ON SHELL OR OFF SHELL

IS THAT THE QUESTION?

Toine 60 : Leuven, March 16 2013

Bernard de Wit
Nikhef Amsterdam
Utrecht University

Instituut voor
Theoretische Fysika

Katholieke
Universiteit
Leuven

April 1977

Preprint–KUL–TF–77/010

FUNCTIONAL METHODS, RENORMALIZATION,
AND SYMMETRY IN QUANTUM FIELD THEORY*

B. de Wit
Instituut-Lorentz, University of Leiden,
Leiden, The Netherlands

Notes taken and edited by A. Van Proeyen**

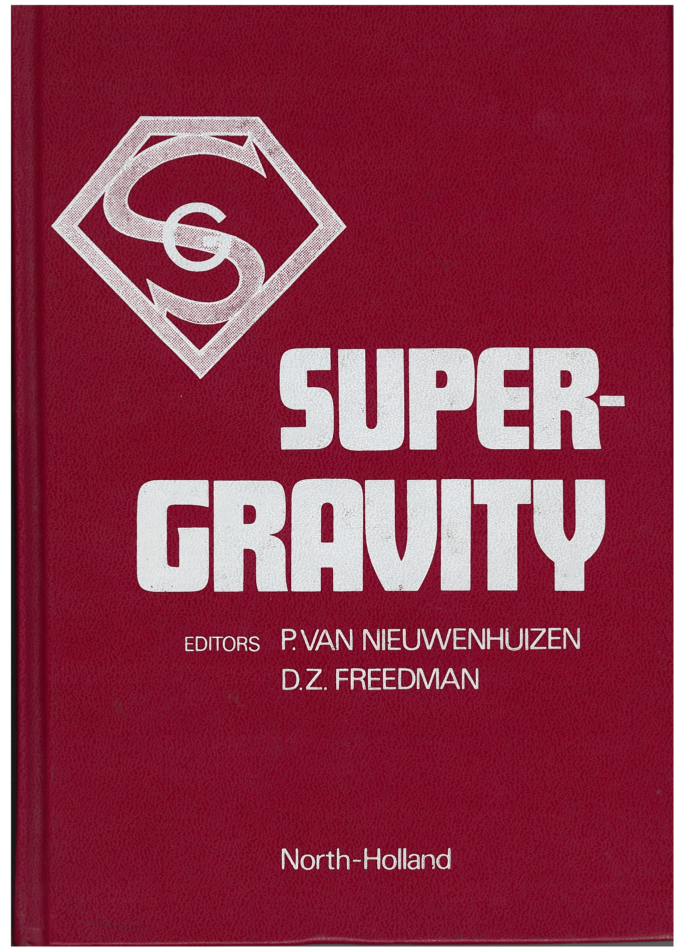
* Lectures given at the Instituut voor Theoretische Fysica, Universiteit Leuven, B-3030 Leuven, België, in the 3rd cycle program "Veldentheorie en Statistische Mechanica", sponsored by the N.F.W.O.

** Aspirant N.F.W.O. (Belgium)

How we met...

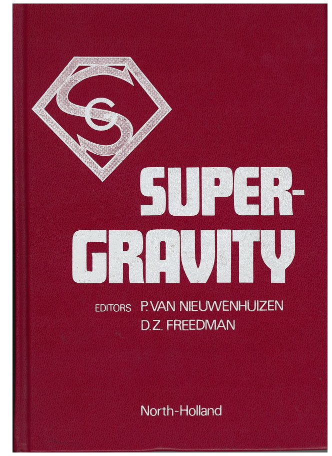
In 1976 I was invited to give a series of lectures at the University in Leuven. Toine was proposed by his advisor, Raymond Gastmans, as a talented student who could take notes. This marked the beginning of our long and fruitful collaboration.

