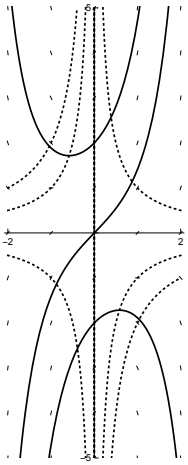
Answers Midterm Exam Calculus
October 17, 2014

Exercise 1: Solve the differential equation: $x\frac{dy}{dx} - (2x+1)e^{-y} = 0$ with initial condition $y(1) = 0$.
Answer: Separation of variables gives: $\int e^y dy = \int (2+\frac{1}{x})dx \Rightarrow e^y = 2x + \log|x| + c$ This can be written as: $y = \log(2x + \log|x| + c)$. If we use that $y(1) = 0$ we get: $\log(2+c) = 0$ from which we deduce that $c = -1$. The solution is therefore: $y(x) = \log(2x + \log|x| - 1)$

Exercise 2: Sketch a direction field for the differential equation $\frac{dy}{dx} = 1 + xy$. (Hint: Calculate the isoclines when $\frac{dy}{dx} = -1$, $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = 1$ and $\frac{dy}{dx} = 2$.) Then use the direction field to sketch solutions with initial condition $y(0) = 0$, $y(0) = 2$ and $y(0) = -2$ both for positive and negative values of x.

Answer: The isoclines $\frac{dy}{dx} = -1$, $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = 1$ and $\frac{dy}{dx} = 2$ correspond respectively to $y = -2/x$, $y = -1/x$, $x = 0$ or $y = 0$ and $y = 1/x$. This leads to the following picture:



where the dotted lines are the isoclines and the full lines are the three solutions.

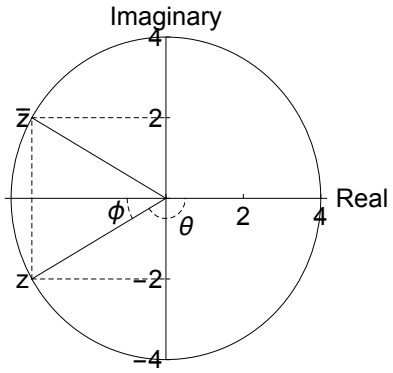
Exercise 3: Give the general solution of the differential equation:

$y''(t) - 4y'(t) + 5y(t) = 1 + e^t$.
Answer: First consider the homogeneous case: $y_h''(t) - 4y_h'(t) + 5y_h(t) = 0$. The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$. The 2 solutions are $\lambda = 2 + i$ and $\lambda = 2 - i$. The solution of the homogeneous problem therefore is $e^{2t}(A\cos t + B\sin t)$ with A and B constants.

For the particular solution, try $y_p(t) = C + De^t$. Then $y_p'(t) = y_p''(t) = De^t$. If we fill $y_p(t)$ in the differential equation, we get: $De^t - 4De^t + 5C + 5De^t = 1 + e^t$. We deduce that $D = 1/2$ and $C = 1/5$. Th general solution of the inhomogeneous differential equation is now: $y(t) = e^{2t}(A\cos t + B\sin t) + \frac{1}{5} + \frac{1}{2}e^t$.

Exercise 4: Write the complex number $z = -2\sqrt{3} - 2i$ in the form $re^{i\phi}$ with r and ϕ real numbers and $r > 0$ and $-\pi < \phi \leq \pi$. Also give the complex conjugate $\bar{z}$ of z. Draw z and $\bar{z}$ in the Argand plane.

Answer: The norm r of z is $\sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4$. For the angle ϕ (see picture), we have $\sin\phi = 2/4 = 1/2$. So $\phi = \frac{\pi}{6}$. The angle θ with the positive real axis is therefore $-\frac{5}{6}\pi$. So $z = 4e^{-\frac{5}{6}\pi i}$. $\bar{z} = -2\sqrt{3} + 2i = 4e^{\frac{5}{6}\pi i}$.



Exercise 5: Solve $I = \int_1^e x^3 \log x \, dx$.

Answer: Use partial integration. $I = \frac{1}{4}x^4 \log x - \int_1^e \frac{1}{4}x^4 \frac{1}{x} dx = \frac{1}{4}e^4 - \int_1^e \frac{1}{4}x^3 =$. $\frac{1}{4}e^4 - \frac{1}{16}x^4|_1^e = \frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16} = \frac{1}{16}(3e^4 + 1)$.

Exercise 6: Classify the equilibria of the system

$$x' = y(x+1)$$
$$y' = x(y+3)$$

and draw the phase plane diagram of the system.

Answer: The equilibria satisfy $x' = 0$ and $y' = 0$. Therefore, the equilibria satisfy ($y = 0$ or $x = -1$) and ($x = 0$ or $y = -3$). We find two equilibria: $(0, 0)$ and $(-1, -3)$. If we linearize around $(0, 0)$ we get (no substitution is needed):

$$x' \approx y$$
$$y' \approx 3x$$

We conclude that $a = 0$, $b = 1$, $c = 3$ and $d = 0$, so $p = 0$, $q = -3$ and $\Delta = 12$. We conclude from this that we have a saddle.

To linearize around the equilibrium $(-1, -3)$, we have to introduce new variables $X = 1 + x$ and $Y = 3 + y$. The differential equations in the new variables are:

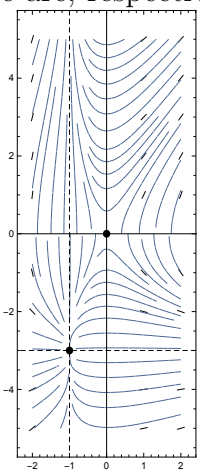
$$X' = x' = y(x+1) = (Y-3)X = XY - 3X$$
$$Y' = y' = x(y+3) = (X-1)Y = XY - Y$$

Linearizing gives:

$$X' \approx -3X$$
$$Y' \approx -Y$$

We conclude that $a = -3$, $b = 0$, $c = 0$ and $d = -1$, so $p = -4$, $q = 3$ and $\Delta = 4$. We conclude from this that we have a stable node.

To draw the phase plane, we calculate the lines where $x' = 0$ and the lines where $y' = 0$. These are, respectively, the lines $y = 0$ and $x = -1$ and the lines $x = 0$ and $y = -3$.



Exercise 7: Solve $I = \int \frac{1}{2-e^y} \frac{e^y}{2\sqrt{e^y-1}} dy$. Hint: Try the substitution $e^y = 1 + u^2$.

Answer: If we take the derivative with respect to u on both sides of the equation $e^y = 1+u^2$, we get $e^y \frac{dy}{du} = 2u$. So $dy = \frac{2u}{1+u^2} du$. Now we perform the given substitution to obtain: $I = \int \frac{1}{2-(1+u^2)} \frac{1+u^2}{2\sqrt{u^2}} \frac{2u}{1+u^2} du$. This simplifies to $I = \int \frac{1}{1-u^2} du$. This integral an be solved using partial fraction. We have that $\frac{1}{1-u^2} = \frac{1}{2}\frac{1}{1-u} + \frac{1}{2}\frac{1}{1+u}$. Therefore, we have: $I = \frac{1}{2}\int \frac{1}{1+u} du + \frac{1}{2}\int \frac{1}{1-u} du = \frac{1}{2}\log(1+u) - \frac{1}{2}\log(1-u) = \frac{1}{2}\log\frac{1+u}{1-u} = \frac{1}{2}\log\frac{1+\sqrt{e^y-1}}{1-\sqrt{e^y-1}}$.

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