ON THE STATISTICAL INTERPRETATION OF BLACK HOLE ENTROPY

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Key facts :

Black hole solutions exist in Einstein's theory of general relativity

Black hole solutions exhibit a horizon: nothing can escape
(at least not classically) from behind the horizon

Schwarzschild radius $R_{\mathrm{S}} = 2\, M\, G_{\mathrm{N}}$

There is astrophysical evidence for black holes in Nature

There is a surprising relation between:

- Laws of black hole mechanics
- Laws of thermodynamics

Black hole mechanics ⟷ Thermodynamics

Bardeen, Carter, Hawking, 1973

$$\delta E = T\,\delta S - p\,\delta V$$

$$\delta M = \frac{\kappa_{\rm s}}{2\pi}\,\frac{\delta A}{4} + \phi\,\delta Q + \Omega\,\delta J$$

Planck units!

$$T \leftrightarrow \frac{\kappa_{\rm s}}{2\pi}$$

Hawking temperature
Surface gravity

$$S \leftrightarrow \frac{A}{4}$$

Bekenstein-Hawking area law

Charged Black Holes

Reissner-Nordstrom black holes:

charged, static, spherically symmetric, solutions to Einstein-Maxwell theory

$P,\ Q$ magnetic/electric

$$G_{\mathrm{N}}\, M^2 > P^2 + Q^2$$ non-extremal, two horizons

$$G_{\mathrm{N}}\, M^2 = P^2 + Q^2$$ extremal, horizons coalesce

$$G_{\mathrm{N}}\, M^2 < P^2 + Q^2$$ naked singularity, physically unacceptable

Extremal black holes have zero temperature

BPS ⟷ Partially supersymmetric

Naturally embedded in extended supergravity

Question: statistical interpretation of black hole entropy ?

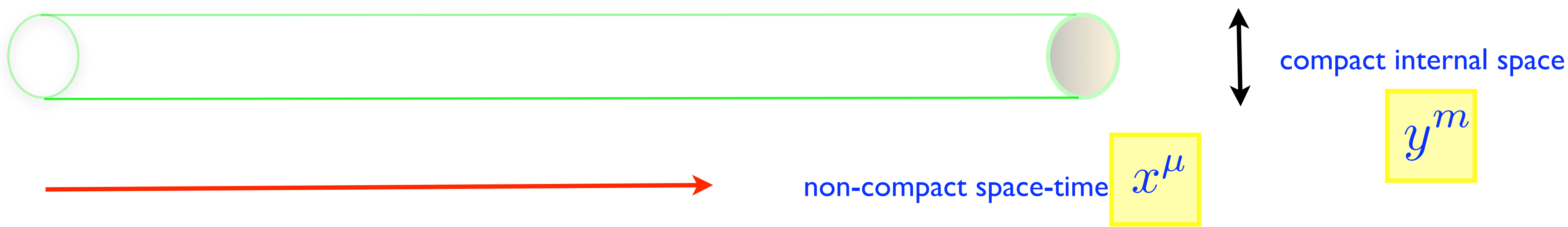
STRING THEORY provides new insights — Strominger, Vafa, 1996

Important: string theory lives in more than 4 space-time dimensions

Compactification of extra dimensions → Kaluza-Klein

- effective field theory determines/describes geometry of extra dimensions
- extended objects may get entangled in non-trivial cycles
- $\dfrac{R_{\rm hor}}{l_{\rm s}} \sim g_{\rm s}\, Q$
- two phases: $g_{\rm s}\, Q \gg 1$ macroscopic black holes