I had been working in supersymmetry since 1974, but this topic was not included in my lectures.
In those days Toine was working for his PhD on quantum gravity and more phenomenological subjects. I remained in contact with him and with Leuven in the years that followed.



In 1979 Toine started to work on supersymmetry and visited Stony Brook. He returned in September of the same year to attend the first supergravity workshop ever! His first paper on supersymmetry was published shortly thereafter.

In the same year Jan-Willem van Holten and I asked Toine to collaborate on unraveling and expanding the off-shell structure of N=2 supergravity that we had just discovered. This was, and still is, a very fruitful area of research. Toine agreed. As a result Toine and I, often with others, wrote more than 12 papers on this topic during the period 1980-1985.



The year 1979 marked Toine's start in the off-shell world!

One special paper and a confession:

Structure of N=2 supergravity - Nucl. Phys. B184 (1981) 77
B. de Wit, J.W. van Holten (CERN) and A. Van Proeyen (Leuven U.)

Where was Toine in those days?
Not in Leuven!

In those days Toine was in the army, calculating fallout patterns for the case of a nuclear attack on Belgium. Military service is mandatory in Belgium!
However, when his superiors were not paying attention he was just working on this paper.

A late apology: we did not include the army in our acknowledgement.

The work with Toine opened the way to many applications: effective actions, gaugings, hidden symmetries, moduli fixing, flat potentials, black holes, etc.

At an early stage Toine already left the familiar world of 4 space-time dimensions, and used the same approach to investigate off-shell issues in 5 and 6 dimensions.

Supersymmetry with 8 supercharges became fashionable and was an ideal testing ground for many new ideas!

One needs a well developed formalism. For supergravity his is provided by the *superconformal calculus*.

- off-shell irreducible supermultiplets
  in superconformal gravity background
- extra superconformal gauge invariances
- gauge equivalence (compensating supermultiplets)

Later on, during 1990-1995, we joined forces again on what happens to duality symmetries upon dimensional reduction, following earlier work by Cecotti, Ferrara, Girardello and others. Especially the following papers were relevant in this respect:

Symmetries of dual quaternionic manifolds - Phys. Lett. B252 (1990) 221
B. de Wit (Utrecht U.), A. Van Proeyen (CERN)

Special geometry, cubic polynomials and homogeneous quaternionic spaces
Commun. Math. Phys. 149 (1992) 307
B. de Wit (CERN), A. Van Proeyen (Leuven U.)

Symmetry structure of special geometries - Nucl. Phys. B400 (1993) 463
B. de Wit (Utrecht U.), F. Vanderseypen, A. Van Proeyen (Leuven U.)

Two relevant operations:

c-map : from 4 to 3 space-time dimensions

r-map : from 5 to 4 space-time dimensions

Mathematically:

real geometry $\longrightarrow$ **Kähler geometry** $\longrightarrow$ **Quaternionkähler geometry**

However, this last work was not done in an off-shell form! Furthermore it was directed towards theories without higher space-time derivatives. The latter appear as corrections to effective actions and are eventually important.

Nowadays we need more precise results, for instance, in the study of black hole entropy and the comparison to black hole microstates beyond the leading order (i.e. large charges - thermodynamic limit). Also in the discussion of possible finiteness of certain supergravities, more information is welcome.

Dimensional reduction is a subtle affair in a more general context. First of all, the geometric features of theories in different dimensions can be completely different. In odd dimensions one can, for instance, have Chern-Simons terms which do not appear in even dimensions. Furthermore, upon dimensional reduction invariants do not necessarily have to remain irreducible!

5D ➜ 4D

Banerjee, dW, Katmadas, 2011

The *4D/5D* connection is exact up to topological features related to the *5D* Chern-Simons terms.

The off-shell *4D/5D* connection also enables one to understand why the field equations of the *5D* and the *4D* theory are different (in the bulk), as has been observed in the literature.

Upon dimensional reduction the invariant Lagrangians with at most two derivatives lead to the corresponding *4D* Lagrangians. There will be no qualitative differences.

