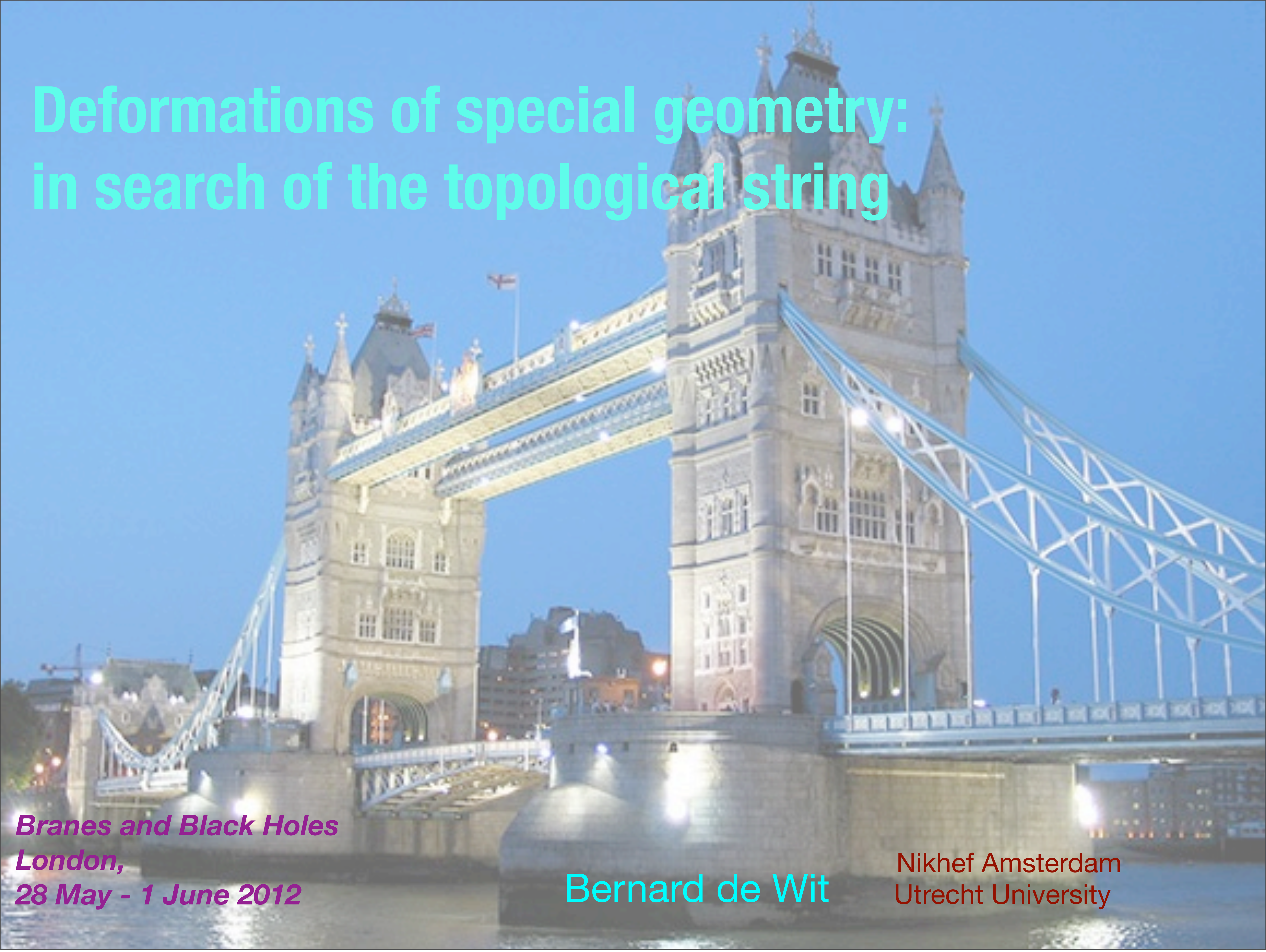
Deformations of special geometry: in search of the topological string



