Midterm Exam UcSciMat11(A)

Thursday October 15, 2015; 13:45-15:30

1. Write your name and studentnumber on each page.
2. You may use the formula sheet.
3. *Motivate* your answers by indicating which arguments and results you use to draw a conclusion.
4. You have 105 minutes for the exam.
5. Good luck!

Exercise 1

Sketch a direction field for the differential equation: $\frac{dy}{dx} = xy$.
Use this direction field to sketch solutions with the initial condition $y(0) = -1$, one with $y(3) = 0$ and one with the initial condition $y(1) = 2$.

Exercise 2

Solve the differential equation (using separation of variables): $\frac{dy}{dx} = y^2$, $y(0) = 1$. What happens at $x = 1$?

Exercise 3

Given are the complex numbers $z = -2 + 2i$ and $w = \sqrt{2} + \sqrt{2}i$. Give (or calculate), respectively,

$$|z|,\ Re(z),\ Im(w),\ \frac{z}{w},\ \arg(z),\ w^2.$$

Write the number w as $re^{i\phi}$ with $r \geq 0$ and $-\pi < \phi \leq \pi$. Also draw the complex numbers z, $\bar{z}$, w and $\bar{w}$ in the Argand plane.

Exercise 4

Solve $\int_0^{\pi} x^2 \cos(x)\, dx$.

Exercise 5

Give the general solution of the differential equation:

$$x''(t) - x'(t) - 20x(t) = 145\sin(2t) - 4000.$$

Exercise 6

Solve $\int 2y \sin(y^2) e^{y^2}\, dy$. Hint: use the substitution $x = y^2$.

Exercise 7

Find the values a and b such that: $2e^{\frac{i\pi}{6}} = a + bi$.

END

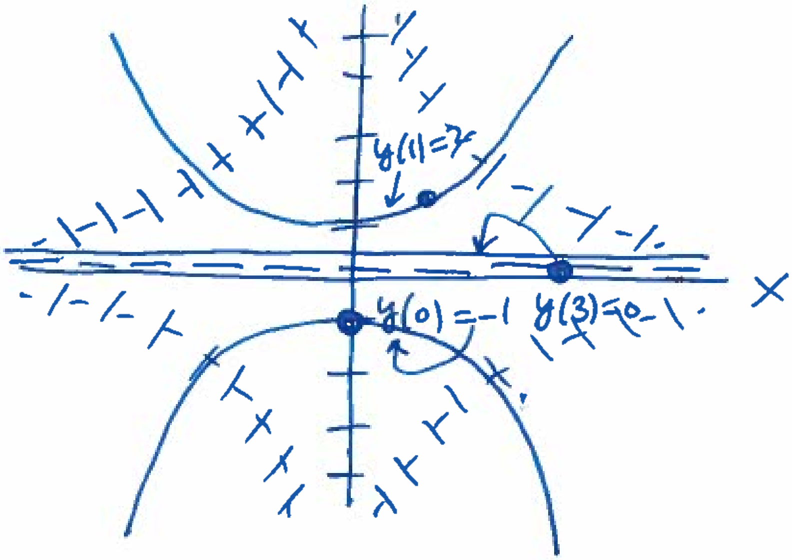
exercise 1

direction field of $\frac{dy}{dx} = xy$

$\frac{dy}{dx} = 0$ if $x = 0$ or $y = 0$ ("$xy = 0$")

$\frac{dy}{dx} = 1$ if $xy = 1 \iff y = \frac{1}{x}$

$\frac{dy}{dx} = -1$ if $xy = -1 \iff y = -\frac{1}{x}$



exercise 2

$$\frac{dy}{dx} = y^2 , \quad y(0) = 1$$

$$\Leftrightarrow \int \frac{dy}{y^2} = \int dx + C$$

$$\Leftrightarrow -\frac{1}{y} = x + C$$

$$\Leftrightarrow \frac{1}{y} = -x - C$$

$$\Leftrightarrow y = \frac{1}{-x - C}$$

$$y(0) = \frac{1}{-0 - C} = -\frac{1}{C} = 1 \Rightarrow C = -1$$

$$\Rightarrow y(x) = \frac{1}{-x + 1} = \frac{1}{1 - x}$$

at $x = 1$: $y(x) \longrightarrow +\infty$ (infinity)

exercise 3

$$z = -2+2i$$
$$w = \sqrt{2} + \sqrt{2}i$$

$$|z| = \sqrt{(-2)^2+2^2} = \sqrt{4+4} = \sqrt{8} \quad (=2\sqrt{2})$$

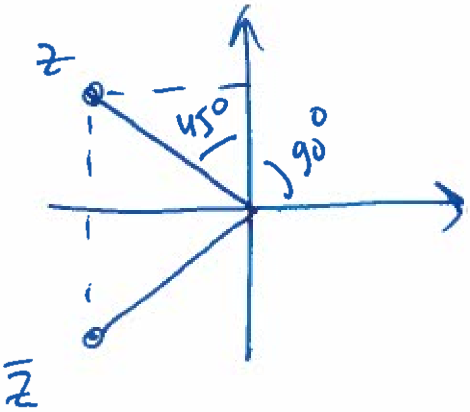
$$Re(z) = -2$$

$$Im(w) = \sqrt{2}$$

$$\frac{z}{w} = \frac{-2+2i}{\sqrt{2}+\sqrt{2}i} = \frac{-2+2i}{\sqrt{2}+\sqrt{2}i} \cdot \frac{\sqrt{2}-\sqrt{2}i}{\sqrt{2}-\sqrt{2}i} = \frac{-2\sqrt{2}+2\sqrt{2}i+2\sqrt{2}i-2\sqrt{2}\overset{=-1}{i^2}}{2+\underbrace{2i-2i}_{=0}\underbrace{-2\overset{=-1}{i^2}}_{=+2}}$$