However, the higher-order derivative term in *5D* leads upon reduction to three separate *4D* invariant Lagrangians.

One of these Lagrangians is known to give no contribution to the entropy and the electric charges of *4D* BPS black holes, by virtue of a non-renormalization theorem.

dW, Katmadas, van Zalk, 2010

Off-shell dimensional reduction

Weyl multiplet, vector multiplet, hypermultiplet

First Kaluza-Klein ansatz with gauge choices:

$$e_M{}^A = \begin{pmatrix} e_\mu{}^a & B_\mu \phi^{-1} \\ 0 & \phi^{-1} \end{pmatrix} , \qquad e_A{}^M = \begin{pmatrix} e_a{}^\mu & -e_a{}^\nu B_\nu \\ 0 & \phi \end{pmatrix} , \qquad b_M = \begin{pmatrix} b_\mu \\ 0 \end{pmatrix}$$

There are compensating transformations to preserve this form.

Upon the reduction the *5D* Weyl multiplet decomposes into the *4D* Weyl multiplet and a Kaluza-Klein vector multiplet.

Continue with the other fields and then consider the action of the five-dimensional supersymmetry transformations.

In this case, there is no conflict between conformal invariance and dimensional reduction.

The supersymmetry transformations turn out to satisfy a uniform decomposition:

$$\delta_{\mathrm{Q}}(\epsilon)\big|_{5D}^{\mathrm{reduced}}\Phi = \delta_{\mathrm{Q}}(\epsilon)\big|_{4D}\Phi + \delta_{\mathrm{S}}(\tilde{\eta})\big|_{4D}\Phi + \delta_{\mathrm{SU}(2)}(\tilde{\Lambda})\big|_{4D}\Phi + \underbrace{\boxed{\delta_{\mathrm{U}(1)}(\tilde{\Lambda}^0)\big|_{4D}}}_{\text{compensating transformation}}\Phi\,.$$

The Kaluza-Klein vector multiplet has a real scalar field, rather than a complex one. Moreover, the R-symmetry remains restricted to $\mathrm{SU}(2)$ rather than extending to $\mathrm{SU}(2)\times\mathrm{U}(1)$!

Explanation: It turns out the the result takes the form of a gauge-fixed version with respect to $\mathrm{U}(1)$!

Note: not a new feature! This was also employed in the consistency proof of the seven-sphere reduction to *N*=*8* supergravity.

dW, Nicolai, 1986

Weyl multiplet 5D ➡ 4D (reducible)

$$\boxed{\begin{aligned}\phi &= 2|X^0| \\ B_\mu &= W_\mu{}^0\end{aligned}}$$

KK vector multiplet missing pseudoscalar

$$V_{Mi}{}^j = \begin{cases} V_{\mu i}{}^j = \mathcal{V}_\mu{}^j{}_i - \frac{1}{4}\varepsilon_{ik}\, Y^{kj\,0}\, |X^0|^2 W_\mu{}^0 \\ V_{5i}{}^j = -\frac{1}{4}\varepsilon_{ik}\, Y^{kj\,0}|X^0|^2 \end{cases}$$

$$T_{a5} = \tfrac{1}{12}\mathrm{i}e_a{}^\mu\Big(\frac{\mathcal{D}_\mu X^0}{X^0} - \frac{\mathcal{D}_\mu \bar{X}^0}{\bar{X}^0}\Big)$$

$$T_{ab} = -\frac{\mathrm{i}}{24\,|X^0|}\Big(\varepsilon_{ij}T_{ab}{}^{ij}\,\bar{X}^0 - F_{ab}^-(B)\Big) + \mathrm{h.c.}$$

