraaf, Verlinde, Verlinde, 1997

$$\mathrm{i}S = \frac{Y^1}{Y^0}$$

More general decomposition:

$$F = -\tfrac{Y^1}{Y^0} Y^a \eta_{ab} Y^b + \boxed{2\mathrm{i}\Omega}$$

real, homogeneous

$\begin{pmatrix} Y^I \\ F_I \end{pmatrix}$ transforms as $\begin{pmatrix} p^I \\ q_I \end{pmatrix}$ under duality rotations (monodromies)

This determines the transformation of Ω

The function F is not invariant!

!

Comments

The entropy function and the incorporation of non-holomorphic terms, although well-motivated and seemingly without alternative, are not entirely based on first principles (i.e. a complete action).

Their validity is in agreement with microscopic derivations (semiclassically), which also confirm the existence of a variational principle, and the presence of the non-holomorphic corrections.

Dijkgraaf, Verlinde, Verlinde, 1997
Cardoso, dW, Käppeli, Mohaupt, 2004,2006
Shih, Strominger, Yin, 2005
Jatkar, Sen, 2005
David, Sen, 2006

Important conceptual implication: Apart from the existence of a free energy function, it indicates the consistency of a possible non-holomorphic deformation of special geometry.

Cardoso, dW, Mahapatra, 2008

As it turns out this can be understood in de context of the following general theorem. *Cardoso, dW, Mahapatra, to appear*

Theorem *(canonical perspective)*

Consider a Lagrangian $\mathcal{L}(q,\dot q)$ *depending on* n *coordinates* q^i *and* n *velocities* $\dot q^i$ *with a corresponding Hamiltonian*

$$\mathcal{H}(q,p) = \dot q^i\, p_i - \mathcal{L}(q,\dot q)$$

Then there exists a description in terms of a complex function $F(x,\bar x)$, *with* $x^i = \frac{1}{2}(q^i + \mathrm{i}\dot q^i)$, *such that:*

$$\boxed{\begin{aligned} 2\,\mathrm{Re}\,x^i &= q^i \\ 2\,\mathrm{Re}\,F_i(x,\bar x) &= p_i \end{aligned}}$$

$$F_i = \frac{\partial F(x,\bar x)}{\partial x^i}$$

The function $F(x,\bar x)$ *is decomposable into a holomorphic and a purely imaginary non-harmonic function:*

$$F(x,\bar x) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar x)$$

usually the 'classical' part

contains non-Wilsonian contributions (and more?)

this decomposition is not unique

Then the Lagrangian and Hamiltonian take the following form:

$$\mathcal{L} = 4[\mathrm{Im}\, F - \Omega]$$

$$\mathcal{H} = -\mathrm{i}(x^i\, \bar{F}_{\bar\imath} - \bar{x}^{\bar\imath}\, F_i) - 4\,\mathrm{Im}[F^{(0)} - \tfrac{1}{2}x^i\, F_i^{(0)}] - 2(2\,\Omega - x^i\Omega_i - \bar{x}^{\bar\imath}\Omega_{\bar\imath})$$

symplectic invariant

vanishes owing to homogeneity

proportional to deformation parameters

The $2n$*-vector* (x^i, F_i) *transforms under canonical (symplectic) reparametrizations as:*

$$\begin{pmatrix} x^i \\ F_i(x,\bar{x}) \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}^i \\ \tilde{F}_i(\tilde{x},\bar{\tilde{x}}) \end{pmatrix} = \begin{pmatrix} U^i{}_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} x^j \\ F_j(x,\bar{x}) \end{pmatrix}$$

non-holomorphic period vector

$$\mathcal{H} = -4\,[\mathrm{Im}\, F - \Omega] + 2\mathrm{Im}\, x^i\, p_i$$

The existence of an Hamiltonian and corresponding canonical transformations is crucial!

Confirmation !!

Cardoso, dW, Käppeli, Mohaupt, 2004, 2006

Non-holomorphic extensions are incorporated by:

$$F \longrightarrow F^{(0)}(Y) + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$$

Non-holomorphic and homogeneous

and are compatible with special geometry.
In particular this preserves the form of the attractor equations.