  $g_{\rm s}\, Q \ll 1$ microscopic black holes



space-time $\otimes$ internal (compact) space

FIELD THEORY $\Phi(x,y)$

massless fields $\Phi(x,y) = \phi(x)\,\eta(y)$

$$\Box_D \Phi = \Box_x \Phi + \Delta_y \Phi = 0$$

mass term

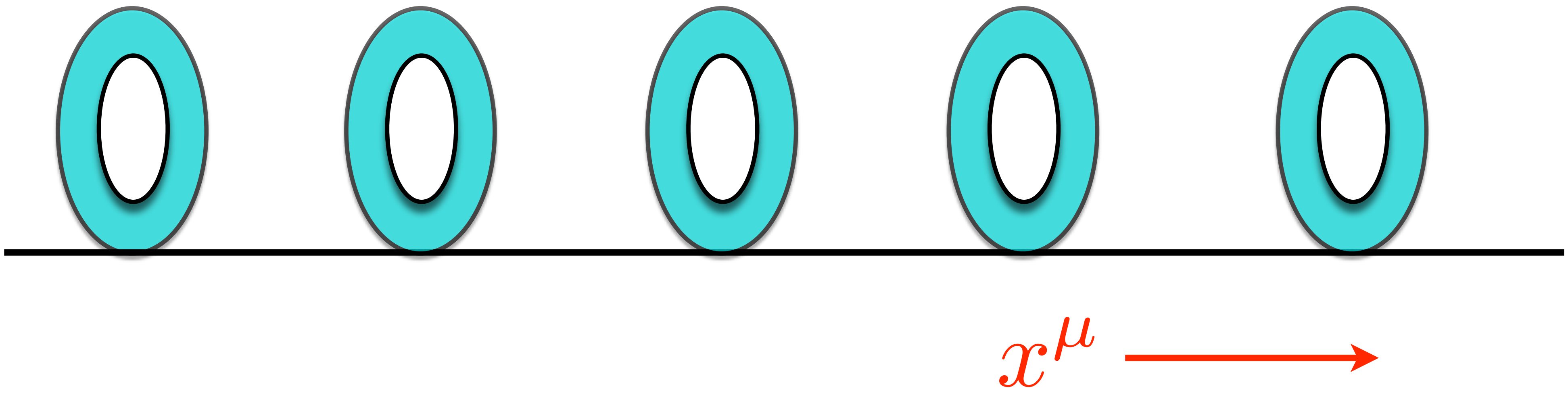
massless $\longleftrightarrow$ $\Delta_y \eta(y) = 0$ $\longleftrightarrow$ harmonic

- generically: masses of order $\frac{1}{L}$

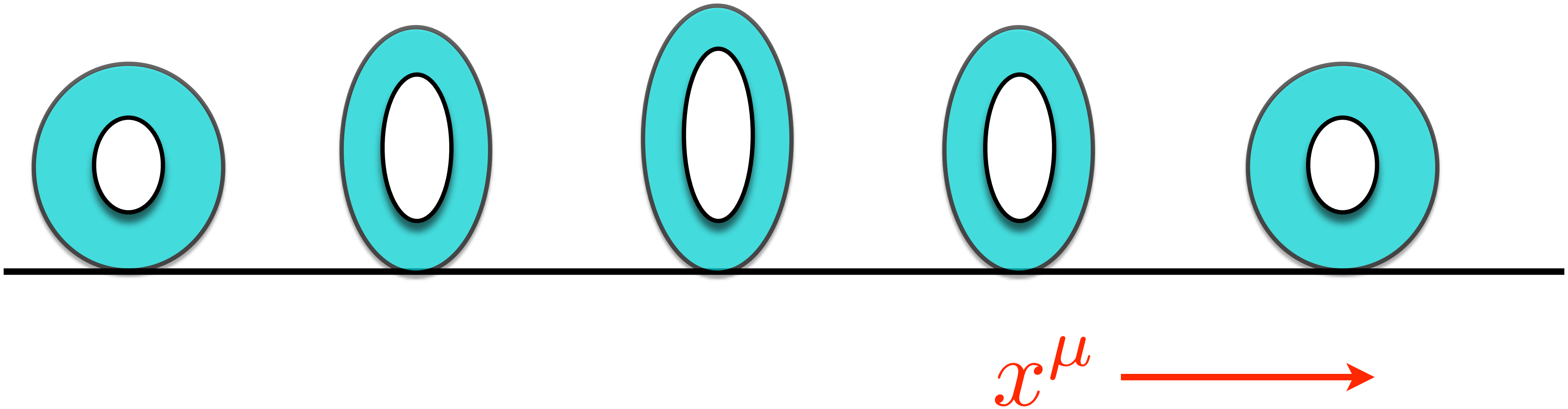
  where L is the size of the extra dimension

- This describes the local degrees of freedom à la Kaluza-Klein

- at each point in space-time there exists an internal manifold of the same topology



