Principles of Magnetic Resonance Imaging.
Final assignment

Write a report about the following exercises. The report should include also a short introduction, a theoretical part, discussions and a summary/conclusion, If you write Matlab code, please add it to the report. Try to answer (and/or discuss) as many questions as you can. Always motivate your answers and mention the theoretical results that you applied.
You can cooperate with a fellow student. You might be asked to answer questions related to your report during a meeting with the teacher to assess your grade.
Send the report in a digital version to my email: `a.sbrizzi` at `umcutrecht.nl`.
The deadline for the submission is on the 16th of April, 2017 at 23.59. Good luck!

Exercise 1: A trajectory for shortened k-space data acquisition

Given a *real* function $f : \mathbb{R}^N \mapsto \mathbb{R}$, denote by $\hat{f}$ its Fourier transform. Show that

$$\hat{f}(\vec{\omega}) = (\hat{f}(-\vec{\omega}))^*$$

where * denotes complex conjugation. In an ideal MRI experiment, the image function is proportional to the proton density of the object, which is thus a real non-negative function. In this case, do we need to scan the whole k-space? Discuss.
In this exercise we assume to be in this ideal situation.
Suppose you wish to scan an object, which measures $L \times L$, with a given spatial resolution $\Delta_x = \Delta_y = \Delta_s$. Furthermore, you decide to scan only *half* k-space. The k-space trajectory you choose is a concentric half-circular trajectory, starting from the center, as shown in Figure 1. How many half circles do you need to be

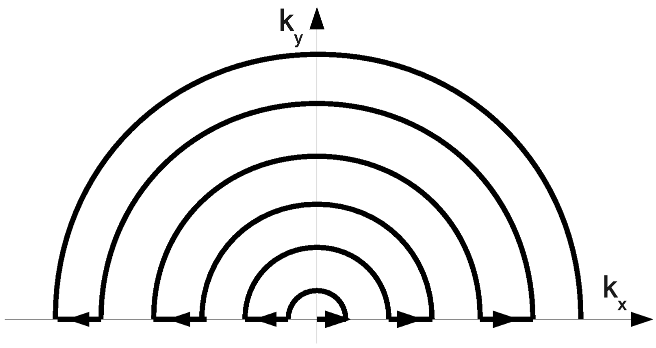

Figure 1: Half-circular k-space trajectory

sure the image you get will not exhibit aliasing? What is the distance between the circles, along the radial direction? Write an analytical formula for the circular parts of the trajectory. Derive the gradients G_x and G_y needed to acquire the data in this way.

The k-space samples are taken along the trajectory with a distance in time equal to Δ_t, i.e. the k-space is discretized in a circular fashion. Modify the above derived formula for the trajectory in such a way that the distance between the samples fulfills the Nyquist criterion, that is $\Delta_k \leq 1/L$.
In practice, the gradients are subject to the constraint $|G_x| \leq G_{\max}$ and $|G_y| \leq G_{\max}$ with $G_{\max}$ a given positive real number. Modify your trajectory in a way that it fulfills also this requirement.
Can you write a matlab-code which, given L, Δ_s, Δ_t, γ and $G_{\max}$, returns the gradients for this trajectory?
Suppose you wish to apply SENSE reconstruction and undersample the (half) k-space by a factor $R = 2$. How can you do it efficiently with this trajectory?
How can you reconstruct the image from the data acquired along this trajectory? Can you apply the Fast Fourier transform?
In practice, the spin magnetization is subject to *relaxation*. This means that the signal decays during the scan. As a consequence, the effect of noise will be more relevant at the end of the acquisition. Can you explain what is the advantage of scanning the k-space along this trajectory?

Exercise 2: MRI of a phantom

Often, MRI scientists make use of phantoms for their experiments: plastic containers with arbitrary forms which are filled with water and/or other substances. Scanning phantoms is much easier than scanning humans: they do not move and they do not complain about the length of the exam!
For this exercise, we focus on two kinds of phantom: a disk-shaped phantom and a composite object.

Disk phantom

Using the properties of the Bessel functions and the polar coordinate sytems (r, θ), it is possible to show that the Fourier transform of a function defined as

$$f(r, \theta) = \begin{cases} 1 & \text{if } r \leq a \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$F(k_r, k_\theta) = a \frac{J_1(2\pi a k_r)}{k_r}$$

where $k_r \equiv \sqrt{k_x^2 + k_y^2}$ and J_1 denotes the so-called Bessel function of the first kind, order 1.
The Matlab function `phantom_kspace` returns the k-space data of a simple disk-shaped phantom given the scan parameters $k_{\max}$ and Δ_k. Its code is listed at the end of this document.
Suppose you want to scan such a phantom, and you set the field of view $L = 16$ and resolution $\Delta_x = \Delta_y = 0.125$. Calculate the signal for this experiment and reconstruct the image.
What do you notice? What are the main differences with respect to the true image of the phantom (Fig. 2-left)? What about the edges of the object? Can you explain?

Composite phantom

You now want to scan a composite phantom, similar to the previous one but with three additional circular compartments (Fig. 2-right) with radii 1.75, 1.75 and 0.75, respectively. The (x, y) coordinates of the center of the three compartments are (3,2), (-3,2) and (0,-3), respectively. The image value inside the compartments is exactly twice as large as the value of the main disk.
Derive the expression for the k-space values (Fourier transform) of this phantom. Implement this in the Matlab function `phantom_kspace`. Compute the signal for $k_{\max}$ and Δ_k as above and plot the reconstructed image.
Multiply the k-space data with the gaussian window function $W(k_r) = \exp(-(k_r/b)^2)$ and reconstruct the image. Try $b = 1, 2, 4, \ldots$. What do you observe? Explain. Can you get rid of the artifacts?

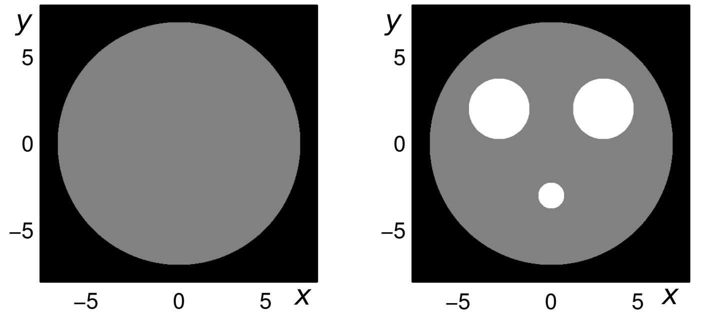

Figure 2: The simple (left) and composite (right) phantoms.

Matlab code for the exercise

```
function signal = phantom_kspace(kmax,dk)

kx = -kmax:dk:(kmax-dk);
ky = kx;
[KX KY] = meshgrid(kx,ky);
kr = sqrt(KX.^2+KY.^2);
%% signal
a = 7; % radius of the phantom
F = (a*besselj(1,2*pi*a*kr)./kr);
F(find(kr==0)) = a^2*pi; % avoid problem with division by 0
signal = F;
%% implement the signal for the composite phantom
% ...
```