Syllabus UCSCIMAT21

Fall 2019

1 Ordinary differential equations (lecture 1a)

1.1 First order ODEs

We consider first-order ordinary differential equations (ODEs) with non-constant coefficients of the following standard form

$$u'(t) + b(t)u(t) = q(t). \tag{1}$$

with initial condition $u(0) = u_0$. The general solution is given by

$$u(t) = e^{-B(t)} \left(C + \int \mathrm{d}s\, q(s) e^{B(s)} \right), \tag{2}$$

where $B(t)$ is an antiderivative of $b(t)$ (i.e., $B'(t) = b(t)$) and $C = u_0 e^{B(0)}$.

read: A.1 (except Wronskian)
exercises: A.1.{1, 3, 5, 11, 20}

1.2 Second order ODEs with constant coefficients

We consider second order ODEs of the form

$$au''(t) + bu'(t) + cu(t) = q(t), \tag{3}$$

where a, b and c are given constants and $u(0) = u_0$, $u'(0) = v_0$ are the initial conitions. We express the solution as a superposition of the *homogeneous* and *particular* solutions: $u(t) = u_h(t) + u_p(t)$.

We obtain the homogeneous solution by plugging in a solution of the form $u(t) = e^{\lambda t}$, yielding the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0.$$

This quadratic equation has two solutions

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the roots are distinct, i.e., $\lambda_- \neq \lambda_+$, the general solution is given by

$$u(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}.$$

If the roots coincide ($\lambda_- = \lambda_+$), we have

$$u(t) = C_1 e^{\lambda_+ t} + C_2 t e^{\lambda_- t}.$$

If the roots are complex and of the form $\lambda_\pm = \alpha \pm \imath\beta$, we can express the solution as

$$u(t) = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right).$$

We obtain the particular solution by the *method of undetermined coefficients*. We try a solution that has a similar form as $q(t)$. For example, when $q(t) = \cos(\pi t)$ we try a solution of the form $u_p(t) = c_0 \cos(\pi t) + c_1 \sin(\pi t)$; when $q(t)$ is a polynomial of order 2, we try $u_p(t) = c_0 + c_1 t + c_2 t^2$, etc. The coefficients can be determined by plugging the trial solution in equation (3).

Finally, we determined any remaining free coefficients (C_1 and C_2 in the homogeneous equation) by plugging in the initial conditions in the full solution $u(t) = u_h(t) + u_p(t)$ and solving the resulting system of equations.

read: A.2 (except Wronskian)
exercises: A.2.{3, 7, 10, 25, 37, 40}

2 Partial differential equations (lecture 1b)

A *partial differential equation* (PDE) is an equation relating derivatives of multi-variate functions. To denote the derivative of a multi-variate function $f(x, y)$ w.r.t. one of its variables we use the partial derivative symbol ∂ in stead of the conventional d. Often, this is abbreviated using a subscript, e.g. f_{xx} denotes the second derivative of f w.r.t. x.

Some examples of PDEs are

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u_t &= \kappa^2 u_{xx} \\ u_t + u_x &= 0 \\ u_{xx} + u_{yy} &= f(x, y) \\ u_{tt} &= c^2(u_{xx} + u_{yy}) \end{aligned}$$

To classify PDEs, we look at the *order* (the highest order derivative that occurs), the *dimensionality* (the number of variables) and the *homogeneity* (whether there is a driving term or not).

To determine the solution, we need to specify additional conditions. A general rule of thumb is that we need one condition for each derivative. For example, consider the one-dimensional wave-equation

$$u_{tt} = c^2 u_{xx}, \tag{4}$$

which is a two-dimensional homogeneous PDE of order 2. We have a second order derivative in x, so we specify two *boundary conditions*:

$$u(t,0) = 0, \tag{5}$$
$$u(t,L) = 0. \tag{6}$$

Note that this implictly defines the domain to be $x \in [0,L]$. We also specify two *initial conditions:* $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$. Note that the initial condition is a function – not a single number as in the case of an ODE.