gauge fixed formulation w.r.t. *U(1)*

vector multiplet 5D ➡ 4D

$$\sigma = -\mathrm{i}|X^0|\,(t-\bar{t})$$

$$W_M = \begin{cases} W_\mu = W_\mu - \frac{1}{2}(t+\bar{t})\,{W_\mu}^0 \\ W_5 = -\frac{1}{2}(t+\bar{t}) \end{cases}$$

$$Y^{ij} = -\frac{1}{2}Y^{ij} + \frac{1}{4}(t+\bar{t})\,Y^{ij\,0}$$

$$\boxed{t = \frac{X}{X^0}}$$

5D vector multiplet action

$$8\pi^2\mathcal{L}_{\mathrm{vvv}} = 3\,E\,C_{ABC}\,\sigma^A\Big[\tfrac{1}{2}\mathcal{D}_M\sigma^B\,\mathcal{D}^M\sigma^C + \tfrac{1}{4}F_{MN}{}^B F^{MNC} - Y_{ij}{}^B Y^{ijC} - 3\,\sigma^B F_{MN}{}^C T^{MN}\Big]$$
$$-\tfrac{1}{8}\mathrm{i}\,C_{ABC}\varepsilon^{MNPPQR}\,W_M{}^A F_{NP}{}^B F_{QR}{}^C$$
$$-E\,C_{ABC}\sigma^A\sigma^B\sigma^C\,\Big[\tfrac{1}{8}R - 4\,D - \tfrac{39}{2}T^{AB}T_{AB}\Big]$$

5D higher-derivative action

$$8\pi^2\,\mathcal{L}_{\mathrm{vww}} = \tfrac{1}{4}\,E\,c_A Y_{ij}{}^A\,T^{CD}\,R_{CDk}{}^j(V)\,\varepsilon^{ki}$$
$$+E\,c_A\sigma^A\,\Big[\tfrac{1}{64}R_{CD}{}^{EF}(M)\,R_{EF}{}^{CD}(M) + \tfrac{1}{96}R_{MNj}{}^i(V)\,R^{MN}{}_i{}^j(V)\Big]$$
$$-\tfrac{1}{128}\mathrm{i}\varepsilon^{MNPQR}\,c_A W_M{}^A\,\,\big[R_{NP}{}^{CD}(M)\,R_{QRCD}(M) + \tfrac{1}{3}\,R_{NPj}{}^i(V)\,R_{QRi}{}^j(V)\big]$$
$$+\tfrac{3}{16}E\,c_A\big(10\,\sigma^A T_{BC} - F_{BC}{}^A\big)R(M)_{DE}{}^{BC}\,T^{DE} + \cdots$$

Chern-Simons terms!

yields the following 4D actions:

$$F(X,(T_{ab}{}^{ij}\varepsilon_{ij})^2) = -\frac{1}{2}\frac{C_{ABC}X^AX^BX^C}{X^0} - \frac{1}{2048}\,\frac{c_A\,X^A}{X^0}\,(T_{ab}{}^{ij}\varepsilon_{ij})^2$$

corresponding to the chiral superspace density invariant

Bergshoeff, de Roo, dW, 1981

$$\mathcal{H}(X,\bar{X}) = \tfrac{1}{384}\mathrm{i}c_A\left(\frac{X^A}{X^0}\,\ln\bar{X}^0 - \frac{\bar{X}^A}{\bar{X}^0}\,\ln X^0\right)$$

corresponding to a full superspace density invariant, described in terms of a Kähler potential. This class of actions does not contribute to the electric charges and the entropy of BPS black holes.

