# Q1

DE: $y'' - 5y' + 6y = e^x + x$

$DE_{hom.}: y'' - 5y' + 6y = 0$

↳ characteristic equation: $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$

$\Rightarrow$ solution of homogeneous equation:

$$y_{hom} = C_1 e^{2x} + C_2 e^{3x}$$

particular solution:
(method of undetermined coefficients) $y_{part} = Ae^x + Bx + C$

plug into DE $\Leftarrow$ $\begin{cases} y'_{part} = Ae^x + B \\ y''_{part} = Ae^x \end{cases}$

$Ae^x - 5(Ae^x + B) + 6(Ae^x + Bx + C) = e^x + x$

$\Rightarrow \begin{cases} 2Ae^x = e^x \\ 6B = 1 \\ 6C - 5B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{6} \\ C = \frac{5}{36} \end{cases}$

$\Rightarrow y_{part} = \frac{1}{2}e^x + \frac{1}{6}x + \frac{5}{36}$

$\Rightarrow$ general solution: $y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2}e^x + \frac{1}{6}x + \frac{5}{36}$

Q2

$$u_t = 4u_{xx} - 3u$$

$$u = e^{-(4\pi^2+3)t} \sin(\pi x)$$

$$u_t = -(4\pi^2+3)\, e^{-(4\pi^2+3)t} \sin(\pi x)$$

$$u_x = e^{-(4\pi^2+3)t}\, \pi \cos(\pi x)$$

$$u_{xx} = -e^{-(4\pi^2+3)t}\, \pi^2 \sin(\pi x)$$

and

$$4u_{xx} - 3u$$

$$= -4\pi^2 e^{-(4\pi^2+3)t} \sin(\pi x) - 3e^{-(4\pi^2+3)t} \sin(\pi x)$$

$$= -(4\pi^2+3)\, e^{-(4\pi^2+3)t} \sin(\pi x)$$

$$= u_t$$ ∎

# Q3

$$\vec{y}' = \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix} y \,, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

characteristic polynomial: $\begin{vmatrix} 3-\lambda & 6 \\ 6 & 3-\lambda \end{vmatrix} = (\lambda-3)^2 - 36$

$= (\lambda - 9)(\lambda + 3)$

$\Rightarrow$ eigenvalues of matrix: $\begin{cases} \lambda_1 = 9 \\ \lambda_2 = -3 \end{cases}$

eigenvectors: $\lambda = 9 \Rightarrow \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_1 = x_2}$

$\swarrow$

$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3 \Rightarrow \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_1 = -x_2}$

$\swarrow$

$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow$ solution of DE: $\boxed{\vec{y} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{9t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}}$

apply ICs: $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 - c_2 = 3 \\ c_1 + c_2 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -2 \end{cases}$$

$$\Rightarrow \vec{y} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{9t} - 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

$$\Rightarrow \begin{cases} y_1(t) = e^{9t} + 2e^{-3t} \\ y_2(t) = e^{9t} - 2e^{-3t} \end{cases}$$

Q4

$$\sum_{n=0}^{\infty} \frac{n!\,(x-\sqrt{2})^n}{100^n}$$

ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(x-\sqrt{2})^{n+1}}{100^{n+1}}}{\frac{n!\,(x-\sqrt{2})^n}{100^n}} \right|$$

$$= |x-\sqrt{2}| \cdot \lim_{n\to\infty} \frac{n+1}{100} = \infty$$

diverge

, if $x \neq \sqrt{2}$

if $x=\sqrt{2}$, then we have convergence

radius of convergence $= 0$

# Q5

$$y'' + 2y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\Rightarrow \quad y'' + 2y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2 \cdot \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + 2 a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{-2 a_n}{(n+2)(n+1)} \qquad n = 0, 1, 2, \cdots$$

$a_0, a_1$ are not given, they are arbitrary in this question

$$\Rightarrow a_2 = \frac{-2 a_0}{2 \cdot 1} = -a_0$$

$$\left[ \; a_3 = \frac{-2 \cdot a_1}{3 \cdot 2} = -\frac{1}{3} a_1 \; \right]$$

# Q6

$f(x) = x^2$, $-\pi \le x \le \pi$ and $2\pi$-periodic

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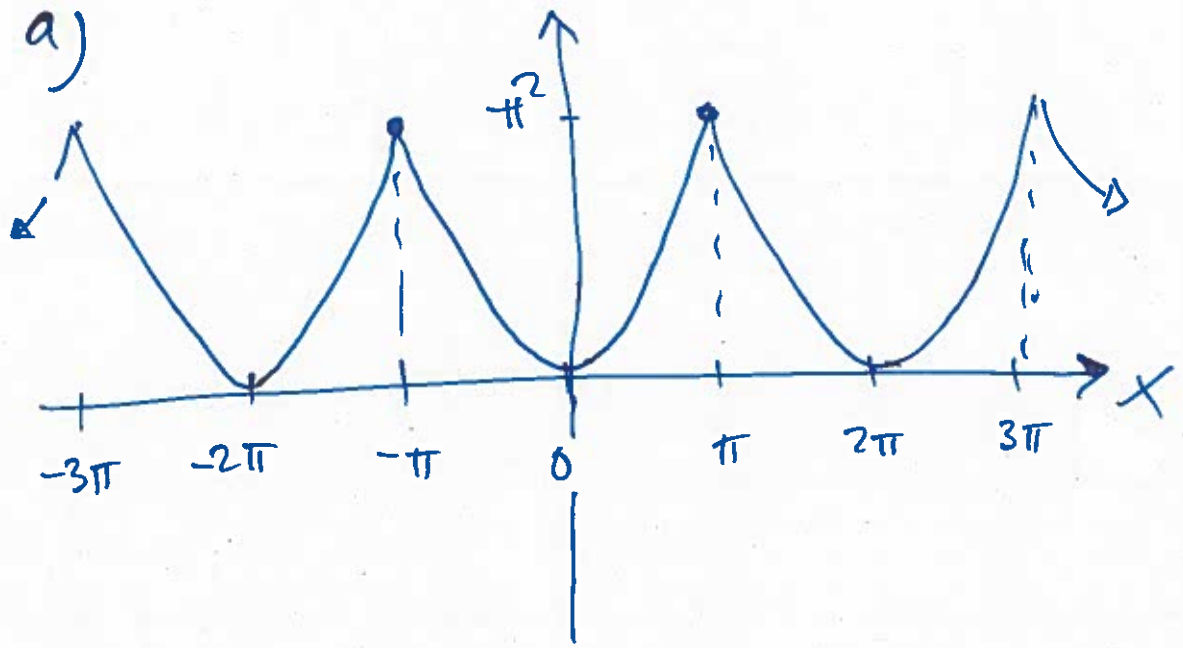
a)