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SPECIAL GEOMETRY

N=2, d=4 vector supermultiplets: *(rigid or local supersymmetry)*

E/M duality (symplectic equivalence) (Wilsonian effective action)

Lagrangian encoded in a holomorphic homogeneous function $F(X)$

period vector $X^I,\ F_I = \partial_I F$

$I = 0, 1, \ldots, n$

dW, Van Proeyen,1984

Cecotti, Ferrara, Girardello, 1989

Calabi-Yau moduli spaces

homology cycles (symplectic equivalence)

holomorphic three-form Ω with periods

$$X^I = \int_{A^I} \Omega \qquad F_I = \int_{B_I} \Omega$$

$I = 0, 1, \ldots, b_{21}$

Strominger,1990

Deformations and Perspectives

Supergravity effective action

Higher-derivative couplings: mainly chiral (‘*F*-terms’)

Non-holomorphic corrections: from integrating out massless modes

Dixon, Kaplunovsky, Louis, 1991

Topological string

Genus-*g* partition functions of a twisted non-linear sigma model with a *CY* target space

Holomorphic anomaly: pinched cycles of the Riemann surface

Bershadsky, Cecotti, Ooguri, Vafa, 1994

The two are related through string theory!

The partition functions capture certain string amplitudes, which can also be reproduced by certain (supergravity) couplings.

Antoniadis, Gava, Narain, Taylor,1993

And, more recently, in the context of BPS black hole entropy.

Ooguri, Strominger, Vafa. 2004

How to explore/understand this relation ?

Central question *can there exist a single (homogeneous) function that encodes both the effective action and the topological string?*

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g}\Upsilon^g\, F^{(g)}(t)$$

$(Y^0)^2 F^{(0)}(t)$

$t^A = Y^A/Y^0$

genus-*g* partition function of a twisted non-linear sigma model with CY target space

Y^0 loop-counting parameter: $Y^0 = g_{\rm top}^{-1}$

To answer this question it is helpful to consider the characteristic features of the above expansion when interpreted as the function that encodes either the effective action, or the topological string partition function.

The key feature here is the behaviour under duality

Cardoso, dW, Mahapatra, 2008

effective action

- the $F^{(g)}$ are NOT invariant
- the periods transform correctly under monodromies
- the duality transformations are Υ-dependent

topological string

- the $F^{(g)}$ are COVARIANT sections
- the periods refer to $F^{(0)}$
- the duality transformations are Υ-independent

Nevertheless, both must be related to the same string amplitudes!

Lagrangian $\mathcal{L}$ ⟷ Hamiltonian $\mathcal{H}$

↑ ↑

$\mathrm{Im}[F(Y,\Upsilon)]$ ⟷ Hesse potential $\mathcal{H}$

Legendre transform

In the spirit of a general theorem (to be presented later)

First test: the FHSV Model

require covariance under S- and T-duality of the periods

expand $\Omega \approx \mathrm{Im}[F(Y,\Upsilon)]$ in Υ starting from:

$$\Omega^{(1)}(S,\bar S,T,\bar T,\Upsilon,\bar\Upsilon) = \frac{1}{256\,\pi}\Big[\tfrac{1}{2}\Upsilon\ln[\eta^{24}(2S)\,\Phi(T)] + \tfrac{1}{2}\bar\Upsilon\ln[\eta^{24}(2\bar S)\,\Phi(\bar T)]$$
$$+ (\Upsilon+\bar\Upsilon)\ln[(S+\bar S)^3(T+\bar T)^a\eta_{ab}(T+\bar T)^b]\Big]$$

This expression is S- and T-duality invariant! *(for real Υ)*

Note: non-holomorphic!