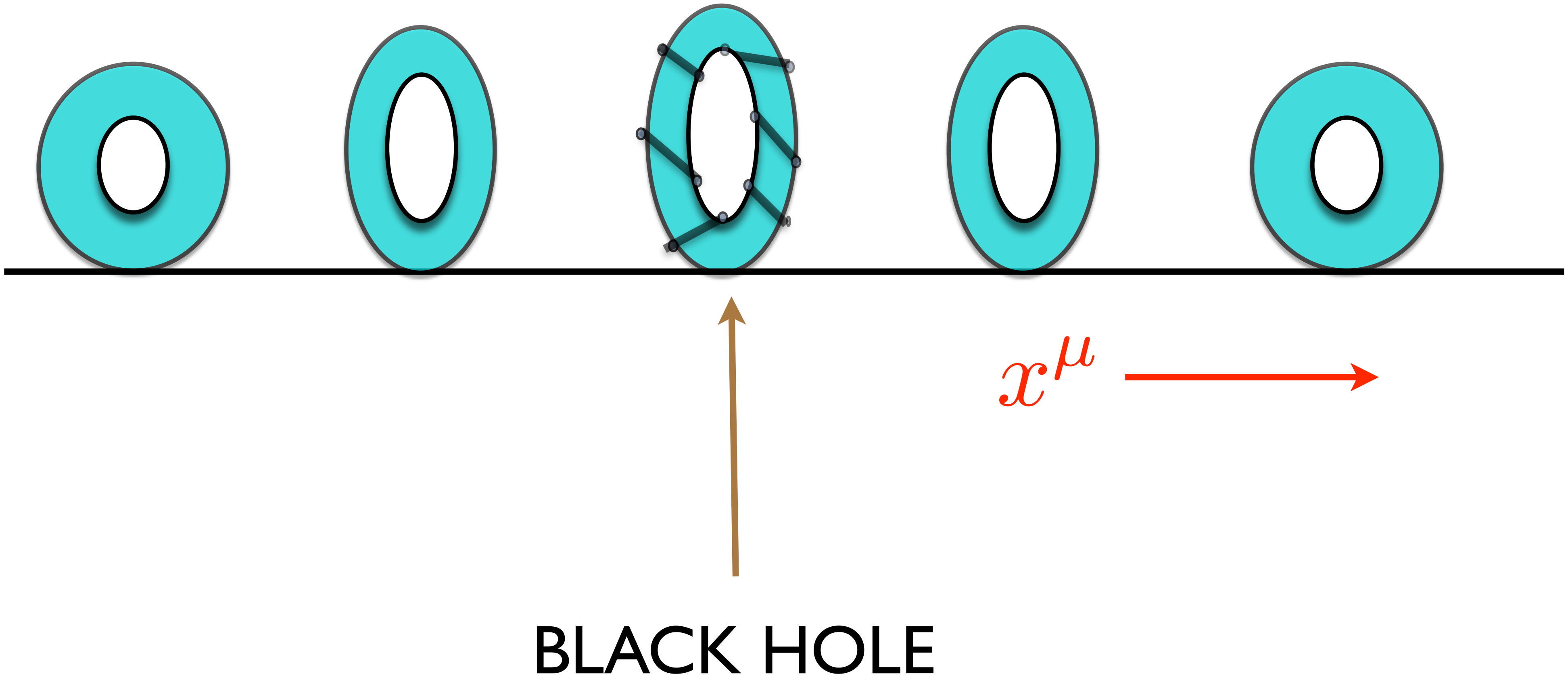
More realistically .... there is no reason why these tori should have the same shape and size !



Shape and size of the internal manifold encoded in the 4-dimensional fields

➔ curved space-time !

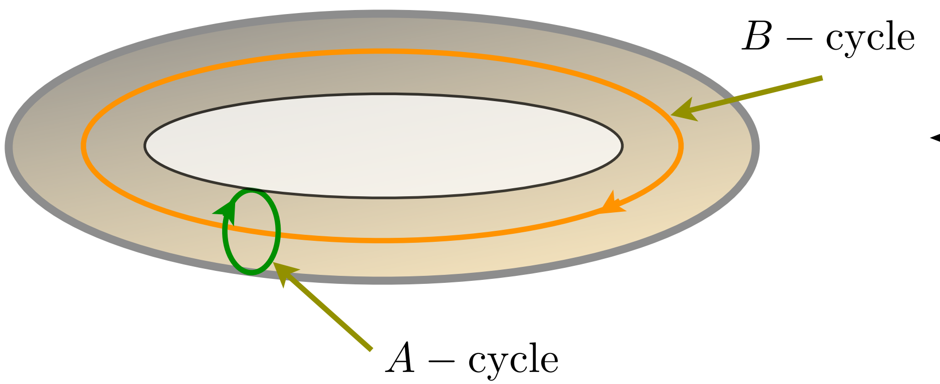
black hole ?


x^μ
BLACK HOLE

How can black holes arise as consistent solutions of the low-energy effective field theory ?

