Morse Homology and Novikov Homology

by

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Abstract

The goal is to define the Morse homology and show the Morse homology theorem, which states that the Morse homology is isomorphic to the singular homology. To do this we first explain the necessary background in Morse theory. We proof the Morse inequalities, these give a relation between the critical points of a Morse function and the Betti numbers. After the Morse homology theorem we explain a generalisation of Morse Homology named Novikov Homology. Here we consider a Morse form instead of a Morse function. We will define the Novikov homology and also explain the Novikov inequalities.

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CHAPTER 1

Introduction

1. Overview

Morse theory and Morse homology give a way to study the topological properties of a manifold by studying Morse functions on a manifold. In the first part of this thesis we will prove the main theorem of this thesis, the Morse homology theorem.

THEOREM 1.1 (Morse Homology Theorem). *Let M be a closed manifold*[(i)] *equipped with a Morse function f and gradient-like vector field v obeying the Morse-Smale condition then we have an isomorphism between the Morse homology and the Singular homology.*

$$HM_k(M) \simeq H_k(M)$$

To get to this result we need to do some preliminary work. A function $f : M \to \mathbb{R}$ is called Morse if all the critical points are non-degenerate, i.e. the Hessian is non-singular. The Morse Lemma then states that near a critical point c the Morse function $f : M \to \mathbb{R}$ looks like

$$f(x) = c - (x_1^2 + ... + x_i^2) + x_{i+1}^2 + ...x_n^2.$$

Here i coincide with the index of the critical point c. The index is defined to be the dimension of the negative definite subspace of the Hessian. Together with a gradient-like vector field v we can show that the manifold has a CW structure. With this we will derive the Morse inequalities, these give a relation between the number of critical point of certain index and the Betti numbers. The strong version of the inequalities are

$$\sum_{i=0}^{j}(-1)^{j-i}c_i \geq \sum_{i=0}^{j}(-1)^{j-i}b_i(M)$$

Where c_i is the number of cirtical points of index i and b_i is the i-th Betti number.

When the vector field v also obeys the Morse-Smale condition, we will define the Morse homology with coefficients modulo two. The k-th complex of the Morse chain complex are all formal sums with critical points of index k,

$$C_k(f, v) := \{\sum a_1 c_i : a_1 \in \mathbb{Z}_2 \text{ and } c_i \in Cr_k l(f)\}.$$

The boundary operator $\delta : C_k(f, v) \to C_{k-1}(f, v)$ is defined as

$$\delta(c) = \sum_{b \in Cr_{k-1}(f)} n(c, b)b$$

the number $n(c, b)$ counts the number of flow paths from c to b modulo two. The Morse-Smale condition makes sure that the boundary operator is well defined and squares to zero.

[(i)]A manifold is called closed if it is compact and an empty boundary

After this we will at Novikov homology. This is concerned with closed one-forms with a gradient like vector field. Morse homology is a special case of this since the closed one form we consider then is df.

An important class of the Novikov homology, is the case of a circle-valued Morse function. We show how a circle-valued function is related to an integral closed one-form. We can construct the Novikov complex in a similar way as the Morse complex only with coefficients in the Novikov ring. The homology that follows from this construction isomorphic to the singular homology with local coefficients. This local system is constructed using the closed one-form.

2. Organization of this thesis

In chapter 2 we start with Morse theory, this chapter concerns with preliminaries for the following chapters. We define what a Morse function is and show that a smooth closed manifold has a CW-decomposition. In this chapter we also prove the Morse inequalities.

Chapter 3 is concerned with the definition of the Morse homology groups. To show that it is well-defined we introduce the space of broken trajectories $\overline{\mathcal{L}}(a, b)$. We show that in the case that the indices differ by 2 this space is a compact one-dimensional manifold.

The Morse homology theorem is proven in chapter 4. This theorem states that the Morse homology is isomorphic to the singular homology. In order to do so, we need to know some results from cellular homology these shall be treated as well.

The generalisation of Morse theory explained in chapter 5. This generalisation is called Novikov theory and we also have the Novikov homology. Novikov theory use a closed one-form to investigate the topology of a manifold.

CHAPTER 2

Morse Theory

In this chapter we will introduce Morse functions on smooth compact manifolds. We will show that locally around a critical point these functions behave as quadratic functions. Using a Morse function together with a vector field satisfying the Morse-Smale condition, we show that the manifold has the homotopy type of a CW-complex. Finally we prove the Morse inequalities, which show that the n-th Betti number is smaller then the number of critical points of index n. This chapter will follow mostly [Mil63] with influences from [AD14] and [BH04].

1. Morse functions

In the first part concerning Morse theory we will consider a smooth function, $f : M \to \mathbb{R}$ where M is a m-dimensional closed manifold. In the rest of this thesis we assume the function to be smooth and the manifold to be closed, unless specifically stated otherwise.

Definition 2.1. *Given V a vector space and let $A : V \times V \to \mathbb{R}$ be a bilinear form.*

- *The bilinear form A is called non-degenerate if*
$$\forall y \in V \quad A(x, y) = 0 \Rightarrow x = 0.$$
- *The index of A is the maximal dimension of the subspace where A is negative definite, i.e.*
$$\operatorname{Ind} A := \dim\{w \in V : A(w, w) < 0\}.$$

Definition 2.2. *Given a function $f : M \to \mathbb{R}$, a critical point of f is a point $x \in M$ such that $df(x) = 0$. The set of all critical points of f is denoted as $Cr(f)$.*

Given a coordinate system $(x_1, ..., x_n)$ around a critical point, x, then the above definition is equivalent to saying that $\frac{\partial f}{\partial x_i}(x) = 0$ for all i.

In the case of a critical point, c, we can define a bilinear form
$$H_f : T_cM \times T_cM \to \mathbb{R}; \quad H_f(X, Y) = \mathcal{L}_X \mathcal{L}_Y f. \quad ^{(i)}$$
This bilinear form is called the Hessian and is symmetric since
$$\mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f = \mathcal{L}_{[X,Y]} f = df \cdot [X, Y] = 0.$$
Because of this symmetry the form is well defined since it doesn't depend on the extension of the vector field. When this bilinear form is non-degenerate then we call the critical point non-degenerate. Again, given a coordinate system $(x_1, ..., x_n)$ around our critical point x, then we can translate the Hessian to the matrix of second order partial derivatives:
$$H_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{ij}.$$
Then non-degeneracy of H_f is equivalent to saying that the corresponding matrix is invertible.

(i) This is the standard definition of the Lie derivative for example see [Lee13]

DEFINITION 2.3 (Morse Function). *A function $f : M \to \mathbb{R}$ is called a Morse function when its the critical points are non-degenerate.*

REMARK 2.4. For every compact manifold there are Morse functions. In fact every smooth function can be approximated by a Morse function. This is beyond the scope of this thesis, but to clarify a bit, a way of proving this is by using the fact that a smooth compact manifold can be embedded in to the euclidian space $\mathbb{R}^k$ for some k. Then define a distance function from a point to the embedded manifold

$$f_p : M \to \mathbb{R}, \quad f_p(x) = \|x - p\|^2.$$

By Sard's Theorem the set of critical values of such function have zero measure, which implies an abundance of Morse functions. We can even proof that every smooth function can be approximated by Morse function.

Consider a Morse function f then at a critical point the bilinear form H_f is well defined. Hence the following definition makes sense.

DEFINITION 2.5. *The index of a critical point x of a Morse function is f is defined as,*

$$\operatorname{Ind}(f, x) := \operatorname{Ind} H_f$$

If it is clear what the Morse function is, we denote the index as $\operatorname{Ind}(x) := \operatorname{Ind} H_f$. Furthermore we define

$$Cr_i(f) := \{x \in Cr(f) : \operatorname{Ind}(x) = i\}.$$

2. Morse's Lemma

The main result in this section is Morse's Lemma. This lemma allows us to describe the function around a critical point. This will prove useful when determining the homotopy type of the manifold and also with the Morse homology theorem.