ITERATE: *determine the corrections to the transformation rules of the moduli and integrate. This yields:*

$$\Omega^{(2)} = -\frac{G_2(2S)}{(Y^0)^2}\frac{\partial\Omega^{(1)}}{\partial T^a}\frac{\partial\Omega^{(1)}}{\partial T_a} - \frac{1}{4(Y^0)^2}\frac{\partial\ln\Phi(T)}{\partial T_a}\frac{\partial\Omega^{(1)}}{\partial T^a}\frac{\partial\Omega^{(1)}}{\partial S} + \mathrm{c.c}$$

which is not invariant and determined up to an invariant function.

The corresponding Hesse potential is S- and T-duality invariant !!

Cardoso, dW, Mahapatra, 2008

Conceptually this determination is different from the one used for the topological string. Here we require the periods to transform covariantly under the dualities.
For the topological string one integrates the holomorphic anomaly equation. This determines the partition functions up to terms that are holomorphic. Indeed, there are definitely overlapping terms in the two results!

Grimm, Klemm, Marino, Weiss, 2007

This overlap is expected on the basis of the fact that the effective action and the topological string have been shown to lead to mutually consistent implications for string S-matrix elements.

Antoniadis, Gava, Narain, Taylor,1993

The above observations did emerge in the semiclassical treatment of **BPS black holes**

electro- and magnetostatic potentials:

$$\chi_I = F_I + \bar{F}_I$$

$$\phi^I = Y^I + \bar{Y}^I$$

in the spirit of *Real special geometry*

Freed, 1999
Cortés,2001
Alekseevsky, Cortés, Devchand, 2002

There exists a Hesse potential, $\mathcal{H}(\phi, \chi, \Upsilon, \bar{\Upsilon})$

with the dualities represented as canonical transformations!

attractor equations for BPS black holes:

$$\frac{\partial \mathcal{H}}{\partial \phi^I} = q_I \qquad \frac{\partial \mathcal{H}}{\partial \chi_I} = -p^I$$

'Hamiltonian form' or the BPS black hole free energy

The Hesse potential can include non-holomorphic corrections (!) and follows from a Legendre transformation of the effective action,

$$4(\mathrm{Im}\, F - \Omega) - 2\chi_I \,\mathrm{Im}\, Y^I = \mathcal{H}(\phi, \chi)$$

$$\mathcal{H}(\phi, \chi) - q_I \phi^I + p^I \chi_I = \frac{\mathcal{S}_{\mathrm{macro}}(p, q)}{\pi}$$

Wald entropy

The validity is in agreement with microscopic derivations (at the semiclassical level). It also confirms the existence of a variational principle, and the presence of the non-holomorphic corrections.

Dijkgraaf, Verlinde, Verlinde, 1997
Cardoso, dW, Käppeli, Mohaupt, 2004,2006
Shih, Strominger, Yin, 2005
Jatkar, Sen, 2005
David, Sen, 2006

Theorem (canonical perspective)

Consider a Lagrangian $\mathcal{L}(q,\dot{q})$ depending on n coordinates q^i and n velocities $\dot{q}^i$ with a corresponding Hamiltonian

$$\mathcal{H}(q,p) = \dot{q}^i\, p_i - \mathcal{L}(q,\dot{q})$$

Then there exists a description in terms of a complex function $F(x,\bar{x})$, with $x^i = \frac{1}{2}(q^i + \mathrm{i}\dot{q}^i)$, such that:

$$\begin{aligned} 2\,\mathrm{Re}\,x^i &= q^i \\ 2\,\mathrm{Re}\,F_i(x,\bar{x}) &= p_i \end{aligned}$$

$$F_i = \frac{\partial F(x,\bar{x})}{\partial x^i}$$

The function $F(x,\bar{x})$ is decomposable into a holomorphic and a purely imaginary non-harmonic function:

$$F(x,\bar{x}) = F^{(0)}(x) + 2\mathrm{i}\Omega(x,\bar{x})$$

usually the ‘classical’ part

contains non-Wilsonian contributions (and more?)