Poincaré duality provides a relation between the wrapping of extended objects around non-trivial cycles and the fields of the low-energy effective action

CYCLES dual to HARMONIC FORMS



harmonic one-forms

HOMOLOGY

CO-HOMOLOGY

FLAT space-time	CURVED space-time

EXTREMAL black hole $\Leftrightarrow$ 1/2 *BPS* black hole

residual supersymmetry leads to stable extrapolation in g_{s}

superstrings ⟵ SUPERSYMMETRY ⟶ supergravity

microscopic | macroscopic

microscopically

relevant degrees of freedom :

massless excitations of the wrapped *5-brane*

11-dimensional space-time $\to \mathrm{Mink}_4 \times \mathrm{CY}_3 \times S^1$

5-brane wrapped around P_4

momentum

p^A q_0

these degrees of freedom are described by a 2-dimensional (super)conformal theory on S^1

with central charge defined by the p^A

Apply Cardy's formula for degeneracy of states at large q_0

MICROSCOPIC ENTROPY

$$S_{\rm macro} = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\,(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A)}$$

triple intersection number — second Chern class

membrane charges : $\hat{q}_0 = q_0 - \frac{1}{2}C^{AB}q_A q_B$ $\qquad C_{AB} = C_{ABC}\,p^C$

$p^0 = 0$

c_{2A} *subleading correction !*

Maldacena, Strominger, Witten, 1997

Vafa, 1997

macroscopic

$\mathrm{CY}_3 \times S^1$: *32 → 8 supersymmetries*

→ *effective field theory N=2 supergravity*

winding	momentum	membrane charges
p^A	q_0	q_A
	p^0	

vector supermultiplets coupled to N=2 supergravity :

scalars: $X^I = X^0,\, X^A$ complex

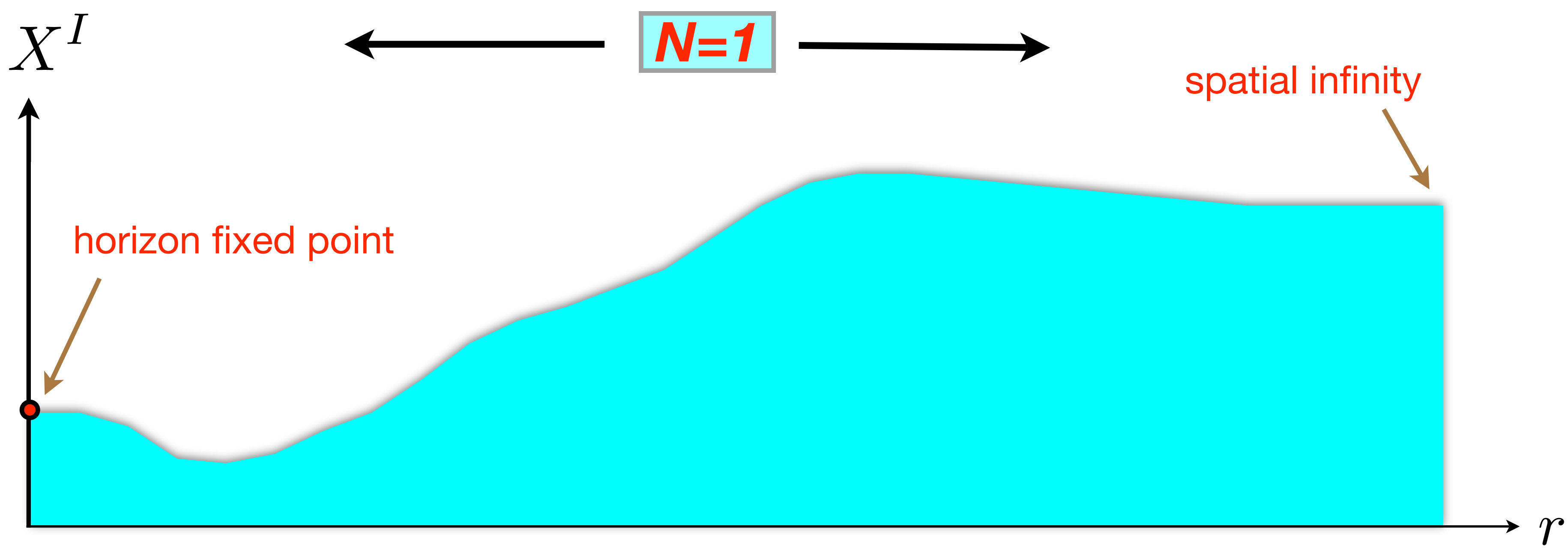
vectors: $W_\mu{}^I = W_\mu{}^0,\, W_\mu^A$ charges $p^0 \;\; q_0 \;\; p^A \; q_A$