We can easily verify that both

$$u(t,x) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right),$$

and

$$u(t,x) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right),$$

satisfy the PDE and boundary conditions for $n = 1,2,\ldots$. We to this by verifying that these solutions satisfy both the PDE (4) *and* the boundary conditions (5 - 6).

These solutions are called the *normal modes* of the system, see figure 1. By the *superposition principle*, any linear combination of these normal modes is also a solution. The most general solution, then, is given by

$$u(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n\cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right). \tag{7}$$

To satisfy the initial conditions, we need to determine the coefficients $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right),$$

and

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L}\sin\left(\frac{n\pi}{L}x\right).$$

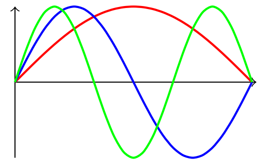

Figure 1: Normal modes at $t = 0$ for $n = 1$, $n = 2$ and $n = 3$.

This is called a *Fourier sine expansion* of f (and g).

Finding the coefficients is rather straightforward when the initial conditions are simple superpositions of sines, such as

$$f(x) = 3\sin\left(\frac{2\pi}{L}x\right) - 5\sin\left(\frac{10\pi}{L}x\right),$$

and

$$g(x) = 0,$$

in which case $a_2 = 3$, $a_{10} = -5$ and all other $a_n = 0$ and $b_n = 0$ for all n. How and under which assumptions we can express a general function f as such a series is the subject of the next section.

read: 1.2, 3.1

exercises: 1.2.{15,17,19}, 3.1.{1,3,5,7}

3 Fourier Series (lecture 4)

We have seen in the previous section that it could be worthwhile studying infinite series of sines and cosines. We ask ourselves how and under what circumstances we can write a given function $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx), \tag{8}$$

where a_n and b_n are constants that need to be determined.

3.1 Continuous and smooth functions

First, we discuss a few properties of functions that are of interest; *periodic*, *(piecewise) continuous* and *(piecewise) smooth* functions.

A function f is *periodic* with period T if we have

$$f(x) = f(x+T),$$

for all x. We generally take T to be the smallest constant such that this holds. From the definition

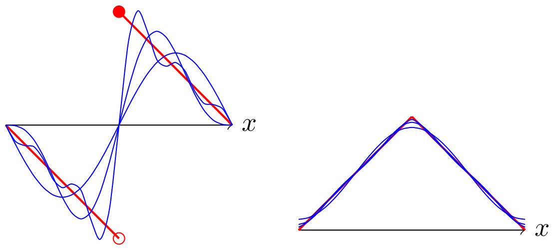

Figure 2: Piecewise continous, periodic functions and their partial Fourier sum for $N = 1, 2$ and 5. Note the difference in the way the partial sum converges to the function.

in equation (8) we see that the function f has to be $2\pi-$*periodic* in order to admit such an expansion.

We only need to specify a $T-$periodic function on the interval $[0, T]$ since it simply repeats itself outside of the interval. This can lead to function with *discontinuities* when $f(0) \neq f(T)$. Take for example the function $(\pi - x)/2$ defined on the interval $x \in [0, 2\pi]$.

A function is *continuous* if at every x the left and right limits exist and are the same. A function is said to be *piecewise continuous* if the left and right limits are different at a only a finite number of points. A function f is *smooth* if both f and f' are continuous and *piecewise smooth* if both f and f' are piece-wise continuous.

An example of a piecewise smooth function is shown in figure 2.

read: 2.1

exercises: 2.1.{2,10}

3.2 The Euler Formulas

We consider a $2p-$periodic function f with Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right).$$