LEMMA 2.6 (Morse's Lemma). *Let c be a non-degenerate critical point of the function $f : M \to \mathbb{R}$. There exists a neighborhood U of c and a diffeomorphism $\varphi : (U, c) \to (\mathbb{R}^n, 0)$ such that*

$$f \circ \varphi^{-1}(x_1, ..., x_n) = f(c) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2, \text{ with } i = \operatorname{Ind}(c).$$

To prove Morse's lemma we need the following lemma.

LEMMA 2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f(0) = 0$, then*

$$f(x) = \sum_{i=1}^{n} x_i g_i(x),$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$.

PROOF. Given the function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(0) = 0$ we can write

$$\begin{aligned} f(x) &= f(x) - f(0) \\ &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx)x_i dt \\ &= \sum_{i=1}^n x_i g_i(x)dt \end{aligned}$$

□

If we look at a function f such that $f(0) = 0$ and 0 is a critical point then we can apply the lemma on $g_i(x)$, since $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$. Doing this gives us,

$$f(x) = \sum_{i,j=1}^n x_i x_j h_{ij}(x).$$

By defining the matrix $(H(x))_{ij} = \frac{1}{2}(h_{ij}(x) + h_{ji}(x))$ the function can be expressed as

$$f(x) = x^t H(x)x.$$

Note that $H(x)$ is symmetric for every x. Now we are ready to prove Morse's Lemma.

PROOF OF MORSE'S LEMMA. We may assume that $f(0) = 0$ and $df(0) = 0$ on a convex neighborhood. The matrix of the hessian at 0 is of the form

$$B = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)_{ij}.$$

This matrix is non singular since we look at a non-degenerate critical point. Therefore there is a coordinate base such that B is a diagonal matrix with entries ± 1. From this matrix B we make a function

$$A : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto x^t B x = \sum_{i=1}^n \delta_i x_i^2$$

with $\delta_i = \left(\frac{\partial^2 f}{\partial x_i \partial x_i}(0)\right) = \pm 1$. Now we want to prove that there exists a diffeomorphism φ such that $f \circ \varphi = A$.

What we will prove is that there exists a family of diffeomorphisms such that

$$f_t \circ \varphi_t = f_0, \quad \text{with } f_t(x) = t \cdot f(x) + (1-t)A(x). \tag{2.1}$$

When we prove such a family exists then for $t = 1$ we obtain our desired result.

Belonging to the diffeomorphisms φ_t we have a family of smooth vector fields v_t defined as the tangent lines of $t \mapsto \varphi_t(x)$ and we have

$$\frac{d\varphi_t}{dt}(x) = v_t(\varphi_t(x)).$$

Differentiating both sides of (2.1) with respect to t gives

$$\left(\frac{df_t}{dt} + df_t \cdot v_t\right) \circ \varphi_t = 0$$

If we have a vector field such that $df_t \cdot v_t = B - f =: g$ then we have proven Morse's Lemma. With lemma 2.7, g can be expressed as,

$$g(x) = \sum_{i,j}^{n} x_i x_j h_{ij}(x) = \langle Hx, x \rangle$$

since $g(0) = 0$ and $dg(0) = 0$.

An adaptation of lemma 2.7 allows us to write

$$df_t(x) \cdot v_t = \langle B_t(x) v_t, x \rangle \quad \text{with} \ (B_t(x))_{ij} = \int_0^1 \frac{\partial^2 f_t}{\partial x_i \partial x_j}(sx) ds. \tag{2.2}$$

This can be seen by writing out the exterior derivative

$$df_t(x) \cdot v_t = \sum_{i=1}^{n} v_t^i \frac{\partial f_t}{\partial x_i}(x)$$

Further note that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial A}{\partial x_i}(0) = 0$ hence evaluating

$$\frac{\partial f_t}{\partial x_i} = t \frac{\partial f}{\partial x_i} + (1-t) \frac{\partial A}{\partial x_i} \tag{2.3}$$

at the point $x = 0$ results in

$$\frac{\partial f_t}{\partial x_i}(0) = 0.$$

Applying lemma 2.7 on the functions $\frac{\partial f_t}{\partial x_i}(x)$ gives that

$$\frac{\partial f_t}{\partial x_i}(x) = \sum_{j=1}^{n} x_j \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(sx) ds.$$

Substituting gives the desired equation (2.2).

Note we have $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 A}{\partial x_j \partial x_i} = 0$ when $i \neq j$. Hence the matrix $B_t(0)$ is a diagonal matrix with on the diagonal

$$(B_t(0))_{ii} = t \frac{\partial^2 f}{\partial x_i \partial x_i} + (1-t) \frac{\partial^2 A}{\partial x_i \partial x_i} = t\delta_i + (1-t)2\delta_i = (2-t)\delta_i$$

Therefore $B_t(0)$ is invertible for all $t \in [0,1]$. From the tube lemma we know there exists an open $0 \in U \subset M$ such that $B_t(x)t$ is invertible for all $x \in U$ and $t \in [0,1]$. Now the vector field is determined by $v_t(x) = (B_t(x))^{-1} Hx$ and we see that

$$\langle B_x^t v_t, x \rangle = \langle Hx, x \rangle,$$

which translates to $df_t \cdot v_t = g = B - f$ and $v_t(0) = 0$. This gives us $f_t \circ \varphi_t = f_0$ and especially $f_1 \circ \varphi_1 = f_0$ which is our desired diffeomorphism. □

From lemma 2.6 we can immediately conclude the following corollary.

Corollary 2.8. *Non-degenerate critical points are isolated.*

Since the manifold is compact there are only finitely many critical points.

Definition 2.9. *A Morse neighborhood of a critical point is a neighborhood U on which there exists a chart such that the function f can be written as in the Morse's lemma.*

3. Vector fields

The definition of Morse homology requires a vector field that obeys the Morse-Smale condition. Such a vector field has some nice properties. Given a critical point c, by flowing along the vector field we will reach another critical point. Along the vector field the index of a critical point decreases. Furthermore such a vector field is used to define the boundary operator ∂ between the Morse complexes.

DEFINITION 2.10. *Let $f : M \to \mathbb{R}$ be a function. We call v a gradient-like vector field if the following conditions are satisfied:*

- *$df(x) \cdot v(x) \leq 0$ with equality if and only if x is a critical point.*
- *In a neighborhood of the critical point this vector field should coincide with the negative gradient of the function.*

REMARK 2.11. A gradient-like vector field always exists for a Morse function on a compact manifold. Indeed for every point in M, there exists a neighborhood U_i together with a diffeomorphism $\varphi : U \to \mathbb{R}^n$ giving injections

$$T_x\varphi : T_xM \to T_\varphi(x)\mathbb{R}^n \quad \forall x \in U_i.$$

For the function

$$f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$$

we can take the negative gradient, i.e. $-\nabla(f \circ \varphi^{-1})$. This vector field can be pulled back, resulting locally in the vector field

$$v_i(x) = (T_x\varphi)^{-1}(\nabla(f \circ \varphi^{-1})(\varphi(x))).$$

We may assume that the critical points are only contained in one open. Using a partition of unity ϕ_i subordinate to the cover $\{U_i\}$, which exists because the manifold is compact. The globally defined vector field

$$v(x) = \sum_i \phi_i(x)v_i(x)$$

is a gradient-like vector field. If c is a critical point then $v(c) = v_i(c)$ for some i and thus equals the negative gradient. Furthermore if p is not a critical point then $v(p)$ is a finite sum of negative numbers, because we look at the negative gradient, and hence negative.

With a vector field there is also the flow of the vector field, $\psi_x(t) : \mathbb{R} \to M$ with $\frac{d\psi_x(t)}{dt} |_{t=t_0}= X(\psi_x(t_0))$. Since the manifold is compact we know that the vector field is complete, meaning that $\psi_x(t)$ is well defined on $\mathbb{R}$ [Lee13, Cor. 9.17.].