dW, Katmadas, van Zalk, 2010

Supersymmetric *4D* Lagrangian arising from dimensional reduction of five-dimensional supergravity:

$$
\begin{aligned}
&e^{-1}\mathcal{L} = \\
&\mathcal{H}_{IJ\bar{K}\bar{L}}\Big[\tfrac{1}{4}\big(F_{ab}^{-\,I}\,F^{-ab\,J} - \tfrac{1}{2}Y_{ij}{}^{I}\,Y^{ij\,J}\big)\big(F_{ab}^{+\,K}\,F^{+ab\,L} - \tfrac{1}{2}Y^{K\,ij}\,Y^{L}{}_{ij}\big) \\
&\qquad +4\,\mathcal{D}_aX^I\,\mathcal{D}_b\bar{X}^K\big(\mathcal{D}^aX^J\,\mathcal{D}^b\bar{X}^L + 2\,F^{-\,ac\,J}\,F^{+\,b}{}_c{}^L - \tfrac{1}{4}\eta^{ab}\,Y_{ij}^J\,Y^{L\,ij}\big)\Big] \\
&+\Big\{\mathcal{H}_{IJ\bar{K}}\Big[4\,\mathcal{D}_aX^I\,\mathcal{D}^aX^J\,\mathcal{D}^2\bar{X}^K - \big(F^{-ab\,I}\,F_{ab}^{-\,J} - \tfrac{1}{2}Y_{ij}^I\,Y^{Jij}\big)\big(\Box_{\rm c}X^K + \tfrac{1}{8}F_{ab}^{-\,K}\,T^{abij}\varepsilon_{ij}\big) \\
&\qquad +8\,\mathcal{D}^aX^IF_{ab}^{-\,J}\big(D_cF^{+\,cb\,K} - \tfrac{1}{2}\mathcal{D}_c\bar{X}^KT^{ij\,cb}\varepsilon_{ij}\big) - \mathcal{D}_aX^I\,Y_{ij}^J\,\mathcal{D}^aY^{K\,ij}\Big] + \text{h.c.}\Big\} \\
&+\mathcal{H}_{I\bar{K}}\Big[4\big(\Box_{\rm c}\bar{X}^I + \tfrac{1}{8}F_{ab}^{+\,I}\,T^{ab}{}_{ij}\varepsilon^{ij}\big)\big(\Box_{\rm c}X^K + \tfrac{1}{8}F_{ab}^{-\,K}\,T^{abij}\varepsilon_{ij}\big) + 4\,\mathcal{D}^2X^I\,\mathcal{D}^2\bar{X}^K \\
&\qquad +8\,\mathcal{D}_aF^{-\,abI}\,\mathcal{D}_cF^{+c}{}_b{}^K - \mathcal{D}_aY_{ij}{}^I\,\mathcal{D}^aY^{K\,ij} + \tfrac{1}{4}T_{ab}{}^{ij}\,T_{cdij}\,F^{-ab\,I}F^{+cd\,K} \\
&\qquad +\big(\tfrac{1}{6}R(\omega,e) + 2\,D\big)Y_{ij}{}^I\,Y^{ij\,K} + 4\,F^{-ac\,I}\,F^{+}{}_{bc}{}^K\,R(\omega,e)_a{}^b \\
&\qquad -\big[\varepsilon^{ik}\,Y_{ij}{}^I\,F^{+ab\,K}\,R(\mathcal{V})_{ab}{}^j{}_k + \text{h.c.}\big] \\
&\qquad -\big[\mathcal{D}_c\bar{X}^K\big(D^cT_{ab}{}^{ij}\,F^{-\,I\,ab} + 4\,T^{ij\,cb}\,D^aF_{ab}^{-\,I}\big)\varepsilon_{ij} + \text{h.c.}\big] \\
&\qquad +8\big(R^{\mu\nu}(\omega,e) - \tfrac{1}{3}g^{\mu\nu}R(\omega,e) + \tfrac{1}{4}T^{\mu}{}_b{}^{ij}\,T^{\nu b}{}_{ij} + \mathrm{i}R(A)^{\mu\nu} - g^{\mu\nu}D\big)\mathcal{D}_\mu X^I\,\mathcal{D}_\nu\bar{X}^K\Big]
\end{aligned}
$$

Surprise: there is one more higher-derivative coupling in *4D* emerging from *5D* !