classical solutions :

generalized (extremal) Reissner-Nordstrom black holes

how to calculate ?

solitonic solutions with residual N=1 supersymmetry (BPS)



N=2	**N=2**
horizon	spatial ∞
Bertotti-Robinson geometry	flat Minkowski space-time
$X^I\vert_{\mathrm{horizon}}$ determined by $p,\ q$	determines black hole mass

ATTRACTOR MECHANISM

Ferrara, Kallosh, Strominger, 1995

ATTRACTOR MECHANISM

behaviour at the horizon is determined by the charges

the $X^I(r)$ *parametrize the* CY_3 (projectively)

which change as a function of r

(special Kähler geometry)

HOW TO CALCULATE ?

N=2 supergravity Lagrangians are encoded in
holomorphic homogeneous functions:

$$F(\lambda X) = \lambda^2 F(X)$$

At the horizon, use rescaled variables $X^I \longrightarrow Y^I$
reduces to an algebraic problem

attractor equations

$$Y^I - \bar{Y}^I = \mathrm{i}\, p^I$$
$$F_I(Y) - \bar{F}_I(\bar{Y}) = \mathrm{i}\, q_I$$

$$F_I(Y) \equiv \partial_I F(Y)$$

$$\frac{\text{AREA}}{G_\mathrm{N}} = 4\pi\, |Z|^2$$ with $$|Z|^2 = p^I F_I(Y) - q_I\, Y^I$$

EXAMPLE $$F(Y) = -\frac{1}{6}\,\frac{C_{ABC}\,Y^A Y^B Y^C}{Y^0}$$

AREA LAW ⇒ entropy ⇒

$$S_{\text{macro}}(p,q) = \pi\,|Z|^2 = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\,C_{ABC}\,p^A p^B p^C}$$

this represents the leading term of the microscopic result :
it scales quadratically in the charges!

Subleading corrections:

extend with one ‘extra’ complex field, originating from pure supergravity.

Υ

homogeneity preserved:

$$F(\lambda Y, \lambda^2 \Upsilon) = \lambda^2 \, F(Y, \Upsilon)$$

dependence leads to terms $\propto \, ({R_{\mu\nu\rho}}^{\sigma})^2$ in effective field theory

attractor mechanism: at the horizon $\Upsilon = -64$

EXAMPLE:

$$F(Y, \Upsilon) = -\frac{1}{6} \, \frac{C_{ABC} \, Y^A Y^B Y^C}{Y^0} - \frac{c_{2A} \, Y^A}{24 \cdot 64 \, Y^0} \, \Upsilon$$

to ensure the validity of the first law of black hole mechanics, one must modify the definition of black hole entropy.

INSTEAD:
use Wald's prescription based on a Noether surface charge.

Wald, 1993

general N=2 formula:

$$S_{\rm macro} = \pi\,|Z|^2 - 256\,{\rm Im}\,F_\Upsilon(Y,\Upsilon)\Big|_{\Upsilon=-64}$$

$|Z|^2 \to \dfrac{\rm AREA}{4\,G_{\rm N}}$ ${\rm Im}\,F_\Upsilon \to$ modification

leads indeed to the microscopic result

$$S_{\rm macro} = 2\pi\sqrt{\frac{1}{6}|\hat q_0|\,(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A)}$$

full agreement!