LEMMA 2.12. *Let $f : M \to \mathbb{R}$ be a function and X a gradient-like vector field. Then for every $x \in M$*

$$\lim_{t\to\infty} \psi_x(t) \text{ and } \lim_{t\to-\infty} \psi_x(t)$$

exist and are both critical points of f.

PROOF. Assume that for a point $x \in M$, $\varphi_t(x)$ does not converge to a critical point. This means that around each critical point $c_i \in Cr(f)$ there exists an open U_i for which there exists a t_i such that when $t > t_i$, $\varphi_t(x) \notin U_i$. Since the vector field is gradient-like there exists a $\epsilon < 0$ such that $df(x') \cdot X(x') < \epsilon$ for each $x' \in M \setminus (\bigcup_i U)$. This gives for $s > T := \max_i\{t_i\}$

$$\begin{aligned}\frac{d(f(\varphi_t(x)))}{dt} &= df(\varphi_t(x)) \cdot \frac{d\varphi_t(x)}{dt} \\ &= df(\varphi_t(x)) \cdot X(\varphi_t(x)) \\ &< \epsilon < 0\end{aligned}$$

Hence

$$f(\varphi_s(x)) < f(\varphi_t(x)) + (s - T)\epsilon.$$

The right hand side goes to minus infinity when s goes to infinity. But this is a contradiction with the fact that $f(M)$ is bounded. Therefore $\lim_{t\to\infty} \psi_x(t)$ is a critical point. The case $s \to -\infty$ is similar. □

DEFINITION 2.13. *Let M to be a compact manifold and ψ a flow on M. Let $a \in M$ be a critical point. We define the stable manifold of a as*

$$W^s(a) = \{x \in M \mid \lim_{t\to+\infty} \psi_x(t) = a\}$$

and the unstable manifold of a as

$$W^u(a) = \{x \in M \mid \lim_{t\to-\infty} \psi_x(t) = a\}.$$

REMARK 2.14. In the case that we regard a Morse function f and a gradient-like vector field the spaces $W^s(a)$ and $W^u(a)$ are in fact submanifolds of the manifold M. Furthermore $W^u(a)$ is homeomorphic to the $\text{Ind}(a)$-dimensional disk.

A direct consequence of the Morse's Lemma is that

$$\dim W^u(a) = \text{codim} W^s(a) = \text{Ind}(a).$$

To explain what it means for a vector field to obey the Morse-Smale condition, we first need to introduce the notion of transversal. Let V be a vector space and A and B be vector subspaces. Then A is transversal to B when $A + B = V$. This will be denoted as $A \pitchfork B$.

DEFINITION 2.15. *Let N_1 and N_2 be submanifolds of the manifold M, N_1 is transversal to N_2 when $T_xN_1 \pitchfork T_xN_2$ for all $x \in N_1 \cap N_2$. This is also denoted as $N_1 \pitchfork N_2$.*

Note that if $N_1 \cap N_2 = \emptyset$ then the submanifolds are also transversal.

Transversality has the following nice property that we will use often

LEMMA 2.16. *Let A and B be submanifolds of the manifold M, with dimensions a,b and m respectively. If $A \pitchfork B$, then $A \cap B$ is again a submanifold of M with dimension $a + b - m$.*

PROOF. See [BH04, p. 132] □

DEFINITION 2.17. *Let f be a Morse function. Then a gradient-like vector field, v is said to satisfy the Morse-Smale condition if all stable and unstable manifolds meet transversally, i.e. $W^u(a) \pitchfork W^s(b)$ $\forall a, b \in Cr(f)$.*

That such vector field exists can be found in [Sma61].

Define the intersection

$$\mathcal{M}(a, b) := W^u(a) \cap W^s(b).$$

These are the points with origin a and destination b. By lemma 2.16 this is again a submanifold of dimension

$$\dim(\mathcal{M}(a, b)) = \dim(W^u(a)) + \dim(W^s(b)) - n$$

We see that $\dim(W^u(a)) = \mathrm{Ind}(a)$ and $\dim(W^s(b)) = n - \mathrm{Ind}(b)$ and hence we can rewrite the expression as

$$\dim(\mathcal{M}(a,b)) = \mathrm{Ind}(a) - \mathrm{Ind}(b).$$

An action $\varphi : G \times X \to X$ is called free when

$$g \cdot x = x \Rightarrow g = e$$

If the action is from a topological group G and a topological space X then the action is called proper when the map

$$\phi : G \times X \to X \times X; \quad (g,x) \mapsto (g \cdot x, x)$$

is proper, meaning that the preimage of a compact is again compact.

THEOREM 2.18. *The action of $\mathbb{R}$ induced by the flow on $\mathcal{M}(a,b)$ is free and proper when $a \neq b$.*

Using this result we can use the quotient manifold theorem [Lee13, Cor.21.10]. This theorem states that the quotient map

$$\pi : M \to M/\mathbb{R} =: \mathcal{L}(a,b)$$

is a smooth submersion and the topology on $\mathcal{L}(a,b)$ is the quotient topology. This quotient manifold has as dimension $\mathrm{Ind}(a) - \mathrm{Ind}(b) - 1$. Since the dimension of the manifold has to be non-negative, this implies that the index decreases along the flow lines.

To prove theorem 2.18 we use a few topological facts.

LEMMA 2.19. *Let X, Y be topological spaces. Then the following holds:*

- *Given $A, B \subset X$ compact subspaces then $A \cup B$ and $A \times B \subset X \times X$ are also compact spaces. Here $X \times X$ is endowed with the product topology*
- *If X is a Hausdorff space then if a $A \subset X$ is a compact then A is also closed.*
- *If $A \subset X$ is compact and $B \subset A$ closed, then B is also compact.*
- *Given a continuous function $f : X \to Y$, if $K \subset X$ compact then $f(K) \subset Y$ is also compact.*

PROOF. These results can be found in chapter 1, paragraph 9 of [Bou66]. □

We will need the following result which we prove on page 10.

LEMMA 2.20. *If $K \subset \mathcal{M}(a,b)$ is compact then the set $H_K := \{t \in \mathbb{R} : tK \cap K \neq \emptyset\}$ is compact.*

PROOF OF THEOREM 2.18. Let K be compact in $M \times M$. Then the projection of K on the first coordinate p_1 is also compact, as the projection is a continuous function. The same result applies to the projection onto the second coordinate p_2. Denote

$$K' := (p_1(K) \cup p_2(K)).$$

Then $K \subset K' \times K'$ is again a compact set by lemma 2.19. If $(x,t) \in \psi^{-1}(K)$ then this means that $(t \cdot x, x) \in K \subset K' \times K'$ and especially $x \in K'$ and $t \in H_{K'}$. Therefore,

$$\psi^{-1}(K) \subset K' \times H_{K'}.$$

Since ψ is a continuous function the inverse of a closed set is closed. The set K is closed, since $K \subset \mathcal{M}(a,b)$ is a compact set in a Haussdorf space. Therefore the inverse image of K, $\psi^{-1}(K)$, is also closed. Moreover it is closed inside the compact set $K' \times H_{K'}$ and therefore also compact. Hence the action is proper.

To see it is free note that the function is strictly descreasing when not in a critical point, i.e. for $x \in \mathcal{M}(a,b)$ we have,

$$\psi_x(t) > \psi_x(s) \text{ if } t < s.$$

So if $\psi_x(t) = x = \psi_x(0)$ then $t = 0$. We conclude that the action is also free. □

PROOF OF LEMMA 2.20. Since H_k is a subset of $\mathbb{R}$ it is enough to show that the set is bounded and closed. Note that $\mathcal{M}(a,b)$ is a Hausdorff space and therefore K is also closed.