The BPS free energy equals:

$$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -\mathrm{i}\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) - 2\mathrm{i}\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon\right)$$

The free energy equals the Hesse potential, $\mathcal{H}(\phi,\chi,\Upsilon,\bar{\Upsilon})$
precisely as indicated by the general theorem, expressed in canonical (covariant) variables, the analogue of the Hamiltonian.

The Hesse potential, $\mathcal{H}(\phi, \chi, \Upsilon, \bar{\Upsilon})$

includes non-holomorphic corrections (!) and follows from a Legendre transformation

$$\boxed{4(\operatorname{Im} F - \Omega) - 2\chi_I \operatorname{Im} Y^I = \mathcal{H}(\phi, \chi)}$$

Dualities are represented by canonical transformations!

electro- and magnetostatic potentials:

$$\phi^I = Y^I + \bar{Y}^I \qquad\qquad \chi^I = F_I + \bar{F}_I$$

in the spirit of *Real special geometry*

Freed, 1999
Cortés,2001
Alekseevsky, Cortés, Devchand, 2002

attractor equations:

$$\boxed{\frac{\partial \mathcal{H}}{\partial \phi^I} = q_I \qquad\qquad \frac{\partial \mathcal{H}}{\partial \chi_I} = -p^I}$$

'Hamiltonian form' of the free energy

$$\boxed{\mathcal{H}(\phi,\chi) - q_I\phi^I + p^I\chi_I = \Sigma = \frac{\mathcal{S}_{\text{macro}}(p,q)}{\pi}}$$

Wald entropy

$\Sigma(\phi,\chi;p,q)$ $\longrightarrow$ *BPS entropy function*

Is there a relation to the topological string?

The approximate relation $\mathcal{H} + p^I\chi_I \approx 4\,\mathrm{Im}\,F + \cdots$
suggested at the time a direct relation with the topological string

$$Z_{\text{BH}}(p,\phi) \approx \mathrm{e}^{4\,\mathrm{Im}F} \approx |Z_{\text{top}}(p,\phi)|^2$$

Ooguri, Strominger, Vafa, 2004

➢ The connection with the topological string is not clear, both because of the non-holomorphic corrections and because the effective action and the topological string are functions of different conjugate variables.

➢ PM: in addition to the *'F-term'* invariants, there are non-holomorphic terms related to non-local terms in the *1PI* effective action and/or to possible *'D-term'* invariants.

(non-renormalization theorem for BPS)

Central question *does there exist a single function that encodes both the effective action and the topological string?*

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g}\Upsilon^g\, F^{(g)}(t)$$

$t^A = Y^A/Y^0$

$$(Y^0)^2 F^{(0)}(t)$$

genus-*g* partition function of a twisted non-linear sigma model with CY target space

Y^0 loop-counting parameter: $Y^0 = g_{\rm top}^{-1}$

To answer this question consider the characteristic features of the above expansion when interpreted either as the function that encodes the effective action, or as the topological string partition function.

The key feature to concentrate on is the behaviour under duality

Cardoso, dW, Mahapatra, 2008

effective action

- the $F^{(g)}$ are NOT invariant
- the periods transform correctly under monodromies
- the duality transformations are Υ-dependent

topological string

- the $F^{(g)}$ are COVARIANT sections
- the periods refer to $F^{(0)}$
- the duality transformations are Υ-independent

Nevertheless both must be related to certain string amplitudes!

$$
\begin{array}{ccc}
\text{Lagrangian } \mathcal{L} & \longleftrightarrow & \text{Hamiltonian } \mathcal{H} \\
\uparrow & & \uparrow \\
\mathrm{Im}[F(Y,\Upsilon)] & \longleftrightarrow & \text{Hesse potential } \mathcal{H}
\end{array}
$$

Legendre transform

Free energy $\mathcal{F}$

In the spirit of the Theorem !