Some characteristic terms:

$$
\begin{aligned}
8\pi^2\,\mathcal{L}_{\rm vww} \;\rightarrow\; & -\tfrac{1}{384}\mathrm{i}c_A t^A\left[\tfrac{2}{3}\mathcal{R}_{ab}\mathcal{R}^{ab} + R(\mathcal{V})^{+i}_{ab\,j}\,R(\mathcal{V})^{+abj}{}_i\right] \\
& -\tfrac{1}{768}\mathrm{i}c_A(t^A-\bar{t}^A)\,(X^0)^{-1}\varepsilon_{ij}T^{cdij}\,R(M)^{ab}{}_{cd}\,F^{-0}_{ab} \\
& +\mathrm{h.c.}
\end{aligned}
$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor

Its structure was not fully known at the time. Neither is it known whether this coupling is subject to some non-renormalization theorem. The *N*=2 supersymmetric Gauss-Bonnet invariant is related to this class.

Butter, dW, Lodato, Kuzenko, in preparation

4D ➜ 3D

Banerjee, dW, Katmadas, in progress

This is related to the so-called ***c-map***, which from the perspective of *type-II string theory* is related to *T*-duality.

The off-shell dimensional reduction method can be used in many situations. For instance, in performing the *c-map* of the topological string. Or, upon reducing with respect to time rather than a spatial coordinate, to study black holes in *3D* Euclidean space with higher-derivative couplings.

Off-shell c-map: 4D ➡ 3D

Banerjee, dW, Katmadas, in progress

The same elements as in the *4D/5D* connection:

✦ R-symmetry: $\mathrm{SU}(2)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}} \longrightarrow \mathrm{SU}(2)_{\mathrm{V}} \times \mathrm{SU}(2)_{\mathrm{A}}$

De Jaegher, dW, Kleijn, Vandoren, 1998
dW, Saueressig, 2006

✦ New gauge connections: $A_\mu^{\pm} \propto T_a{}^{\pm} + \cdots$

✦ Include local phase factor to generate corresponding gauge transformations.

✦ Uniform decomposition rule of local supersymmetry.

✦ Spinor conversion: $\psi^i\big|_{4D} \rightarrow \psi^{i\,p}\big|_{3D}$
subject to the condition $C^{-1}\bar{\psi}_{i\,p} = \varepsilon_{ij}\,\varepsilon_{pq}\,\psi^{j\,q}$

✦ The Weyl multiplet decomposes into the *3D* Weyl multiplet and a vector multiplet.

3D Multiplets

c-map

Weyl multiplet
$(14+2)\oplus(8+8)$

$$\boxed{e_\mu{}^a \quad \psi_\mu{}^{i\,p} \quad V_\mu{}^i{}_j \leftrightarrow A_\mu{}^p{}_q \qquad A \quad \chi^{i\,p} \quad D} \quad \boxed{\times(-1)}$$

Vector multiplet
$8\oplus 8$

$$X^p{}_q \quad \Omega^{i\,p} \quad F^\mu \quad Y^i{}_j \qquad (D_\mu F^\mu = 0)$$