We can find expressions for the coefficients by using the following *orthogonality relations*

$$\begin{aligned}
\int_{-p}^{p} \cos\left(\frac{n\pi}{p}x\right) \cos\left(\frac{m\pi}{p}x\right) &= p\delta_{mn}, \\
\int_{-p}^{p} \sin\left(\frac{n\pi}{p}x\right) \sin\left(\frac{m\pi}{p}x\right) &= p\delta_{mn}, \\
\int_{-p}^{p} \cos\left(\frac{n\pi}{p}x\right) \sin\left(\frac{m\pi}{p}x\right) &= 0,
\end{aligned}$$

for $m, n > 0$ and where $\delta_{mn} = 0$ when $m \neq n$ and $\delta_{mn} = 1$ when $m = n$. We find the Euler formulas

$$a_0 = \frac{1}{2p} \int_{-p}^{p} \mathrm{d}x\, f(x), \tag{9}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} \mathrm{d}x\, f(x) \cos\left(\frac{n\pi}{p}x\right), \tag{10}$$

$$b_n = \frac{1}{p} \int_{-p}^{p} \mathrm{d}x\, f(x) \sin\left(\frac{n\pi}{p}x\right), \tag{11}$$

We can now find the Fourier coefficients for the saw-tooth function introduced earlier. We find $a_n = 0$ and $b_n = \frac{1}{n}$.

To formalize things a little, we introduce the *partial sum*

$$s_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right).$$

The *Fourier Representation Theorem* states that for any piecewise-smooth function f, we have

$$\lim_{N\to\infty} s_N(x) = \frac{f(x_-) + f(x_+)}{2},$$

i.e., the series converges to $f(x)$ if f is continuous at x and to the average of the left and right limits otherwise.

We can use the partial sum to study how the Fourier series converges. Consider for example the saw-tooth function in figure 2 and its partial Fourier sum for $N = 1$, 2 and 5. We observe that the error seems to get uniformly smaller expect around the discontinuity. This is called the *Gibbs phenomenon* and it highlights a fundamental difference between the Fourier series of a continuous and piece-wise continuous function. If f is continuous, the Fourier series converges *uniformly* to the function. For a piece-wise continuous function, the Fourier series converges only point-wise. The distinction is clearly seen in figure 2.

read: 2.2, 2.3

exercises: 2.2.{5,7,9,15,17,24},2.3.{1,2}

computer: 2.2.25 (optional)

3.3 Evaluating infinite sums and operations on Fourier series

We can use this theorem directly to evaluate infinite sums by identifying a given infinite sum as the Fourier series of a particular function f and subsequently evaluating the function.

Going back to the saw-tooth function, we have

$$\frac{\pi - x}{2} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin(nx)}{n}.$$

Now, evaluate this at $x = \pi/2$ to get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Given the Fourier series of one function, can we directly find the Fourier series of a related function? Consider again the saw-tooth function $f(x) = \frac{\pi - x}{2}$ and construct from it a new function $g(x) = f(x) + f(\pi - x)$. We can now express g as

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} + \frac{\sin(n(\pi - x))}{n}.$$

This does not have the form of a Fourier series expansion, but using the identity $\sin(nx - n\pi) = (-1)^n \sin(nx)$, we can re-write this as

$$g(x) = 2 \sum_{n=1}^{\infty} \frac{\sin\left((2n+1)x\right)}{2n+1}.$$

3.4 Fourier sine and cosine series

A natural question is whether we can expand a function using only sines or cosines. An important observation is that when f is *odd* (i.e., $f(x) = -f(x)$) we need only sine terms. Using the following identities

$$\begin{aligned} \int_{-a}^{a} \mathrm{d}x \, \mathrm{odd}(x) &= 0, \\ \int_{-a}^{a} \mathrm{d}x \, \mathrm{even}(x) &= 2 \int_{0}^{a} \mathrm{d}x \, \mathrm{even}(x), \end{aligned}$$

and noting that

$$\begin{aligned} \text{odd} \times \text{odd} &= \text{even}, \\ \text{even} \times \text{odd} &= \text{odd}, \\ \text{even} \times \text{even} &= \text{even}, \end{aligned}$$

we see that the coefficients a_n will be zero for an odd function. Similarly, the coefficients b_n will be zero for an *even* function (i.e., when $f(-x) = f(x)$). The Euler formulas can be simplified using these identities to integrate over half the interval.