First test: the FHSV Model

Cardoso, dW, Mahapatra, 2008

expand Ω in Υ and start from:

$$\Omega^{(1)}(S,\bar S,T,\bar T,\Upsilon,\bar\Upsilon) \;=\; \frac{1}{256\,\pi}\Big[\tfrac{1}{2}\Upsilon\ln[\eta^{24}(2S)\,\Phi(T)] + \tfrac{1}{2}\bar\Upsilon\ln[\eta^{24}(2\bar S)\,\Phi(\bar T)] + (\Upsilon+\bar\Upsilon)\ln[(S+\bar S)^3(T+\bar T)^a\eta_{ab}(T+\bar T)^b]\Big]$$

This expression is S- and T-duality invariant! *(for real Υ)*

Note: non-holomorphic!

ITERATE: *determine the corrections to the transformation rules of the moduli and integrate. This yields:*

$$\Omega^{(2)} = -\frac{G_2(2S)}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial T_a} - \frac{1}{4(Y^0)^2}\,\frac{\partial\ln\Phi(T)}{\partial T_a}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial S} + \mathrm{c.c}$$

which is not invariant and determined up to an invariant function.

The corresponding free energy is then shown to be S- and T-duality invariant !!

Perform the Legendre transform (by iteration) :

new covariant coordinates: $Y^I \longrightarrow \mathcal{Y}^I$

Solve the equations:

$$
\begin{gathered}
2\,\mathrm{Re}\,Y^I = \phi^I = 2\,\mathrm{Re}\,\mathcal{Y}^I \\
2\,\mathrm{Re}\,F_I(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = \chi_I = 2\,\mathrm{Re}\,F_I^{(0)}(\mathcal{Y})
\end{gathered}
$$

This will involves an infinite series!
The original holomorphic structure is not respected!

Can one identify the topological string partition function?

Is the topological string contained in Ω *?*
Note: $F^{(0)}$ *enters only in the definition of the duality covariant moduli*

For instance, the expansion of the old variables into the new ones takes the form:

$$
\begin{aligned}
Y^I \;\approx\; & \mathcal{Y}^I - 2\,(\Omega^I - \Omega^{\bar I}) \\
& +2\mathrm{i}(F+\bar F)^{IJK}(\Omega_J - \Omega_{\bar J})(\Omega_K - \Omega_{\bar K}) + 8\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar J})\,(\Omega_J - \Omega_{\bar J}) \\
& -\tfrac{4}{3}\mathrm{i}\left[(F-\bar F)^{IJKL} + 3\mathrm{i}(F+\bar F)^{IJM}(F+\bar F)_M{}^{KL}\right] \\
& \qquad\qquad \times(\Omega_J - \Omega_{\bar J})(\Omega_K - \Omega_{\bar K})(\Omega_L - \Omega_{\bar L}) \\
& -8\mathrm{i}\left[2\,(F+\bar F)^{IJ}{}_K\mathrm{Re}(\Omega^{KL} - \Omega^{K\bar L}) + \mathrm{Re}(\Omega^{IK} - \Omega^{I\bar K})(F+\bar F)_K{}^{JL}\right] \\
& \qquad\qquad \times(\Omega_J - \Omega_{\bar J})(\Omega_L - \Omega_{\bar L}) \\
& -32\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar J})\,\mathrm{Re}(\Omega_{JK} - \Omega_{J\bar K})\,(\Omega^K - \Omega^{\bar K}) \\
& -8\mathrm{i}\,\mathrm{Im}(\Omega^{IJK} - 2\,\Omega^{IJ\bar K} + \Omega^{I\bar J\bar K})(\Omega_J - \Omega_{\bar J})\,(\Omega_K - \Omega_{\bar K}) + \cdots\ .
\end{aligned}
$$

where in each of the terms the old variables have been replaced into the new ones.