Cardoso, de Wit, Mohaupt, 1998

violates the Bekenstein-Hawking area law !

$$\frac{A(p,q)}{4\,G_{\rm N}} = \frac{C_{ABC}\,p^A p^B p^C + \frac{1}{2} c_{2A} p^A}{C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A}\; S_{\rm macro}(p,q)$$

LARGE black holes: $C_{ABC}\,p^A p^B p^C \neq 0$

finite classical area and entropy $S = \mathcal{O}(Q^2)$ $R_{\rm hor} \gg l_{\rm s}$

SMALL black holes: $C_{ABC}\,p^A p^B p^C = 0$

vanishing classical area and finite entropy $S = \mathcal{O}(Q)$ $R_{\rm hor} \approx l_{\rm s}$

Indicative of a more general situation: BPS entropy function

entropy function $\Sigma(Y, \bar{Y}, p, q)$

stationary $\partial_Y \Sigma(Y, \bar{Y}, p, q) = 0 \;\rightarrow\; Y^*(p,q)$ (attractor equations)

$$S_{\rm macro} = \pi\, \Sigma\Big|_{Y^*(p,q)}$$

variational principle !

How to further explore the relation between macroscopic and microscopic descriptions ?

→ Consider heterotic black holes

Heterotic black holes

on T^6*, dual to type-IIA on* $K3 \times T^2$

also for CHL black holes *Chaudhuri, Hockney, Lykken,1995*

N=4 supersymmetry, T- and S-duality

classical result : $S_{\mathrm{macro}} = \dfrac{A}{4\,G_{\mathrm{N}}} = \pi\sqrt{q^2p^2-(q\cdot p)^2}$

$q^2 \quad p^2 \quad p\cdot q$ T-*duality invariant bilinears*

Cvetic, Tseytlin, 1995

Bergshoeff et. al.,1996

two types of BPS states :

1/4 BPS *'dyonic'* $q^2p^2-(p\cdot q)^2>0$

1/2 BPS *'electric'* $q^2p^2-(p\cdot q)^2=0$

↓ zero classical area

microscopic results

1/4 BPS states

dyonic degeneracies
$$d_k(p,q) = \oint_{3-\text{cycle}} \mathrm{d}\Omega \, \frac{\mathrm{e}^{\mathrm{i}\pi[\rho\, p^2 + \sigma\, q^2 + (2\upsilon - 1) p\cdot q]}}{\Phi_k(\Omega)}$$

$k = 10, 6, 4, 2, 1$

$\Omega = \begin{pmatrix} \rho & \upsilon \\ \upsilon & \sigma \end{pmatrix}$ period matrix of g=2 Riemann surface

formally S-duality invariant

Dijkgraaf, Verlinde, Verlinde, 1997

Shih, Strominger, Yin, 2005

Jatkar, Sen, 2005

1/2 BPS states

electric degeneracies
$$d_k(q) = \oint_{1-\text{cycle}} \mathrm{d}\sigma \; \sigma^{k+2} \, \frac{\mathrm{e}^{\mathrm{i}\pi\, q^2 \sigma}}{f^{(k)}(-1/\sigma)}$$

related to perturbative string states

Dabholkar, Harvey, 1989

Asymptotic growth (*1/4 BPS*)

for 'large' dyonic charges $q^2,\, p^2,\, p\cdot q$

dilaton S can remain finite

Cardoso, dW, Käppeli, Mohaupt, 2004 $(k=10)$

Jatkar, Sen, 2005 $(k=6,4,2,1)$

saddle-point approximation reproduces the macroscopic results based up to terms inversely proportional to the charges !

Corresponding entropy function:

Cardoso, de Wit, Mohaupt, 1999

$$\Sigma(S,\bar S,p,q) = -\frac{q^2 - \mathrm{i}p\cdot q(S-\bar S) + p^2|S|^2}{S+\bar S} + 4\,\Omega(S,\bar S,\Upsilon,\bar\Upsilon)$$

$$\Omega = \frac{1}{128\pi}\Big[\Upsilon\,\log\eta^{12}(S) + \bar\Upsilon\,\log\eta^{12}(\bar S) + \tfrac{1}{2}(\Upsilon+\bar\Upsilon)\,\log(S+\bar S)^6\Big]$$

non-holomorphic

this reproduces instanton corrections and non-holomorphic terms!

CONCLUSIONS

- *The statistical interpretation of black hole entropy as inspired by string theory can successfully describe, both qualitatively and quantitatively, the black hole degrees of freedom.*

- *At the moment there exists a large variety of models where this interpretation applies, in both four- and five-dimensional space-times.*

- *Many open questions remain. For instance, can these results be extended beyond the leading and subleading level? Or can one also consider more `realistic' black holes? And what about non-BPS and/or non-extremal black holes?*

- *In other words, do we really understand why things work so well?*