this decomposition is not unique

Then the Lagrangian and Hamiltonian take the following form:

$$\mathcal{L} = 4[\mathrm{Im}\, F - \Omega]$$

$$\mathcal{H} = -\mathrm{i}(x^i\, \bar{F}_{\bar\imath} - \bar{x}^{\bar\imath}\, F_i) - 4\,\mathrm{Im}[F^{(0)} - \tfrac{1}{2}x^i\, F_i^{(0)}] - 2(2\,\Omega - x^i\Omega_i - \bar{x}^{\bar\imath}\Omega_{\bar\imath})$$

symplectic invariant

vanishes owing to homogeneity

proportional to deformation parameters

The $2n$*-vector* (x^i, F_i) *transforms under canonical (symplectic) reparametrizations as:*

$$\begin{pmatrix} x^i \\ F_i(x,\bar{x}) \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}^i \\ \tilde{F}_i(\tilde{x},\bar{\tilde{x}}) \end{pmatrix} = \begin{pmatrix} U^i{}_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} x^j \\ F_j(x,\bar{x}) \end{pmatrix}$$

non-holomorphic period vector

$$\mathcal{H} = -4\,[\mathrm{Im}\, F - \Omega] + 2\mathrm{Im}\, x^i\, p_i$$

The existence of an Hamiltonian and corresponding canonical transformations is crucial!

Confirms earlier insights!

Cardoso, dW, Käppeli, Mohaupt, 2004, 2006

Under the symplectic reparametrizations, the complex function is integrable!

Non-holomorphic extensions are consistently incorporated by:

$$F \longrightarrow F^{(0)}(Y) + 2\mathrm{i}\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

Non-holomorphic and homogeneous

compatible with special geometry.

preserves the form of the attractor equations.

Important conceptual implication: there exist consistent non-holomorphic deformations of special geometry.
Or: *every system of this class can be viewed as a non-holomorphically derformed special geometry system.*

Cardoso, dW, Mahapatra, 2008, to appear

How to make further progress?
Consider the more generic case.

Perform the Legendre transform (by iteration) :

new covariant coordinates: $Y^I \longrightarrow \mathcal{Y}^I$

Solve the equations:

$$2\,\mathrm{Re}\,Y^I = \phi^I = 2\,\mathrm{Re}\,\mathcal{Y}^I$$
$$2\,\mathrm{Re}\,F_I(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = \chi_I = 2\,\mathrm{Re}\,F_I^{(0)}(\mathcal{Y})$$

This will involve an infinite series!
The original holomorphic structure is not respected!

Can one identify the topological string partition function?

Is the topological string contained in Ω *?*
Note: $F^{(0)}$ *enters only in the definition of the duality covariant moduli*

For instance, the expansion of the old variables into the new ones takes the form:

$$
\begin{aligned}
Y^I \approx\ & \mathcal{Y}^I - 2\,(\Omega^I - \Omega^{\bar I}) \\
& +2\mathrm{i}(F+\bar F)^{IJK}(\Omega_J - \Omega_{\bar J})(\Omega_K - \Omega_{\bar K}) + 8\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar J})\,(\Omega_J - \Omega_{\bar J}) \\
& -\tfrac{4}{3}\mathrm{i}\left[(F-\bar F)^{IJKL} + 3\mathrm{i}(F+\bar F)^{IJM}(F+\bar F)_M{}^{KL}\right] \\
& \qquad\qquad \times(\Omega_J - \Omega_{\bar J})(\Omega_K - \Omega_{\bar K})(\Omega_L - \Omega_{\bar L}) \\
& -8\mathrm{i}\left[2\,(F+\bar F)^{IJ}{}_K \mathrm{Re}(\Omega^{KL} - \Omega^{K\bar L}) + \mathrm{Re}(\Omega^{IK} - \Omega^{I\bar K})(F+\bar F)_K{}^{JL}\right] \\
& \qquad\qquad \times(\Omega_J - \Omega_{\bar J})(\Omega_L - \Omega_{\bar L}) \\
& -32\,\mathrm{Re}(\Omega^{IJ} - \Omega^{I\bar J})\,\mathrm{Re}(\Omega_{JK} - \Omega_{J\bar K})\,(\Omega^K - \Omega^{\bar K}) \\
& -8\mathrm{i}\,\mathrm{Im}(\Omega^{IJK} - 2\,\Omega^{IJ\bar K} + \Omega^{I\bar J\bar K})(\Omega_J - \Omega_{\bar J})\,(\Omega_K - \Omega_{\bar K}) + \cdots\ .
\end{aligned}
$$