[here: $F \equiv F^{(0)}(\mathcal{Y})$]

Construct the Hesse potential (in terms of covariant moduli) for the *FSHV*-Model:

$$
\begin{aligned}
\mathcal{H}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) \;\approx\; & -|Y^0|^2(S+\bar{S})(T+\bar{T})^2 + 4\,\Omega^{(1)}(S,\bar{S},T,\bar{T}) \\
& - \Bigg[\frac{4\,\hat{G}_2(2S,2\bar{S})}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T_a}\,\frac{\partial\Omega^{(1)}}{\partial T^a} \\
& \qquad + \frac{1}{(Y^0)^2}\,\frac{\partial \log\big[\Phi(T)\,[(T+\bar{T})^2]^4\big]}{\partial T_a}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial S} + \text{c.c.}\Bigg] \\
& + \frac{4\,(T+\bar{T})^a(T+\bar{T})^b}{|Y^0|^2\,(S+\bar{S})(T+\bar{T})^2}\left(2\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial \bar{T}^b} - \eta_{ab}\,\frac{\partial\Omega^{(1)}}{\partial T_c}\,\frac{\partial\Omega^{(1)}}{\partial \bar{T}^c}\right) \\
& + \frac{8\,(S+\bar{S})}{|Y^0|^2\,(T+\bar{T})^2}\,\frac{\partial\Omega^{(1)}}{\partial S}\,\frac{\partial\Omega^{(1)}}{\partial \bar{S}}
\end{aligned}
$$

S- and T-duality invariant!

holomorphic dependence on the topological string coupling

non- holomorphic dependence

The topological string is indeed contained in the Hesse potential and is characterized by its holomorphic dependance on the topological string coupling constant.

Grimm, Klemm, Marino, Weiss, 2007

However, there are other invariant terms, which do not depend holomorphically on the topological string coupling constant. At present their meaning is not clear, but they do in principle contribute to the BPS black hole entropy at the sub-subleading level.

Let us therefore continue by studying the more generic aspects, irrespective of a particular model.

Upon substitution, the expression for the Hesse potential in terms of the new variables $\mathcal{Y}^I$ will take the form:

$$\begin{aligned}\mathcal{H} \;=\; & \mathcal{H}|_{\Omega=0} + 4\,\Omega - 4\,N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar I}\Omega_{\bar J}) + 8\,\boxed{N^{IJ}\Omega_I\Omega_{\bar J}} \\ & +16\,\mathrm{Re}(\Omega_{IJ} - \Omega_{I\bar J})N^{IK}N^{JL}\big(\Omega_K\Omega_L + \Omega_{\bar K}\Omega_{\bar L} - 2\,\Omega_K\Omega_{\bar L}\big) \\ & -\tfrac{16}{3}(F+\bar F)_{IJK}N^{IL}N^{JM}N^{KN}\,\mathrm{Im}(\Omega_L\Omega_M\Omega_N - 3\,\Omega_L\Omega_M\Omega_{\bar N}) \\ & +\mathcal{O}(\Omega^4)\end{aligned}$$

non-holomorphic square

where $\mathcal{H}(\mathcal{Y},\bar{\mathcal{Y}})\Big|_{\Omega=0} = \mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar F_I(\bar{\mathcal{Y}})]$

and $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}(\mathcal{Y},\bar{\mathcal{Y}}), \quad \Omega(\mathcal{Y},\bar{\mathcal{Y}})$

$$F \equiv F^{(0)}(\mathcal{Y})$$

From the fact that the Hesse potential transforms as a function under symplectic transformations, one arrives at the following result.

The quantity $\Omega(\mathcal{Y},\bar{\mathcal{Y}})$ *does not transform as a function under symplectic transformations !*

Rather it transforms non-linearly (proven by iteration!):

$$\begin{aligned}\tilde{\Omega} \;=\;& \Omega - \mathrm{i}\big(\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J - \bar{\mathcal{Z}}_0^{\bar I\bar J}\,\Omega_{\bar I}\Omega_{\bar J}\big)\\ &+\tfrac{2}{3}\big(F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N + \mathrm{h.c.}\big)\\ &-2\big(\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L + \mathrm{h.c.}\big) + 4\,\Omega_{I\bar J}\,\mathcal{Z}_0^{IK}\Omega_K\,\bar{\mathcal{Z}}_0^{\bar J\bar L}\Omega_{\bar L}\\ &+\mathcal{O}(\Omega^4)\end{aligned}$$

where all variations are encoded in $\mathcal{Z}_0^{IJ} = \dfrac{\partial\mathcal{Y}^I}{\partial\tilde{\mathcal{Y}}_K}\,Z^{KJ}$

This suggests a systematic pattern!