Now we will show that H_k is bounded. To this end, note that since K is compact, $f(K)$ is also compact. Now since $b \notin K$ there exists an $\epsilon > 0$ such that $f(K) > f(b) + \epsilon$. Therefore the open $B := f^{-1}(-\infty, b+\epsilon)$ does not intersect K. Now for every $x \in K$ there exists a $t_x \in \mathbb{R}$ such that $\varphi_{t_x}(x) \in B$. Because $\mathcal{M}(a,b)$ is Hausdorff there exist opens $V \ni \varphi_{t_x}(x)$ and $U \ni x$ such that $V \cap U = \emptyset$. Now we can look at the open

$$A := \varphi_{t_x}^{-1}(V \cap B) \cap U$$

this is an open containing x and such that $\varphi_t(A) \cap K = \emptyset$ for $t > t_x$. For all $x \in K$ there exists an open A_x together with a t_A such that

$$\varphi_t(A_x) \cap K = \emptyset \text{ for } t > t_A$$

Now use the compactness of K to extract a finite subcover and take the maximum of the t_A to get an upper bound for H_K. We can obtain the lower bound in a similar way.

To show H_K is closed is equivalent to showing the complement, $\mathbb{R} \setminus H_K$, is open. To do this assume $t_0 \notin H_K$ so that $K \cap t_0 \cdot K = \emptyset$. Take $y \in t_0 \cdot K$. Because K is closed there is an open U that does not intersect K. Look at $\psi^{-1}(U)$. This is an open in $\mathbb{R} \times \mathcal{M}(a,b)$. This set contains $(0,y)$ and with the product topology there exists $V \subset \mathcal{M}(a,b)$ and $(-\epsilon,\epsilon) \subset \mathbb{R}$ such that

$$(0,y) \in (-\epsilon,\epsilon) \times V \subset \varphi^{-1}(U)$$

We use the set V as an open around y because

$$\varphi_t(V) \cap K = \emptyset \quad \forall t \in (-\epsilon,\epsilon).$$

For every point $y \in t_0 \cdot K$ there exists such an open V and ϵ. Now we can use compactness of $t \cdot K$ to get a finite subcover and take the minimum over the ϵ. Then for $t_0 - \min(\epsilon) < t < t_0 + \min(\epsilon)$ it holds that $K \cap t \cdot K = \emptyset$. Hence the complement is open and therefore H_K is closed.

We have shown that H_K is closed and bounded in $\mathbb{R}$ and therefore compact. □

REMARK 2.21. It is also possible to see the space of trajectories as

$$\mathcal{L}(a,b) := \mathcal{M}(a,b) \cap f^{-1}(\alpha)$$

where $\alpha \in (f(b), f(a))$. This is immediately to be seen a submanifold since $\mathcal{M}(a,b) \pitchfork f^{-1}(\alpha)$. But then it is not trivial that this definition is independent of α. In fact there are diffeomorphisms between different α. Such a diffeomorphism can be constructed using the flow and the fact that there are no critical points in $\mathcal{M}(a,b)$.

4. Homotopy Type

We consider the sublevel sets $M^a := f^{-1}((-\infty, a])$. We will investigate how the homotopy type changes when we change a. First we consider the interval $[b,c]$ on which f doesn't have a critical value. Then M^b is a deformation retract of M^c. Secondly, in case the interval does contain a critical value, we will see that homotopywise this is the same as adding a k-handle. Finally we show inductively that the entire manifold is homotopic to a CW-complex.

DEFINITION 2.22. *Let X be a topological space and A a subspace. A (strong) deformation retract of X on A is a homotopy between Id_X and the map $r : X \to A$ for which $r(A) = Id_A$.*

THEOREM 2.23. *Suppose that the interval $[a,b] \subset \mathbb{R}$ contains no critical values of the function $f : M \to \mathbb{R}$ and suppose $f^{-1}[a,b]$ is compact. Then the sets M^a and M^b are diffeomorphic. Furthermore, M^a is a deformation retract of M^b, so that the inclusion $M^a \to M^b$ is a homotopy equivalence.*

PROOF. The function does not contain a critical value in $[a,b]$. Therefore X is non zero on $f^{-1}[a,b]$, meaning that $df \cdot X < 0$. Define the smooth function $\rho : M \to \mathbb{R}$ in such a way that

$$\rho(x) = -\frac{1}{df(x) \cdot X(x)}$$

on $f^{-1}[a,b]$ and that it vanishes outside a compact neighborhood of $f^{-1}[a,b]$. Then the vector field $\tilde{X} = \rho X$ has compact support and hence defines an one-parameter group of diffeomorphisms $\phi_t : M \to M$. For these diffeomorphisms we have $\tilde{X}(\phi_t(x)) = \frac{d}{dt}\phi_t(x)$. For an, $x \in M$ we get a path

$$\gamma_x : \mathbb{R} \to \mathbb{R}; \quad \gamma_x(t) = f(\phi_t(x))$$

Then differentiating γ_x with respect to t yields

$$\begin{aligned} \frac{d}{dt}\gamma(t) &= df(\phi_t(x)) \cdot \frac{d}{dt}\phi_t(x) \\ &= df(\phi_t(x)) \cdot \tilde{X}(\phi_t(x)) \end{aligned}$$

If $\phi_t(x) \in f^{-1}[a,b]$ then

$$\tilde{X}(\phi_t(x)) = -\frac{X(\phi_t(x))}{df(\phi_t(x)) \cdot (X)(\phi_t(x))}.$$

This implies

$$\frac{d}{dt}\gamma(t) = -1 \text{ and } \gamma(t) - \gamma(0) = \int_0^t \frac{d}{ds}\gamma(s)ds = -t.$$

Take the map $\phi_{b-a} : M \to M$. Note that ϕ_{b-a} restricts to a diffeomorphism from M^b to M^a. Its inverse is given by ϕ_{a-b}. When $x \in f^{-1}(b)$, then $\phi_{b-a}(x) \in f^{-1}(a)$ since

$$f(\phi_{b-a}(x)) = \gamma(b-a) = \gamma(0) - b + a = b - b + a = a.$$

To show that M^a is deformation retract we consider the homotopy

$$r_t(x) = \begin{cases} x & \text{if } f(x) \leq a \\ \phi_{t(a-f(x))}(x) & \text{if } a \leq f(x) \leq b \end{cases}$$

This continuous function is the homotopy between $r_0 = Id_{M^b}$ and r_1, which is the retraction from M^b to M^a. Hence M^a is a deformation retract of M^b. □

The above theorem shows that M^a and M^b are homotopy equivalent if there are no critical values between a and b. The next theorem expresses what happens when we do cross a critical value.

THEOREM 2.24. *Let $f : M \to \mathbb{R}$ be a function and $a \in Cr_k(f)$ non-degenerate. Set $\alpha = f(a)$. Suppose that for some sufficiently small $\epsilon > 0$, the set $f^{-1}[\alpha - \epsilon, \alpha + \epsilon]$ is compact and does not contain any critical point of f other than a. Then for every sufficiently small $\epsilon > 0$, the homotopy type of the space $M^{\alpha+\epsilon}$ is that of $M^{\alpha-\epsilon}$ with a cell of dimension k attached (the unstable manifold of a).*

Proof. Take a neighborhood U of the critical point, such that the function can be written as in lemma 2.6. On this neighborhood there is a chart such that

$$f = \alpha - (x_1^2 + ... + x_k^2) + (x_{k+1}^2 + ... + x_n^2).$$

Lets denote $x_1^2 + ... + x_k^2 = \chi_-(x)$ and $x_{k+1}^2 + ... + x_n^2 = \chi_+(x)$. Then the function can be written as

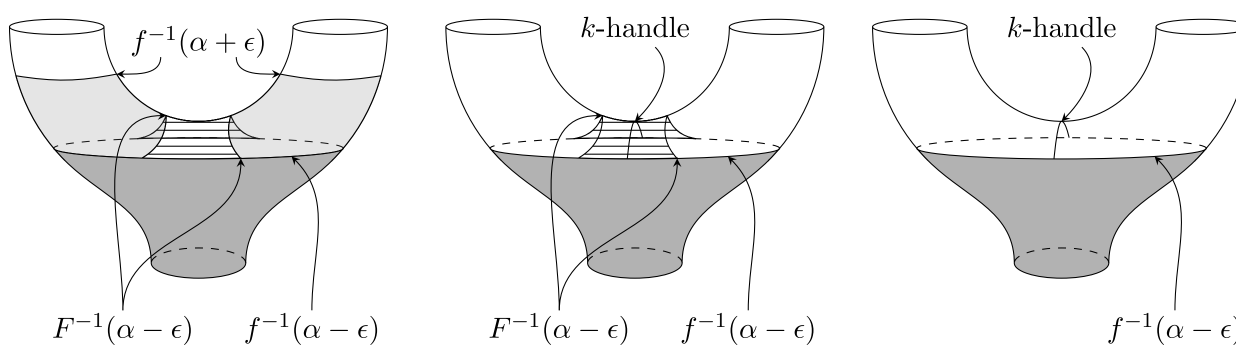
$$f(x) = \alpha - \chi_-(x) + \chi_+(x). \tag{4.1}$$