If we are given a function that is only defined on half the interval, as is the case with many boundary value problems, we can simplify Euler's by integrating over only half the domain.

read: 2.4
exercises: 2.4.{9,13,15}

3.5 Approximation error

When using the partial sum $s_N(x)$ to approximate a function $f(x)$, we may wonder how big the error is. To this end we introduce the *mean square error*

$$E_N = \frac{1}{2p} \int_{-p}^{p} \mathrm{d}x \, \left(f(x) - s_N(x)\right)^2. \tag{12}$$

Using the Fourier representation Theorem, we find that the integrand $(f(x) - s_N(x))$ tends to zero as $N \to \infty$, except at the points of discontinuity. The integral of a function that is zero everywhere, except at a few points, still yields zero however. We conclude that $\lim_{N\to\infty} E_N = 0$.

Expanding the brackets of the integrand in E_N and using the orthogonality relations stated earlier, we find the following expression for the mean square error

$$E_N = \frac{1}{2p} \int_{-p}^{p} \mathrm{d}x \, f(x)^2 - a_0^2 - \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2).$$

A remarkable observation is that $E_{N+1} \le E_N$, so it never hurts to add more Fourier coefficients.

Using the fact that $E_N \to 0$ we derive *Parseval's Identity*

$$\frac{1}{2p} \int_{-p}^{p} \mathrm{d}x \, f(x)^2 = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{13}$$

This identity implies that when a function is *square integrable*, its Fourier coefficients are *square summable.*

Applying this identity to the saw-tooth function we saw earlier we can obtain the following result

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}.$$

read: 2.5
exercises: 2.5.{1,3}

4 One-dimensional wave and heat equation (lecture 5)

We illustrate the main concepts involved in solving PDEs by considering the one-dimensional wave and heat equations

$$u_{tt} = c^2 u_{xx}, \tag{wave equation}$$

and

$$u_t = \kappa^2 u_{xx}, \tag{heat equation}$$

each supplemented with appropriate boundary and initial conditions.

4.1 Separation of variables

We start with the wave equation and take $u(t,0) = u(t,L) = 0$ as boundary conditions and $u(0,x) = f(x)$, $u_t(0,x) = g(x)$ as initial conditions. The key idea of the method of *separation of variables* is to look for solutions of the form

$$u(t,x) = X(x)T(t).$$

Plugging this in the wave equation and re-ordering some terms we find

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

The only way this equality can hold for all t and x is if both sides are constant so we get

$$X''(x) - kX(x) = 0,$$

and

$$T''(t) - c^2 kT(t) = 0,$$

where k is the (unknown) *separation constant*. We set out to solve these two equations, starting with X. via the characteristic equation we find

$$X(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}.$$

Using the boundary conditions $X(0) = X(L) = 0$ we find (after some effort) that $A = -B$ and $k = -(n\pi/L)^2$, so that

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right),$$

for $n = 1, 2, \ldots$ are the fundamental solutions. Now that the separation constant is fixed we readily find

$$T_n(t) = a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right).$$

Multiplying the solutions back together and summing over n, we arrive that the solution presented earlier in (7) where the coefficients are given by

$$a_n = \frac{2}{L}\int \mathrm{d}x\, f(x) \sin\left(\frac{n\pi}{L}x\right),$$

and

$$b_n = \frac{2}{cn\pi}\int \mathrm{d}x\, g(x) \sin\left(\frac{n\pi}{L}x\right).$$

These expressions can be derived by using the simplified Euler formula for the sine expansion.

For the heat equation we can follow the same procedure, the only difference lies in the solution for T. Note also that we need only one initial condition; $u(0,x) = f(x)$. We find the general solution to be

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi\kappa}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right),$$

where a_n are the Fourier sine coefficients of $f(x)$.

We see that this solution decays exponentially in time. In the limit as $t \to \infty$ we obtain the *steady-state solution*, $u_s(x)$, which in this case is identically zero. We will use this steady-state solution to handle non-zero boundary conditions.