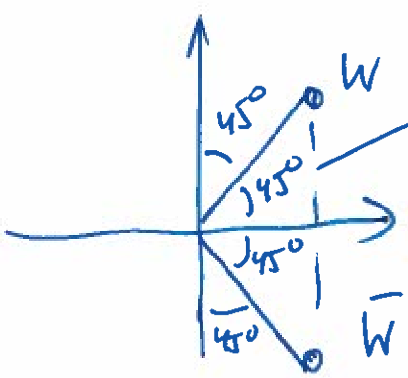
$$= \frac{4\sqrt{2}\cdot i}{4} = \sqrt{2}\cdot i$$

$$w^2 = (\sqrt{2}+\sqrt{2}i)(\sqrt{2}+\sqrt{2}i) = 2+2i+2i+\underbrace{2i^2}_{=-2} = 4i$$



$$\Rightarrow \arg(z) = 90° + 45° = 135° = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$|w| = \sqrt{(\sqrt{2})^2+(\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$$



$$\varphi = 45° = \frac{\pi}{4}$$

$$\Rightarrow w = 2e^{\frac{\pi}{4}i}$$

exercise 4

$$\int_0^{\pi} x^2 \cos(x)\,dx$$

partial integration

$$= \left(x^2 \sin(x)\right)_{x=0}^{x=\pi} - \int_0^{\pi} 2x \sin(x)\,dx$$

$$= \underbrace{\pi^2 \sin(\pi)}_{=0} - \underbrace{0^2 \sin(0)}_{=0} - 2\int_0^{\pi} x \sin(x)\,dx$$

$$= -2\int_0^{\pi} x \sin(x)\,dx$$

partial integration

$$= -2\left\{ \left(-x\cos(x)\right)_{x=0}^{x=\pi} - \int_0^{\pi} (-\cos(x))\,dx \right\}$$

$$= -2\left\{ \underbrace{(-\pi \underbrace{\cos(\pi)}_{=-1}}_{=+\pi} \underbrace{--}_{=+} \underbrace{0\underbrace{\cos(0)}_{=1}}_{=0} + \underbrace{\int_0^{\pi}}_{(--)} \cos(x)\,dx \right\}$$

$$= -2\left\{\pi + \int_0^{\pi} \cos(x)\,dx\right\}$$

$$= -2\left\{\pi + \sin(x)\Big|_{x=0}^{x=\pi}\right\}$$

$$= -2\left\{\pi + \underbrace{\sin(\pi)}_{=0} - \underbrace{\sin(0)}_{=0}\right\}$$

$$= -2\pi$$

exercise 5

homogeneous equation: $x''_{hom} - x'_{hom} - 20\, x_{hom} = 0$

$$x_{hom}(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$\lambda_1 = 5$ and $\lambda_2 = -4$

particular solution: $x_{part}(t) = B_1 \cos(2t) + B_2 \sin(2t) + B_3$

try

$$x_{part}'(t) = -2B_1 \sin(2t) + 2B_2 \cos(2t)$$

$$x_{part}''(t) = -4B_1 \cos(2t) - 4B_2 \sin(2t)$$

differential equation with $145 \sin(2t) - 4000$

$\Longrightarrow$

$$-4B_1 \cos(2t) - 4B_2 \sin(2t) + 2B_1 \sin(2t) - 2B_2 \cos(2t)$$
$$-20B_1 \cos(2t) - 20B_2 \sin(2t) - 20B_3 = 145 \sin(2t) - 4000$$

$$\Longrightarrow \begin{cases} B_3 = 200 \\ 24B_1 + 2B_2 = 0 \\ 2B_1 - 24B_2 = 145 \end{cases} \Longrightarrow \begin{array}{l} B_1 = \frac{1}{2} \\ B_2 = -6 \end{array}$$

$$\Longrightarrow X(t) = X_{hom}(t) + X_{part}(t) = A_1 e^{5t} + A_2 e^{-4t} + \tfrac{1}{2}\cos(2t) - 6\sin(2t) + 200$$

exercise 6

$$\int 2y \sin(y^2) e^{y^2} \, dy$$

$x=y^2$

$$\overset{}{=} \int \sin(x) e^x \, dx \overset{\text{define}}{=} I$$

$dx = 2y\,dy$

$$= -\cos(x) e^x \underset{(--)}{+} \int \cos(x) e^x \, dx$$

partial integration

$$= -\cos(x) e^x + \sin(x) e^x - \underbrace{\int \sin(x) e^x \, dx}_{\overset{!!!}{=} I}$$

$$\Rightarrow I = -I + \sin(x) e^x - \cos(x) e^x$$

$$\Rightarrow 2I = (\sin(x) - \cos(x)) e^x$$

$$\Rightarrow I = \frac{1}{2} (\sin(x) - \cos(x)) e^x$$

$$\overset{x=y^2}{=} \frac{1}{2} (\sin(y^2) - \cos(y^2)) e^{y^2}$$

exercise 7

$$2e^{\frac{i\pi}{6}} = a + bi$$

$$\parallel$$

$$2\left(\cos\left(\frac{\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6}\right)\right)$$

$$= 2\left(\frac{1}{2}\sqrt{3} + i \cdot \frac{1}{2}\right)$$

$$= \sqrt{3} + i$$

$$\Rightarrow \quad a = \sqrt{3} \quad \text{and} \quad b = 1$$