Moreover there exists an ϵ small enough such that the neighborhood U contains the ball of radius $\sqrt{2\epsilon}$ and such that $f^{-1}[\alpha - \epsilon, \alpha + \epsilon]$ contains no critical points beside a. Now the k-cell is given as

$$e^k := \{x \in U : \chi_-(x) \leq \epsilon \text{ and } \chi_+(x) = 0\}.$$

Now we still need to show that $M^{\alpha+\epsilon}$ is of the same homotopy type as $M^{\alpha-\epsilon}$ together with this handle.

The problem is that $f^{-1}[\alpha - \epsilon, \alpha + \epsilon]$ is not contained in the neighborhood U around the critical point a. Therefore we can not make an explicit homotopy. To overcome this problem we will construct a function $F : M \to \mathbb{R}$. This F lowers the values in small open around a without creating new critical points. This is illustrated in the figures below. First we retract $f^{-1}(-\infty, \alpha + \epsilon]$ onto $F^{-1}(-\infty, \alpha - \epsilon]$ as shown in figure 1a. Then we construct a homotopy from $F^{-1}(\alpha - \epsilon)$ to $M^{\alpha-\epsilon} \cup e_k$ as shown in figure 1b by the horizontal lines. The result in figure 1c shows the k-handle and $f^{-1}(\alpha - \epsilon)$. In these figures we have $k = 1$.



(a) Step 1 **(b)** Step 2 **(c)** Step 3

Figure 1

To construct this F first define a smooth function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mu(0) > \epsilon$, $\mu(s) = 0$ for $s \geq 2\epsilon$ and $-1 < \mu'(s) \leq 0$ for every s. Note that μ is supported on U. Then define F as follows

$$F(x) = \begin{cases} f(x) & \text{if } x \notin U \\ f(x) - \mu(\chi_-(x) + 2\chi_+(x)+) & \text{if } x \in U \end{cases}$$

Together with 4.1, the function F can be written as

$$F(x) = \alpha - \chi_-(x) + \chi_+(x) - \mu(\chi_-(x) + 2\chi_-(x)).$$

This function is again smooth. Moreover the function F has the following properties

- Firstly,

$$F^{-1}(-\infty, \alpha + \epsilon] = f^{-1}(-\infty, \alpha + \epsilon] \tag{4.2}$$

since outside the ellipsoid $\chi_-(x) + 2\chi_+(x) \leq 2\epsilon$ the functions coincide. Inside the ellipsoid we have

$$F(x) \leq f(x) = \alpha - \chi_-(x) + \chi_+(x) \leq \alpha + \frac{1}{2}\chi_-(x) + \chi_+(x) \leq \alpha + \epsilon.$$

With the fact that $F \leq f$ we get that

$$F^{-1}[\alpha - \epsilon, \alpha + \epsilon] \subset f^{-1}[\alpha - \epsilon, \alpha + \epsilon].$$

Since $f^{-1}[\alpha-\epsilon, \alpha+\epsilon]$ is assumed to be compact, the set $F^{-1}[\alpha-\epsilon, \alpha+\epsilon]$ is also compact.

- The critical points of F are the same as the critical points of f. Again outside the open U the functions agree and therefore also the critical points. On U the derivative equals

$$dF = \frac{\partial F}{\partial \chi_-} d\chi_- + \frac{\partial F}{\partial \chi_+} d\chi_+$$

Now $\frac{\partial F}{\partial \chi_-} = -1 - \mu'(\chi_- + 2\chi_+) < 0$ and $\frac{\partial F}{\partial \chi_+} = 1 - 2\mu'(\chi_- + 2\chi_+) \geq 1$. Therefore dF can only be zero when $d\chi_- = d\chi_+ = 0$ this happens only at the point a. But since $F(a) = f(a) - \mu(0) < \alpha - \epsilon$ we see that $F^{-1}[\alpha - \epsilon, \alpha + \epsilon]$ does not contain any critical points.

Observe that the conditions for theorem 2.23 are satisfied and so the $F^{-1}(-\infty, \alpha + \epsilon]$ retracts onto $F^{-1}(-\infty, \alpha - \epsilon]$. Then with (4.2) we have the homotopy equivalence

$$f^{-1}(-\infty, \alpha + \epsilon] \simeq F^{-1}(-\infty, \alpha + \epsilon]$$

Denote $F^{-1}(-\infty, \alpha - \epsilon] = M^{\alpha-\epsilon} \cup H$, with

$$H := \overline{F^{-1}(-\infty, \alpha - \epsilon] \setminus f^{-1}(-\infty, \alpha - \epsilon]}.$$

Note that $H \subset U$ because the only part where F differs from f is where μ is defined and that is on U. Furthermore if $q \in e^k$ then $F(q) \leq F(p) \leq \alpha - \epsilon$ since $\frac{\partial F}{\partial \mathbf{x}_-} < 0$. This means that, because $f(q) \geq \alpha - \epsilon$, the k-cell is contained in H, as depicted in the figure below.

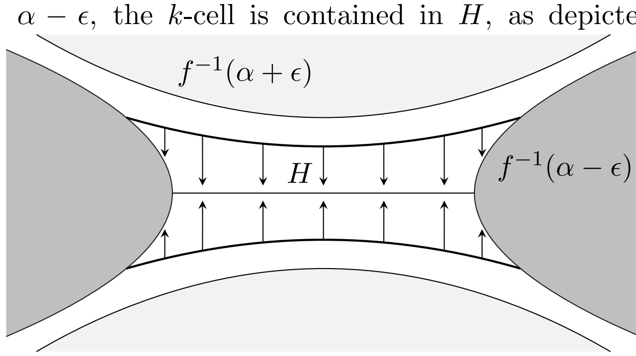


Figure 2. Deformation Retract

To show that $M^{\alpha-\epsilon} \cup H$ retracts onto $M^{\alpha-\epsilon} \cup e^k$. We distinguish three cases:

(1) When $\chi_-(x) \leq \epsilon$ the homotopy is described by

$$(u_1, ..., u_k, tu_{k+1}, ..., tu_n)$$

(2) When $\epsilon \leq \chi_-(x) \leq \chi_+(x) + \epsilon$ the homotopy is given by

$$(x_1, ..., x_k, s_t x_{k+1}, ..., s_t x_n), \text{ where } s_t = t + (1-t)\left(\frac{\chi_-(x) - \epsilon}{\chi_+(x)}\right)^{1/2}$$

We now clarify the formula for s_t. Given $u = (u_-, u_+) \in U$, we want to project it onto $f^{-1}(\alpha - \epsilon)$. Since $f = \alpha - \chi_-(x) + \chi_+(x) = \alpha - \epsilon$, if follows that $\chi_+(x) = \chi_-(x) - \epsilon$, meaning that the length of u_+ must become the length of $u_- - \epsilon$. This results in the s_t.

Note that if $\chi_+(x) \to 0$ then we have that $\chi_-(x) \to \epsilon$. Note that this homotopy agrees with the homotopy in the case, i.e. when $\chi_-(x) = \epsilon$.

(3) The last case is when the point is already in $M^{c-\epsilon}$, here we will let the retract be constant.