Consider now heat equation with boundary conditions $u(t,0) = T_1$ and $u(t,L) = T_2$ and split the solution in two parts:

$$u(t,x) = u_1(x) + u_2(t,x),$$

plugging this in the equation we see that $u_1(x)$ obeys $u_1'' = 0$ with boundary conditions $u_1(0) = T_1$ and $u_2(L) = T_2$ and u_2 obeys a heat equation with homogeneous boundary conditions ($u_2(0) = u_2(L) = 0$). This is very similar to splitting the solution into a particular and homogeneous solution when solving an inhomogeneous ODE. We find

$$u_1(x) = \frac{T_2 - T_1}{L}x + T_1,$$

and u_2 as before.

read: 3.3, 3.5

exercises: 3.3.{1,2,4,7,12,13}, 3.5.{1,2,3,9}

4.2 Other boundary conditions

So far we have seen boundary conditions that prescribe the value of the solution on the boundaries, these are called *Dirichlet boundary conditions*. Instead, we can prescribe the value of the derivative of the solution on the boundary, these are called *Neumann boundary conditions*. Consider now the heat equation with Neumann boundary conditions $u_x(t,0) = u_x(t,L) = 0$ and initial condition $u(0,x) = f(x)$. Following the separation of variables, we find that

$$u(t,x) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi\kappa}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right),$$

where a_n are the Fourier cosine coefficients of $f(x)$.

A third type of boundary condition is the *Robin boundary condition*, which prescribes a linear combination of the value of the solution and its derivative. As an example, consider the heat equation with mixed boundary conditions $u(t,0) = 0, u_x(t,L) + u(t,L) = 0$. Separating variables and applying the left boundary condition we find

$$X(x) = \sin(\mu x),$$

where $k = -\mu^2$. Plugging in the right boundary condition yields

$$\mu\cos(\mu L) + \sin(\mu L) = 0.$$

We don't have an explicit formula for μ, as we did before, but we know that there are infinitely many solutions to this equation. We denote these by μ_n and express the general solution as

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-(\kappa\mu_n)^2 t} \sin(\mu_n x).$$

To satisfy the initial condition, we need to find coefficients a_n such that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x),$$

which is called a *generalized Fourier series*. Luckily, the functions $\sin(\mu_n x)$ obey an orthogonality relation similar to the one we saw before and we proceed to find the coefficients as

$$a_n = \frac{1}{s_n}\int_0^L \mathrm{d}x\, f(x)\sin(\mu_n x),$$

where $s_n = \int_0^L \mathrm{d}x\, \sin(\mu_n x)^2$.

read: 3.6

exercises: 3.6.{1,2,5,16,17}

4.3 Uniqueness*

How do we now that the solutions we find with separation of variables is unique? We start with the wave equation. Suppose we have two solutions, $u(t,x)$ and $v(t,x)$ that both satisfy the wave equation with initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$, etc. It is readily verified that their difference $w(t,x) = u(t,x) - v(t,x)$ satisfies the wave equation with zero initial conditions. So if we can show that the wave equation with zero initial condition has only the trivial solution $w = 0$ we have established uniqueness since it implies that $u = v$.

To show this we the energy of the system at time t

$$e(t) = \frac{1}{2}\int_0^L \mathrm{d}x\, c^2 w_x(t,x)^2 + w_t(t,x)^2.$$

Differentiating and integrating by parts we find

$$e'(t) = c^2 w_t(t,x) w_x(t,x)\big|_{x=0}^{L} + \frac{1}{2}\int_0^L \mathrm{d}x\, w_t(t,x)\left(w_{tt}(t,x) - c^2 w_{xx}(t,x)\right). \quad (14)$$

Using the boundary conditions we find that the first term equals zero ($w(t,0) = 0$ implies that $w_t(t,0) = 0$ etc.). The second term equals zero because w satisfies the wave equation. We conclude that $e'(t) = 0$. Since $e(0) = 0$ due to the initial conditions we conclude that $w_t(t,x) = 0$ and $w_x(t,x) = 0$. This means that $w(t,x)$ is constant, but because of the boundary conditions this can only mean that $w = 0$.