where in each of the terms the old variables have been replaced into the new ones.

[*here:* $F \equiv F^{(0)}(\mathcal{Y})$]

indices raised and lowered with

$$N_{IJ} = 2\,\mathrm{Im}[F^{(0)}(\mathcal{Y})]$$

Construct the Hesse potential (in terms of covariant moduli) for the *FSHV*-Model:

$$\mathcal{H}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) \;\approx\; -|Y^0|^2 (S+\bar{S})(T+\bar{T})^2 + 4\,\Omega^{(1)}(S,\bar{S},T,\bar{T})$$
$$-\left[\frac{4\,\hat{G}_2(2S,2\bar{S})}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T_a}\,\frac{\partial\Omega^{(1)}}{\partial T^a} + \frac{1}{(Y^0)^2}\,\frac{\partial \log\left[\Phi(T)\,[(T+\bar{T})^2]^4\right]}{\partial T_a}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial S} + \text{c.c.}\right]$$
$$+\frac{4\,(T+\bar{T})^a (T+\bar{T})^b}{|Y^0|^2\,(S+\bar{S})(T+\bar{T})^2}\left(2\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial \bar{T}^b} - \eta_{ab}\,\frac{\partial\Omega^{(1)}}{\partial T_c}\,\frac{\partial\Omega^{(1)}}{\partial \bar{T}^c}\right)$$
$$+\frac{8\,(S+\bar{S})}{|Y^0|^2\,(T+\bar{T})^2}\,\frac{\partial\Omega^{(1)}}{\partial S}\,\frac{\partial\Omega^{(1)}}{\partial \bar{S}}$$

S- and T-duality invariant!

holomorphic dependence on the topological string coupling

non- holomorphic dependence

The topological string is indeed contained in the Hesse potential and is characterized by its holomorphic dependance on the topological string coupling constant.

Grimm, Klemm, Marino, Weiss, 2007

However, there are other invariant terms, which do not depend holomorphically on the topological string coupling constant. At present their meaning is not clear, but they do in principle contribute to the BPS black hole entropy at the sub-subleading level.

Let us therefore continue by studying the more generic aspects, irrespective of a particular model.

Upon substitution, the expression for the Hesse potential in terms of the new variables $\mathcal{Y}^I$ will take the form:

$$
\begin{aligned}
\mathcal{H} \;=\;& \mathcal{H}|_{\Omega=0} + 4\,\Omega - 4\,N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar I}\Omega_{\bar J}) + 8\,N^{IJ}\Omega_I\Omega_{\bar J} \\
&+16\,\mathrm{Re}(\Omega_{IJ} - \Omega_{I\bar J})N^{IK}N^{JL}\big(\Omega_K\Omega_L + \Omega_{\bar K}\Omega_{\bar L} - 2\,\Omega_K\Omega_{\bar L}\big) \\
&-\tfrac{16}{3}(F+\bar F)_{IJK}N^{IL}N^{JM}N^{KN}\,\mathrm{Im}(\Omega_L\Omega_M\Omega_N - 3\,\Omega_L\Omega_M\Omega_{\bar N}) \\
&+\mathcal{O}(\Omega^4)
\end{aligned}
$$