This proves that $M^{\alpha-\epsilon} \cup e^k$ is a deformation retract of $F^{-1}(-\infty, \alpha+\epsilon]$ and hence a deformation retract of $M^{\alpha+\epsilon}$ which completes the proof. □

Remark 2.25. The proof of theorem 2.24 can be adapted to the case where $f^{-1}(c)$ contains more than one critical point. Since M is compact there are only finitely many critical points. For this situation F will be defined similar only with more opens. Then in a similar way we construct the deformation retract.

So now we know what happens homotopy-wise to the manifold when we pass a critical point. This, together with the fact that the homotopy doesn't change when there aren't any critical points, suggests the following theorem

Theorem 2.26. *Let f be a Morse function on M. Then M has the homotopy type of a CW-complex, with one cell of dimension k for each critical point of index k.*

Let us begin with the definition of a CW-complex.

Definition 2.27. *A CW-complex is defined as follows.*

- *Begin with a discrete set $X^{(0)}$, this is called the zero-skeleton.*
- *The n-skeleton is formed from the $n-1$-skeleton by attaching n-cells e^n through a map $f : S^{n-1} \to X^{(n-1)}$. The n-cell, e^n is homeomorphic to $D^n \setminus S^{n-1}$. Then we set $X^{(n)} = X^{(n-1)} \bigcup D^n_i / \sim$. The equivalence relation is defined as $x \sim f(x)$.*
- *The set $X = \bigcup X^{(n)}$ is endowed with the weak topology.*

Since we work with manifolds of finite dimension the second part of the definition is in fact superfluous.

From this definition it is clear that simple induction is not enough. Namely the attaching maps need to go to the $(n-1)$-skeleton. To prove theorem 2.26 we use the following results which shall not be proven here.

Theorem 2.28. *Every map $f : X \to Y$ between CW-complexes is homotopic to a cellular map. [Hat10, Thm. 4.8.]*

Lemma 2.29. *Let φ_0 and φ_1 be homotopic maps from the sphere S^{n-1} to X. Then the identity map of X extends to a homotopy equivalence*

$$\Phi : X \cup_{\varphi_0} D^n \to X \cup_{\varphi_1} D^n$$

[Mil63, Lemma 3.6]

LEMMA 2.30. *Let $\varphi : S^{n-1} \to X$ be an attaching map. Let $f : X \to Y$ be a homotopy equivalence, this extends to a homotopy equivalence $F : X \cup_{\varphi} D^n \to Y \cup_{f\varphi} D^n$. [Mil63, Lemma 3.7]*

PROOF OF THEOREM 2.26. Since M is compact we have a finite amount of critical values $c_1 < ... < c_m$. Let $a \neq c_i$ such that there is a homotopy equivalence ϕ between M^a and a CW-complex X. Such an a exists since the critical points belonging to $f^{-1}(c_1)$ have index 0 and equal a disjoint union of 0-cells. Let c the smallest critical value bigger than a. The previous theorems in this section give that $M^{c-\epsilon}$ is homotopy equivalent to M^a through ψ and that $M^{c+\epsilon}$ is homotopy equivalent to $M^{c-\epsilon} \cup_{f_1} D^{k_1} \cup_{f_2} D^{k_2} \cup ... \cup_{f_n} D^{k_n}$.

Then with lemma 2.30 we see that

$$M^{c-\epsilon} \cup_{f_1} D^{k_1} \cup_{f_2} D^{k_2} \cup ... \cup_{f_n} D^{k_n} \simeq M^a \cup_{\psi f_1} D^{k_1} \cup_{\psi f_2} D^{k_2} \cup ... \cup_{\psi f_n i} D^{k_n}.$$

Applying lemma 2.30 again shows that

$$M^a \cup_{\psi f_1} D^{k_1} \cup_{\psi f_2} D^{k_2} \cup ... \cup_{\psi f_n i} D^{k_n} \simeq X \cup_{\phi\psi f_1} D^{k_1} \cup_{\phi\psi f_2} D^{k_2} \cup ... \cup_{\psi\psi f_n} D^{k_n}$$

Then with theorem 2.28 we see that $\phi\psi f_i$ is homotopic to a cellular map $g_i : S^{k_i - 1} \to X^{(k_i-1)}$. Lemma 2.29 tells us that

$$X \cup_{\phi\psi f_1} D^{k_1} \cup_{\phi\psi f_2} D^{k_2} \cup ... \cup_{\psi\psi f_n} D^{k_n} \simeq X \cup_{g_1} D^{k_1} \cup_{g_2} D^{k_2} \cup ... \cup_{g_n} D^{k_n}$$

Now the attaching maps are cellular, as is needed to construct the CW-complex. We conclude that $M^{c+\epsilon}$ is homotopy equivalent to a CW-complex and we can use induction to finish the proof. □

5. Intermezzo: Singular Homology

In the next section we will prove the Morse inequalities. In order to do this we require a few results from (relative) singular homology. Singular homology is a homotopy invariant of a manifold. It gives a way to represent the 'shape' of the manifold in the form of groups. There are different kinds of homology. In this section we treat singular homology and as an extension relative singular homology. This part is a quick overview to refresh memories and to define notation. To get a better grasp of the subject I recommend [Hat10].

Let X be a topological space. Define the k-simplex,

$$\Delta^k = [v_0, ..., v_k] := \{(x_0, ..., x_k) \in \mathbb{R}^{k+1} : x_i \geq 0 \text{ and } \sum_i x_i = 1\}.$$

Then we can consider all continuous maps from such simplices into our manifold, i.e. maps

$$\sigma^k : \Delta^k \to X.$$

Having all such maps we can look at the freely generated abelian group

$$C_k(X) := \{\sum_i a_i \sigma_i^k : \ a_i \in \mathbb{Z}\}.$$

Note that these sums are finitely supported. Construct a sequence of complexes and connect them with a boundary operator δ as follows:

$$\ldots \to C_k(X) \overset{\delta_k}{\to} C_{k-1}(X) \to \ldots \to C_0(X). \tag{5.1}$$

This δ is defined as a linear operator:

$$\delta_k(\sigma^k) := \sum_{i=0}^{k} (-1)^k \sigma^k|_{[\hat{i}]}$$

A sequence (3.1) is called a chain complex. Here $[\hat{i}] := \{x \in \Delta^k : x_i = 0\}$ and similarly we can already define $[\hat{i}, \hat{j}] := \{x \in \Delta^k : x_i = x_j = 0\}$. The boundary operator δ obeys the condition that it squares to zero i.e. $\delta^2 = 0$. As can easily be verified, first note that

$$\sigma \mid_{[\hat{i}]}|_{[\hat{j}]} = \begin{cases} \sigma \mid_{[\hat{j},\hat{i}]} & \text{if } j < i \\ \sigma \mid_{[\hat{i},\hat{j+1}]} & \text{if } i \leq j. \end{cases}$$

It follows that

$$\begin{aligned} \delta^2(\sigma^k) &= \sum_{i=0}^{k} (-1)^k \sum_{j<i} (-1)^j (\sigma^k \mid_{[\hat{j},\hat{i}]}) + \sum_{i=0}^{k} (-1)^k \sum_{j \geq i}^{k-1} (-1)^j (\sigma^k \mid_{[\hat{i},\hat{j+1}]}) \\ &= \sum_{i=0}^{k} \sum_{j<i} (-1)^{i+j} (\sigma^k \mid_{[\hat{j},\hat{i}]}) + \sum_{i=0}^{k} \sum_{j>i}^{k} (-1)^{i+j-1} (\sigma^k \mid_{[\hat{i},\hat{j}]}) = 0. \end{aligned}$$

This reveals that the image of δ_{k+1} lies in the kernel of δ_k. Therefore, the homology groups,

$$H_k(X) := \frac{\operatorname{Ker} \delta_k}{\operatorname{Im} \delta_{k+1}}$$

are well defined.