We can follow a similar derivation for the heat equation using the energy $e(t) = \frac{1}{2\kappa}\int_0^L \mathrm{d}x\, w(t,x)^2$. An alternative proof is offered by the *maximum principle*, which states that when $m \leq w(t,0), w(t,L), w(0,x) \leq M$, we have that $m \leq w(t,x) \leq M$. In words, this says that the solution of the heat equation does not have any local maxima or minima. For zero boundary and initial conditions we again find that $w = 0$. A proof of the maximum principle can be found in section 3.11 of the book.

4.4 PDEs in rectangular domains

We can pose the wave equation on a rectangular domain $[0,a]\times[0,b]$ as

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right),$$

with four boundary conditions on $x = 0$, $x = a$ and $y = 0$, $y = b$ and initial conditions $u(0,x,y) = f(x,y)$, $u_t(0,x,y)$. Applying separation of variables now yields three separate equations

$$\begin{aligned} X'' - kX &= 0, \\ Y'' - lY &= 0, \\ T'' - (k+l)T &= 0, \end{aligned}$$

where k and l are the separation constants. We can solve for X and Y separately using the boundary conditions. When imposing homogeneous Dirichlet boundary conditions, for example, this leads to the normal modes

$$u_{mn}(t,x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\times \left(a_{mn}\cos(\lambda_{mn}t) + b_{mn}\sin(\lambda_{mn}t)\right),$$

with $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$. This leads to a *double Fourier series* of f and g. We can find the coefficients

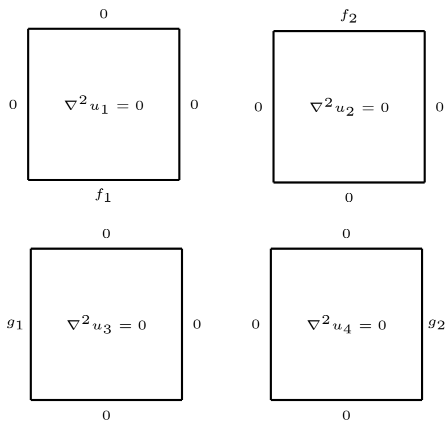

Figure 3: A Dirichlet boundary value problem with inhomogeneous boundary conditions can be split in to four parts.

a_{mn} and b_{mn} by exploiting the orthogonality of the sine functions.

4.5 Laplace's equation

The heat equation on a rectangular domain is treated in a similar fashion. The steady state solution now obeys the *Laplace equation*

$$u_{xx} + u_{yy} = \nabla^2 u = 0,$$

where we introduce more interesting boundary conditions of the form $u(x,0) = f_1(x)$, $u(x,b) = f_2(x)$, $u(0,y) = g_1(y)$, $u(a,y) = f_2(y)$. We don't aim to solve this problem directly, but rather split it into four parts $u = u_1 + u_2 + u_3 + u_4$ as illustrated in figure 4.7.

To solve for u_1 we proceed as follows. Separation of variables yields two ODEs

$$X'' + kX = 0, \quad Y'' - kY = 0.$$

Using the boundary conditions at $x = 0$ and $x = a$ we find $X_n(x) = \sin(\mu_n x)$ with $\mu_n = n\pi/a$. As general solution for the second ODE we find $Y(y) = A\cosh(\mu_n y) + B\sinh(\mu_n y)$. Imposing $Y(b) = 0$ we find

$$Y_n(y) = \sinh(\mu_n(b-y)).$$

Thus, the general solution is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x)\sinh(\mu_n(b-y)).$$

Plugging in the remaining boundary condition $u_1(x,0) = f_1(x)$ leads to a Fourier sine series expansion of f_1.

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with four boundary conditions on $x = 0$, $x = a$ and $y = 0$, $y = b$ and initial conditions $u(0,x,y) = f(x,y)$, $u_t(0,x,y)$. Applying separation of variables now yields three separate equations

$$\begin{aligned} X'' - kX &= 0, \\ Y'' - lY &= 0, \\ T'' - (k+l)T &= 0, \end{aligned}$$

where k and l are the separation constants. We can solve for X and Y separately using the boundary conditions. When imposing homogeneous Dirichlet boundary conditions, for example, this leads to the normal modes

$$u_{mn}(t,x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\times \\ \left(a_{mn}\cos(\lambda_{mn}t) + b_{mn}\sin(\lambda_{mn}t)\right),$$

with $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$. This leads to a *double Fourier series* of f and g. We can find the coefficients a_{mn} and b_{mn} by exploiting the orthogonality of the sine functions.