non-holomorphic square

where $\mathcal{H}(\mathcal{Y},\bar{\mathcal{Y}})\Big|_{\Omega=0} = \mathrm{i}[\bar{\mathcal{Y}}^I F_I(\mathcal{Y}) - \mathcal{Y}^I \bar F_I(\bar{\mathcal{Y}})]$

and $N_{IJ} = 2\,\mathrm{Im}[F^{(0)}(\mathcal{Y})]$

$$F \equiv F^{(0)}(\mathcal{Y})$$

From the fact that the Hesse potential transforms as a function under symplectic transformations, one arrives at the following result.

The quantity $\Omega(\mathcal{Y},\bar{\mathcal{Y}})$ *does not transform as a function under symplectic transformations !*

Rather it transforms non-linearly (proven by iteration!):

$$\begin{aligned}\tilde{\Omega} \;=\;& \Omega - \mathrm{i}\big(\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J - \bar{\mathcal{Z}}_0^{\bar{I}\bar{J}}\,\Omega_{\bar{I}}\Omega_{\bar{J}}\big)\\ &+\tfrac{2}{3}\big(F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N + \mathrm{h.c.}\big)\\ &-2\big(\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L + \mathrm{h.c.}\big) + 4\,\Omega_{I\bar{J}}\,\mathcal{Z}_0^{IK}\Omega_K\,\bar{\mathcal{Z}}_0^{\bar{J}\bar{L}}\Omega_{\bar{L}}\\ &+\mathcal{O}(\Omega^4)\end{aligned}$$

where all variations are encoded in $\mathcal{Z}_0^{IJ} = \dfrac{\partial\mathcal{Y}^I}{\partial\tilde{\mathcal{Y}}_K}\,Z^{KJ}$

dW, 1996

This suggests a systematic pattern!

(we have verified this including terms of fourth order)

DECOMPOSITION OF THE HESSE POTENTIAL

$$\mathcal{H} = \mathcal{H}_0 + 4\,\mathcal{H}_1 - \tfrac{8}{3}\mathrm{i}(\mathcal{H}_2 - \bar{\mathcal{H}}_2) + 16\,\mathcal{H}_3 + \mathcal{H}_4 + \cdots$$

where $\mathcal{H}_0 = \mathcal{H}\Big|_{\Omega=0} = \mathrm{i}[\bar{\mathcal{Y}}^I\, F_I(\mathcal{Y}) - \mathcal{Y}^I\, \bar{F}_I(\bar{\mathcal{Y}})]$

and $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$ are each symplectic functions, which are not harmonic, not even when Ω is harmonic.

In the following, let us assume that Ω *is harmonic.*

$$\Omega(\mathcal{Y}, \bar{\mathcal{Y}}; \Upsilon, \bar{\Upsilon}) = \omega(\mathcal{Y}; \Upsilon) + \mathrm{h.c.}$$

and homogeneous of second degree.

The duality transformations preserve the possible harmonicity of Ω!

$$\begin{aligned}\tilde{\Omega} \;=\; \Omega + \Big[& -\mathrm{i}\mathcal{Z}_0^{IJ}\,\Omega_I\Omega_J \\ & +\tfrac{2}{3}F_{IJK}\,\mathcal{Z}_0^{IL}\Omega_L\,\mathcal{Z}_0^{JM}\Omega_M\,\mathcal{Z}_0^{KN}\Omega_N \\ & -2\,\Omega_{IJ}\,\mathcal{Z}_0^{IK}\Omega_K\mathcal{Z}_0^{JL}\Omega_L \\ & -\tfrac{\mathrm{i}}{3}F_{IJKL}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\ & +\tfrac{4\mathrm{i}}{3}\,\Omega_{IJK}(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K \\ & +\mathrm{i}\,F_{IJR}\,\mathcal{Z}_0^{RS}\,F_{SKL}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J(\mathcal{Z}_0\Omega)^K(\mathcal{Z}_0\Omega)^L \\ & -4\mathrm{i}\,F_{IJK}\mathcal{Z}_0^{KP}\,\Omega_{PQ}\,(\mathcal{Z}_0\Omega)^I(\mathcal{Z}_0\Omega)^J\,(\mathcal{Z}_0\Omega)^Q \\ & +4\mathrm{i}\,\mathcal{Z}_0^{IP}\,\Omega_{PQ}\,\mathcal{Z}_0^{QR}\,\Omega_{RK}\,(\mathcal{Z}_0\Omega)^K\;+\mathrm{h.c.}\Big]\end{aligned}$$