We can also define the relative singular homology. To this end let $A \subset X$ be a subspace of X. We can look at $C_n(X)/C_n(A)$ with the same boundary operator as before. Then from the chain complex

$$\ldots \to \frac{C_k(X)}{C_k(A)} \xrightarrow{\delta_k} \frac{C_{k-1}(X)}{C_{k-1}(A)} \to \ldots \to \frac{C_0(X)}{C_0(A)}$$

the relative homology follows:

$$H_k(X, A) := \frac{\operatorname{Ker} \delta_k}{\operatorname{Im} \delta_{k+1}}$$

The intuitive idea behind the relative homology is that we disregard what is happening in the subspace A. In the case that X is a CW-complex and A is a subcomplex then

$$H_k(X, A) \simeq H_k(X/A, A/A)$$

Furthermore we can look at the short exact sequences of the form

$$0 \to C_k(A) \to C_k(X) \to C_k(X)/C_k(A) \to 0$$

This gives a long exact sequence called 'long exact sequence of the relative homology'.

$$\ldots \to H_{k+1}(A) \to H_{k+1}(X) \to H_{k+1}(X, A) \to H_k(A) \to H_k(X) \to \ldots$$

The following lemma is commonly referred to as the excision lemma.

Lemma 2.31 (Excision). *Let $U \subset A \subset X$ be topological spaces such that the closure of U is contained in the interior of A then there is an isomorphism*

$$H_\bullet(X - U, A - U) \simeq H_\star(X, A)$$

Equivalently, if we have subspaces A, B whose interior cover X then we have

$$H_\star(B, A\cap B) \simeq H_\star(X, A)$$

To see the equivalence set $U = X - B$

PROOF. [Hat10] □

6. Morse Inequalities

The Morse inequalities give relations between critical points of index k and the k-th Betti number. This section isn't needed for the following chapters. But still, the Morse inequalities are useful tools. Knowing that a Morse function exists with its critical points limits the shape of the manifold.

For this part we will need the *Betti numbers.* We define

$$b_n(X, Y) = n\text{th Betti number of } (X, Y) = \text{rank of } H_n(X, Y).$$

Here $H_n(X, Y)$ is the n-th relative homology group as explained in the previous section.

By a tuple of spaces $(X_n, ..., X_0)$ we mean topological spaces such that $X_n \supset ... \supset X_0$. If the tuple consists of three spaces we call it a triple. If it consists of two it is called a pair.

DEFINITION 2.32. *Let S be a function from pair of spaces to the real line. We call S subadditive if for all triples (X, Y, Z) the inequality*

$$S(X, Z) \leq S(X, Y) + S(Y, Z)$$

holds. If equality holds then S is called additive.

Given a pair (X, Y) of which $Y = \emptyset$, we will write $S(X) := S(X, Y)$.

With a triple (X, Y, Z) we can construct the long exact sequence of relative homology

$$\cdots \to H_n(Y, Z) \stackrel{f_n}{\to} H_n(X, Z) \stackrel{g_n}{\to} H_n(X, Y) \stackrel{h_n}{\to} H_{n-1}(Y, Z) \to \dots . \tag{6.1}$$

This exact sequence shows that b_n is subadditive, i.e when (X, Y, Z) is a triple then we have $b_n(X, Z) \leq b_n(X, Y) + b_n(Y, Z)$, because

$$\begin{aligned} \text{Rank}(H_n(X, Z)) &= \text{Rank Ker}\, g_n + \text{Rank Im}\, g_n \\ &= \text{Rank Im}\, f_n + \text{Rank Ker}\, h_n \\ &\leq \text{Rank}\, H_n(Y, Z) + \text{Rank}\, H_n(X, Y) \end{aligned}$$

LEMMA 2.33. *If S is subadditive and we have a tuple of spaces $(X_n, ..., X_0)$ then*

$$S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}).$$

We obtain equality for an additive S

PROOF. This can be proven using induction. □

THEOREM 2.34 (Weak Morse Inequality). *Given a manifold equipped with a Morse function and gradient-like vector field. For each $k \in \{0, ..., n\}$, if we define $c_k := |Cr_k(f)|$ the inequality*

$$b_k(M, M_0) \leq c_k$$

holds.

We assume that there exists numbers $a_0 < ... < a_n$ such that M_i^a contains i critical points. Then M^{a_i} is seen to be homotopy equivalent to $M^{a_{i-1}} \cup e^{n_i}$ for $i \in 1, ..., n$ using results from the previous section. Then we may assume that the a_i exists by looking at a small pertubation of the Morse function.

Proof. Look at the tuple $(M^{a_n}, ..., M^{a_0})$ then we get with some homotopy theory from the previous section and the excision lemma that

$$H_\bullet(M^{a_i}, M^{a_{i-1}}) = H_\bullet(M^{a_{i-1}} \cup e^{m_i}, M^{a_{i-1}}) = H_\bullet(e^{m_i}, \dot{e}^{m_i}) = \begin{cases} \mathbb{Z} & \text{if } \bullet = m_i \\ 0 & \text{otherwise} \end{cases}.$$

This shows that $b_k(e^{m_i}, \dot{e}^{m_i}) = 1$ if k equals the index of the critical point which corresponds with e^{m_i}, and 0 otherwise. Then with Lemma 2.33 we can say that

$$b_k(M, M_0) \leq \sum_{i=1}^{n} b_n(M^{a_i}, M^{a_{i-1}}) = c_k.$$

□

There exists stronger inqualities namely the strong Morse inequalities.

Theorem 2.35 (Strong Morse Inequalities). *Given a manifold M with Morse function and gradient-like vector field. Then we have the inequality*

$$\sum_{i=0}^{j}(-1)^{j-i}c_i \geq \sum_{i=0}^{j}(-1)^{j-i}b_i(M, M_0) \tag{6.2}$$

for every $j \in \{0, ..., n\}$.

To see that these inequalities are indeed stronger we can add the cases of (6.2) for $j = k$ and $j = k - 1$

$$\sum_{i=0}^{k}(-1)^{k-i}c_i + \sum_{i=0}^{k-1}(-1)^{k-i-1}c_i \geq \sum_{i=0}^{k}(-1)^{k-i}b_i(M, M_0) + \sum_{i=0}^{k-1}(-1)^{k-i-1}b_i(M, M_0)$$
$$c_k \geq b_k(M, M_0)$$

Proof. Regarding again the long exact sequence (6.1) we obtain

$$\text{Rank Im}\, h_{n+1} = b_n(Y, Z) - \text{Rank Im}\, f_n.$$

Similar equalities can be obtained for f_n. Repeating this process gives us

$$\text{Rank Im}\, h_{n+1} = \sum_{i=0}^{n}(-1)^{n-i}\left(b_i(Y, Z) - b_i(X, Z) + b_i(X, Y)\right).$$

When we define the notation

$$B_n(X, Y) = b_n(X, Y) - b_{n-1}(X, Y) + ... \pm b_0(X, Y)$$

and observe that $\operatorname{Rank}\operatorname{Im} h_{n+1} \geq 0$ then B_n is subadditive. Therefore we can apply lemma 2.33 and obtain

$$\begin{aligned} B_n(M, M_0) &\leq \sum_{i=1}^{n} B_n(M^{a_i}, M^{a_{i-1}}) \\ &= \sum_{i=1}^{n} \sum_{j=0}^{n} (-1)^{n-j} b_n(M^{a_i}, M^{a_{i-1}}) \\ &= \sum_{j=0}^{n} (-1)^{n-j} c_j \end{aligned}$$

This gives us (6.2). □

What we have derived are in fact the relative Morse inequalities but if we take $M_0 = \emptyset$ then we get the regular Morse inequalities.

CHAPTER 3

Morse Homology

In this chapter we define the Morse homology with coefficients modulo two. For this we construct the chain complex and define the boundary operator. A crucial point with defining the Morse homology lies with the boundary operator, this should square to zero. This boils down to the fact that a compact one-dimensional manifold has an even number of boundary points.

1. Morse Homology

Here we define the Morse complex, using the critical points of a Morse function, and the corresponding boundary operator. After this we define the Morse homology, and already see a few examples.