b) $f$ is an even function $\Rightarrow b_n$'s $= 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

← period $2\pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}x^3\Big|_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{4}{n^2}\cos(n\pi) = \begin{cases} -\frac{4}{n^2} & \text{if } n \text{ odd} \\ +\frac{4}{n^2} & \text{if } n \text{ even} \end{cases}$$

← integration by parts

$$\Rightarrow f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos(x)}{1^2} - \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} - \frac{\cos(4x)}{4^2} \cdots\right]$$

$$= \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) - \frac{4}{9}\cos(3x) + \frac{1}{4}\cos(4x) \cdots\Big]$$

↑ factor 2 drops because $\frac{a_0}{2}$

# Q7

$$\begin{cases} u_t = u_{xx}, & x \in (0,3) \\ u|_{x=0} = 2, \ (u_x + u)|_{x=3} = 0 \\ u|_{t=0} = f(x) \end{cases}$$

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a) $u(x,t) = u_1(x) + u_2(x,t)$

$$\begin{cases} \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} \\ u_2(0,t) = 0 \\ u_{2,x}(3,t) + u_2(3,t) = 0 \end{cases} \qquad \begin{cases} \frac{d^2 u_1}{dx^2} = 0 \\ u_1(0) = 2, \ \frac{du_1}{dx}(3) + u_1(3) = 0 \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \underbrace{\frac{\partial u_1}{\partial t}(x)}_{=0\ \checkmark} + \frac{\partial u_2}{\partial t}(x,t) = \frac{\partial u_2}{\partial t} = \underbrace{\frac{\partial^2 u_1}{\partial x^2}}_{=0\ \checkmark} + \frac{\partial^2 u_2}{\partial x^2} \quad \checkmark$$

BCs:

$$\begin{cases} u(0,t) = u_1(0) + u_2(0,t) = 2 + 0 = 2 \quad \checkmark \\ \frac{\partial u}{\partial x}(3,t) + u(3,t) = \underbrace{\frac{\partial u_2}{\partial x}(3,t)}_{=0} + \underbrace{\frac{\partial u_1}{\partial x}(3) + u_2(3,t) + u_1(3)}_{=0} = 0 \quad \checkmark \end{cases}$$

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b) $\begin{cases} \frac{\partial^2 u_1}{\partial x^2} = 0 \\ u_1(0) = 2, \ \frac{\partial u_1}{\partial x}(3) + u_1(3) = 0 \end{cases} \Longrightarrow Ax + B = u_1(x)$

$u_1(0) = B = 2$

$\Rightarrow u_1(x) = 2 + Ax$

$$\frac{\partial u_1}{\partial x} + u_1\Big|_{x=3} = A + Ax + B\Big|_{x=3} = A + Ax\Big|_{x=3} + 2 = 0 \Rightarrow 4A = -2$$

$$\Rightarrow \quad u_1(x) = -\frac{1}{2}x + 2$$

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c) Separation of variables:

$$\Rightarrow \begin{cases} X'' + kX = 0 \\ X(0) = 0, \; X'(3) + X(3) = 0 \end{cases} \quad \text{and} \quad T' = kT$$

$k > 0 \Rightarrow$ trivial solutions (we do not want these)

$k = 0 \Rightarrow$ " " " "

$k < 0$, say $k = -\mu^2 < 0 \Rightarrow X(x) = A\cos(\mu x) + B\sin(\mu x)$

using the BCs: $X(0) = A = 0$

$$X'(x) = \mu A \sin(\mu x) + B\mu\cos(\mu x)$$

$$X'(3) = -\mu A \sin(3\mu) + B\mu\cos(3\mu)$$

$$X'(3) + X(3) = \underbrace{-\mu A}_{=0}\sin(3\mu) + B\mu\cos(3\mu) + B\sin(3\mu) = 0$$

$$\Rightarrow \quad \mu\cos(3\mu) + \sin(3\mu) = 0$$

$$\boxed{\tan(3\mu) = -\mu}$$

# Q8

a) Parseval's identity:

P = period

$$\frac{1}{2P}\int_{-P}^{P}(f(x))^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Coefficients of Fourier series

b) root test for series: $\sum_{n=0}^{\infty} a_n x^n$

$$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}}$$

convergent series if $|x| < R$