Familiar structure. In the context of the topological string, $\mathcal{Z}_0^{IJ}$ is known as a 'propagator': $\mathcal{Z}_0^{IJ} \propto \Delta^{IJ}$

In that case the partition function is treated as a wave function in a Hilbert space based on quantizing $H_3(X)$

Aganagic, Bouchard, Klemm, 2008

The symplectic function $\mathcal{H}_1$ is 'almost' harmonic in this case: it is the real part of a function that consists of only purely holomorphic derivatives of F and Ω contracted with the non-holomorphic tensor N^{IJ}

$$\begin{aligned}
\mathcal{H}_1 \;=\; & \Omega \\
& + \Big[- N^{IJ}\,\Omega_I \Omega_J \\
& \quad + 2\,\Omega_{IJ}\,N^{IK}\Omega_K N^{JL}\Omega_L \\
& \quad + \tfrac{2}{3}\mathrm{i}F_{IJK}\,(N\Omega)^I\,(N\Omega)^J\,(N\Omega)^K \\
& \quad - \tfrac{\mathrm{i}}{3}\,\big(F_{IJKL} + 3\mathrm{i}\,F_{IJR}\,N^{RS}\,F_{SKL}\big)\,(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\Omega)^L \\
& \quad - \tfrac{4}{3}\,\Omega_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K \\
& \quad - 4\mathrm{i}\,F_{IJK}N^{KP}\,\Omega_{PQ}\,(N\Omega)^I(N\Omega)^J\,(N\Omega)^Q \\
& \quad - 4\,N^{IP}\,\Omega_{PQ}\,N^{QR}\,\Omega_{RK}\,(N\Omega)^K + \mathrm{h.c.}\Big] + \cdots
\end{aligned}$$

This function is symplectically covariant, and harmonic in Υ. Hence it decomposes into the real part of a holomorphic function of Υ.

Furthermore it is homogeneous of second degree in holomorphic variables $\mathcal{Y}^I$ and of zeroth degree in non-holomorphic variables $\bar{\mathcal{Y}}^I$.

And it satisfies (part of) the holomorphic anomaly equation corresponding to the disconnecting pinchings of the Riemann surface.

The remaining component of the non-holomorphic anomaly can be generating by including non-holomorphic terms that are invariant under dualities up to terms that are (non-) holomorphic terms. Examples of such terms are the special Kähler potential and the logarithm of the determinant of N_{IJ}, respectively, $\ln\det[N_{IJ}]$ and $\ln\big[\mathrm{i}[\bar{Y}^I F_I - Y^I \bar{F}_I]\big]$.

Conclusions:

Non-holomorphic deformations of special geometry are consistent in the proposed framework.

The information encoded in the function that encodes the effective action with higher derivatives, and in that comprises the topological string partition functions does not fully overlap, but there are common sectors.

The reason is that the Hesse potential, which is a proper symplectic function, decomposes into several symplectic functions. Precisely one of them exhibits the structure that is known from the topological string. The scenario that is pointed out is subtle and restrictive.

Not many cases are explicitly known on both sides. Case studies, whenever possible, should be useful to further calrify the sitution.

Cardoso, dW, Mahapatra, to appear