4.7 Laplace's equation

The heat equation on a rectangular domain is treated in a similar fashion. The steady state solution now obeys the *Laplace equation*

$$u_{xx} + u_{yy} = \nabla^2 u = 0,$$

where we introduce more interesting boundary conditions of the form $u(x,0) = f_1(x)$, $u(x,b) = f_2(x)$, $u(0,y) = g_1(y)$, $u(a,y) = f_2(y)$. We don't aim to solve this problem directly, but rather split it into four parts $u = u_1 + u_2 + u_3 + u_4$ as illustrated in figure 4.7.

To solve for u_1 we proceed as follows. Separation of variables yields two ODEs

$$X'' + kX = 0, \quad Y'' - kY = 0.$$

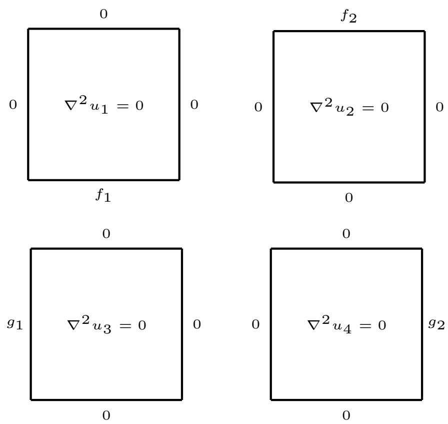

Figure 4: A Dirichlet boundary value problem with inhomogeneous boundary conditions can be split in to four parts.

Using the boundary conditions at $x = 0$ and $x = a$ we find $X_n(x) = \sin(\mu_n x)$ with $\mu_n = n\pi/a$. As general solution for the second ODE we find $Y(y) = A\cosh(\mu_n y) + B\sinh(\mu_n y)$. Imposing $Y(b) = 0$ we find

$$Y_n(y) = \sinh(\mu_n(b-y)).$$

Thus, the general solution is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x)\sinh(\mu_n(b-y)).$$

Plugging in the remaining boundary condition $u_1(x,0) = f_1(x)$ leads to a Fourier sine series expansion of f_1.

4.8 Poisson's Equation

Here we consider a nonhomogeneous version of Laplace's equation

$$\nabla^2 u = f(x,y),$$

which we can't solve directly using separation of variables. As with nonhomogeneous ODEs, though, we can split it into two parts; an inhomogeneous equation with homogeneous boundary conditions and a homogeneous equation with nonhomogeneous boundary conditions, as illustrated in figure 4.8.

We've already seen how to solve the second part in the previous section. To solve the first part, we consider the solutions we found when solving the wave and heat equations with homogeneous boundary conditions

$$\phi_{mn}(x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}x\right).$$

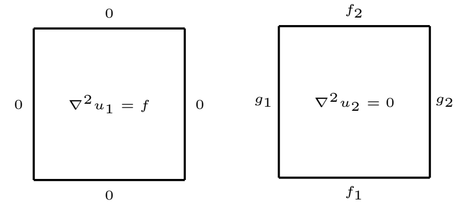

Figure 5: A Poisson problem with inhomogeneous boundary conditions can be split in to two parts.

When applying the Laplace operator to these solutions we find that they have the remarkable property

$$\nabla^2 \phi_{mn} = \lambda_{mn}\phi_{mn},$$

where $\lambda_{mn} = -\pi^2\left((m/a)^2 + (n/b)^2\right)$. We say that ϕ_{mn} is an *eigenfunction* of the Laplace operator (with homogeneous Dirichlet boundary conditions) and λ_{mn} is the corresponding *eigenvalue*. The idea, now, is to look for solution of the Poisson problem of the form

$$u_1(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\phi_{mn}(x,y).$$

Plugging this in the equation leads to a double Fourier sine series expansion of f

$$f(x,y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} a_{mn}\phi_{mn}(x,y)$$