$$\updownarrow$$

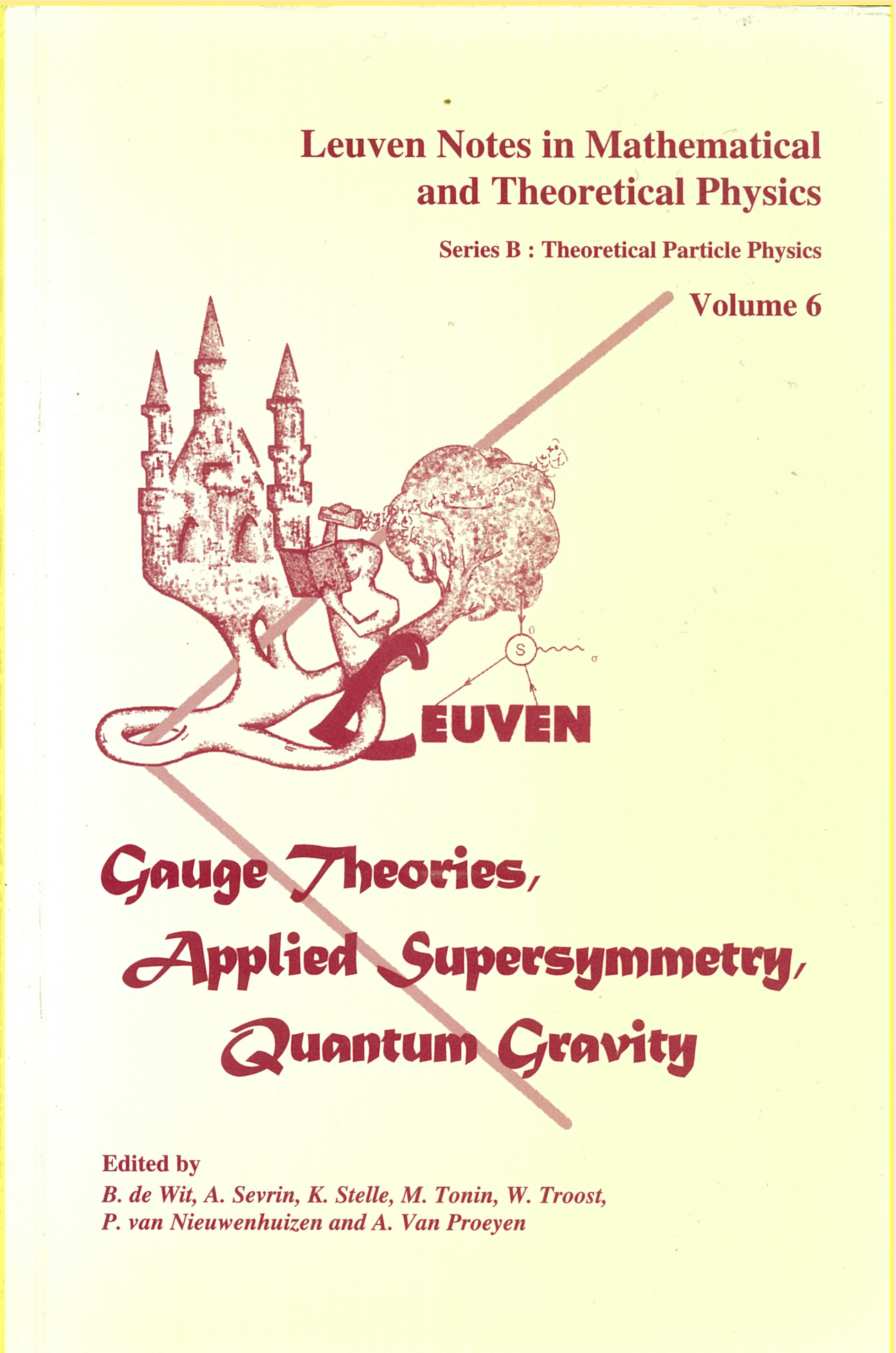
Tensor multiplet
$8\oplus 8$

$$L^i{}_j \quad \varphi^{i\,p} \quad E^\mu \quad G^p{}_q \qquad (D_\mu E^\mu = 0)$$

Hypermultiplets follow by dualizing the vector/tensor of the tensor and the vector multiplets to a scalar.

