Midterm UCSCIMAT21

Monday October 14, 2019; 13:45-15:30

1. Write your name and student number on each page.
2. You may use the formula sheet.
3. *Motivate* your answers by indicating which arguments and results you use to draw a conclusion.
4. You have 105 minutes for the exam.
5. Please use the numbered sheets for each question separately.
6. Good luck!

Question 1 (20 points)

Use the method of *undetermined coefficients* to find the general solution[1] of the differential equation (DE):

$$y'' - 5y' + 6y = \mathrm{e}^x + x.$$

Question 2 (5 points)

Verify that the function $u(x,t) = \mathrm{e}^{-(4\pi^2+3)t}\sin(\pi x)$ satisfies the one-dimensional *heat equation* with damping:

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} - 3u.$$

Question 3 (10 points)

Solve the linear DE system:

$$\vec{y}\,' = \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}\vec{y} \quad \text{with} \quad \vec{y}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Question 4 (5 points)

Find the *radius* and *interval of convergence* of the following *series*:

$$\sum_{n=0}^{\infty} \frac{n!(x-\sqrt{2})^n}{100^n}.$$

Question 5 (15 points)

Find the *power series solution* centered at 0 of the DE: $y'' + 2y = 0$.
Write down the *recurrence relation* for the coefficients and compute the *first three* terms of the series.

[1] You can use C_1 and C_2 for the two constants.

Question 6 (15 points)

The function $f(x)$ with period 2π is defined as:

$$f(x) = x^2, \quad \text{for } -\pi \leq x \leq \pi.$$

a) Draw the graph of the function in the interval $-3\pi \leq x \leq 3\pi$.

b) Find the Fourier series of f.

Question 7 (20 points)

Consider the one-dimensional *heat equation*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 3, \ t > 0 \tag{1}$$

with the boundary conditions (BCs) and initial condition (IC):

$$u(0,t) = 2, \quad \frac{\partial u}{\partial x}(3,t) + u(3,t) = 0, \quad u(x,0) = f(x). \tag{2}$$

a) Show that the solution of PDE (1) with conditions (2) can be written as

$$u(x,t) = u_1(x) + u_2(x,t),$$

where the steady-state solution $u_1(x)$ satisfies $\frac{\partial^2 u_1}{\partial x^2} = 0$ with $u_1(0) = 2$, $\frac{\partial u_1}{\partial x}(3) + u_1(3) = 0$ and $u_2(x,t)$ satisfies the heat equation (1), but now with the boundary conditions $u_2(0,t) = 0$ and $\frac{\partial u_2}{\partial x}(3,t) + u_2(3,t) = 0$.

b) Compute the *steady-state* solution $u_1(x)$.

c) Show that $u_2(x,t)$ is given by[2]:

$$u_2(x,t) = \sum_{n=1}^{\infty} a_n e^{-\mu_n^2 t} \sin(\mu_n x)$$

and give an expression for μ_n (you do not need to give the coefficients μ_n explicitly, only state which relation they satisfy).

Question 8 (10 points)

a) Describe what is meant by *Parsevals' identity* (either by giving the formula or in words).

b) What is the so-called *root test* for series (give a formula)?

[2] Make use of the method of *separation of variables*.