(we have verified this including terms of fourth order)

DECOMPOSITION OF THE HESSE POTENTIAL

$$\mathcal{H} = \mathcal{H}_0 + 8\,\mathcal{H}_1 - \tfrac{8}{3}\mathrm{i}(\mathcal{H}_2 - \bar{\mathcal{H}}_2) + 16\,\mathcal{H}_3 + \mathcal{H}_4 + \cdots$$

where $$\mathcal{H}_0 = \mathcal{H}\Big|_{\Omega=0} + 4\,\Omega + \mathcal{O}(\Omega^2)$$

and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 + \cdots$ are symplectic functions,
which are not harmonic, not even when Ω is harmonic.

In the following, let us assume that Ω is harmonic.

The duality transformations preserve the possible harmonicity of Ω!

$$
\begin{aligned}
\tilde{\Omega} \;=\; \Omega + \Big[& - \mathrm{i}\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J \\
& + \tfrac{2}{3}F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N \\
& -2\,\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L \\
& -\tfrac{\mathrm{i}}{3}F_{IJKL}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\
& +\tfrac{4\mathrm{i}}{3}\,\Omega_{IJK}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K \\
& +\mathrm{i}\,F_{IJR}\,\mathcal{Z}_0^{RS}\,F_{SKL}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\
& -4\mathrm{i}\,F_{IJK}\mathcal{Z}_0^{KP}\,\Omega_{PQ}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J\,(\mathcal{Z}_0\Omega)^Q \\
& +4\mathrm{i}\,\mathcal{Z}_0^{IP}\,\Omega_{PQ}\,\mathcal{Z}_0^{QR}\,\Omega_{RK}\,(\mathcal{Z}_0\Omega)^K \;+\mathrm{h.c.}\Big]
\end{aligned}
$$

Familiar structure. In the context of the topological string, $\mathcal{Z}_0^{IJ}$ is known as a ‘propagator’: $\mathcal{Z}_0^{IJ} \propto \Delta^{IJ}$

In that case the partition function is treated as a wave function in a Hilbert space based on quantizing $H_3(X)$

Aganagic, Bouchard, Klemm, 2008

The symplectic function $\mathcal{H}_0$ is 'almost' harmonic in this case: it is the real part of a function that consists of only purely holomorphic derivatives of F and Ω contracted with the non-holomorphic tensor N^{IJ}

$$
\begin{aligned}
\mathcal{H}_0 \;=\; & \mathcal{H}\Big|_{\Omega=0} + 4\Omega \\
& +4\Big[- N^{IJ}\,\Omega_I\Omega_J \\
& \qquad +2\,\Omega_{IJ}\,N^{IK}\Omega_K N^{JL}\Omega_L \\
& \qquad +\tfrac{2}{3}\mathrm{i}F_{IJK}\,(N\Omega)^I\,(N\Omega)^J\,(N\Omega)^K \\
& \qquad -\tfrac{\mathrm{i}}{3}\,\big(F_{IJKL} + 3\mathrm{i}\,F_{IJR}\,N^{RS}\,F_{SKL}\big)\,(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\Omega)^L \\
& \qquad -\tfrac{4}{3}\,\Omega_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K \\
& \qquad -4\mathrm{i}\,F_{IJK}N^{KP}\,\Omega_{PQ}\,(N\Omega)^I(N\Omega)^J\,(N\Omega)^Q \\
& \qquad -4\,N^{IP}\,\Omega_{PQ}\,N^{QR}\,\Omega_{RK}\,(N\Omega)^K \;+\mathrm{h.c.}\Big]
\end{aligned}
$$

Conclusions:

Non-holomorphic deformations of special geometry are consistent in the proposed framework.

The Hesse potential, which is a proper symplectic function, decomposes into several symplectic functions. One of them has a structure that resembles that of the topological string partition function.

On the other hand, certain aspects are perhaps generic, and further details have to be worked out.

More work is needed!

Cardos, dW, Mahapatra, to be published