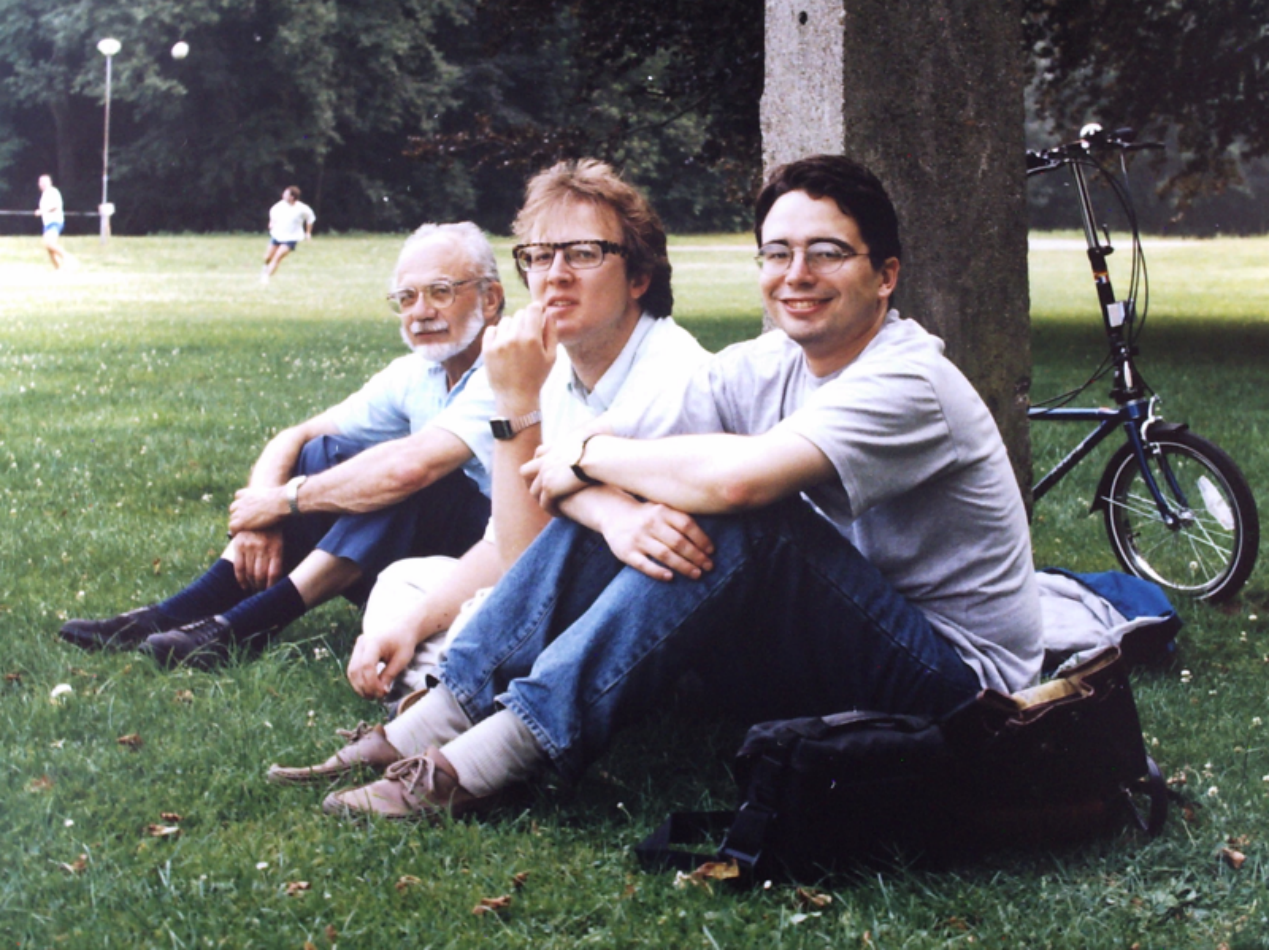
The Weyl multiplet has previously been determined at the linearized level but not from higher dimensions.

Bergshoeff, Hohm, Rosseel, Townsend, 2010



July 10-14 1995

A European Network meeting
in Leuven
with the traditional soccer match



The Referee

Experts?

Scepticism?

Happy Birthday Toine!

and many happy returns!