Let M be a closed manifold and f a Morse function with critical points a, b, furthermore let v be a gradient-like vector field obeying the Morse-Smale condition. To define Morse homology recall the definitions of the following submanifolds

$$W^u(a) \cap W^s(b) = \mathcal{M}(a,b), \text{ and } \mathcal{L}(a,b) = \mathcal{M}(a,b)/\mathbb{R}.$$

When the index differs 1 of two critical points then $\dim \mathcal{L}(a,b) = 0$. Thus the cardinality of $\mathcal{L}(a,b)$ is a finite number of points since we work on a compact manifold. Thus we can define

$$n(a,b) = n(a,b,v) := (\mid \mathcal{L}(a,b) \mid + 2 \cdot \mathbb{Z}) \in \mathbb{Z}_2$$

These are the number of paths coming from a and going to b regarded modulo two.

Definition 3.1. *We define the Morse chain complex of (f, v) by*

$$C_k = C_k(f,v) := \{\sum_i a_i c_i : c_i \in Cr_k(f) \text{ and } a_i \in \mathbb{Z}_2\}.$$

We define the boundary operator

$$\begin{aligned} \delta_k : C_k &\to C_{k-1} \\ c_k &\mapsto \sum_{c_{k-1} \in Cr_{k-1}(f)} n(c_k, c_{k-1}) c_{k-1}. \end{aligned}$$

Remark. Because we use the coefficient group $\mathbb{Z}_2$ the orientation doesn't matter. Should we use the coefficient group $\mathbb{Z}$ then the definition would have an extra sign in the definition of $n(a,b)$. This sign represents information about the orientation.

This gives us the sequence

$$C_n \to \ldots \to C_{k+1} \stackrel{\delta_{k+1}}{\to} C_k \stackrel{\delta_k}{\to} C_{k-1} \to \ldots \to C_0.$$

Theorem 3.2. *The boundary operator squares to zero, i.e. $\delta_{k-1} \circ \delta_k = 0$.*

The proof of this theorem is on page 32. With theorem 3.2 the Morse homology groups are well defined.

DEFINITION 3.3. *For a Morse function f and a gradient-like vector field obeying the Morse-Smale condition we define the k-th Morse homology group*

$$HM_k(M) = HM_k(M, f, v) := \frac{\operatorname{Ker} \delta_k}{\operatorname{Im} \delta_{k+1}}$$

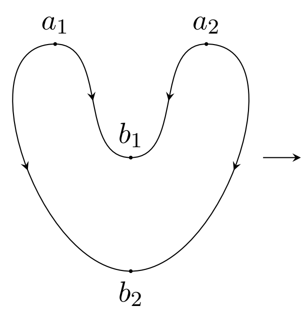
REMARK 3.4. The Morse homology theorem states that the Morse homology is isomorphic to the singular homology. As a corollary we get that for a different Morse function, f' and transversal gradient-like vector field v' the Morse homologies are isomorphic,

$$HM_k(M, f, v) \simeq HM_k(M, f', v').$$

This justifies the notation $HM_k(M)$.

REMARK 3.5. We have considered the Morse homology with coefficients in $\mathbb{Z}_2$. But is is also possible to define it with coefficients in $\mathbb{Z}$. To do this one needs to keep in mind the orientation.

EXAMPLE 3.6. We begin with a simple example of the deformed circle with the height function. Here a_1, a_2 are critical points of index 1 and b_1, b_2 are critical points of index 0.



Calculating the homology

$$0 \to C_1 = \langle a_1, a_2 \rangle \to C_0 = \langle b_1, b_2 \rangle$$

with

$$\delta_1(a_1) = \delta_1(a_2) = b_1 + b_2$$

shows us that

$$HM_1 = \operatorname{Ker} \delta = \langle a_1 + a_2 \rangle = \mathbb{Z}_2$$
$$HM_0 = C_0 / \operatorname{Im} \delta = \langle b_1, b_2 \rangle / \langle b_1 + b_2 \rangle = \mathbb{Z}_2$$

EXAMPLE 3.7. The function

$$f(x, y) = \sin(x) + \sin(y)$$

is a Morse function with 4 non-degenerate critical points, as can be easily verified.

Then the negative gradient is

$$(-\cos(x), -\cos(y)).$$

In Figure 1 we have depicted the torus as the square. Here the critical points, a,b,c and d have index $2, 1, 1$ and 0 respectively. Also the vector field is illustrated in Figure 1.

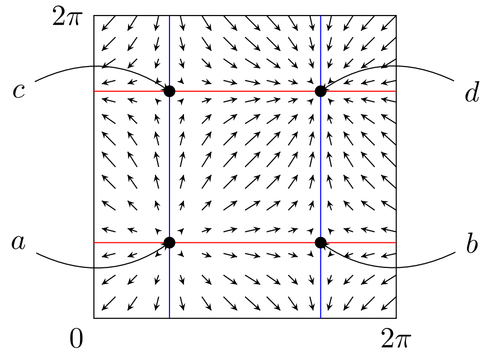


Figure 1. Gradient torus

We claim that this vector field is Morse-Smale. Since $W^u(a)$ and $W^s(b)$ are two dimensional we only need to check that $W^s(c_1) \pitchfork W^u(c_2)$ and $W^s(c_2) \pitchfork W^u(c_1)$. For the first case observe that the stable manifold of c_1 is the left blue line and that the unstable manifold is the right blue line. Their intersection is empty and hence are they vacuously transverse. So we can construct the Morse Homology for this case.

Now we construct the Morse complexes

$$C_2 = \langle a \rangle \to C_1 = \langle b, c \rangle \to C_0 = \langle d \rangle$$

This time the boundary operator behaves as

$$\begin{aligned} \delta_2(a) &= n(a,b)b + n(a,c)c = 2b + 2c = 0 \\ \delta_1(b) = \delta_1(c) &= n(c,d)d = 2d = 0 \end{aligned}$$

and we obtain

$$\begin{aligned} HM_2(T) &= \operatorname{Ker} \delta_2 = \langle a \rangle = \mathbb{Z}_2 \\ HM_1(T) &= \frac{\operatorname{Ker} \delta_1}{\operatorname{Im} \delta_2} = \operatorname{Ker} \delta_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ HM_0(T) &= \frac{\operatorname{Ker} \delta_0}{\operatorname{Im} \delta_1} = \mathbb{Z}_2 \end{aligned}$$

One may notice that the Morse homology of the circle and torus, with given Morse functions, coincide with the singular homology. This is not a coincidence and is true in general. The next chapter will treat this phenomenon.

2. The space of broken trajectories

In the previous section we defined the Morse Homology. To show that this is well defined we need to show that the boundary operator squares to zero. For this we define the space of broken trajectories from a to b and endow with a topology such that it is a compactification of $\mathcal{L}(a, b)$.

In order to show that the Morse homology is well defined, we need to show $\operatorname{Im}\delta_{k+1} \subset \operatorname{Ker}\delta_k$. Take $a \in C_{k+1}$. Then

$$\begin{aligned}\delta^2(a) &= \delta(\sum_{c\in Cr_k} n(a,c)c) \\ &= \sum_{b\in Cr_{k-1}}\sum_{c\in Cr_k} n(a,c)n(c,b)b.\end{aligned}$$

Hence $\delta^2 = 0$ if and only if

$$\sum_{c\in C_{k-1}} n(a,c)n(c,b) = 0 \quad \forall b \in Cr_{k-1}. \tag{2.1}$$

It will be shown that $\sum_{c\in C_{k-1}} n(a,c)n(c,b)$ equals the number of points in the boundary of the one-dimensional manifold $\mathcal{L}(a,b)$. Since the boundary of an one-dimensional manifold consists of an even number of points, it will follows that $\delta^2 a = 0$.

Figure 2 should help to visualise the idea. Here a, c_1,c_2 and b are critical points of index $k+1$, k, k and $k-1$ respectively. Given a path between a and b we can 'move' it to the right untill we approach the critical point c_2. Also, we can more it the other way around when at a critical point, for example c_1, that is connected to a and b, so that there exists paths from a to b that are arbitrarily close to c_1.

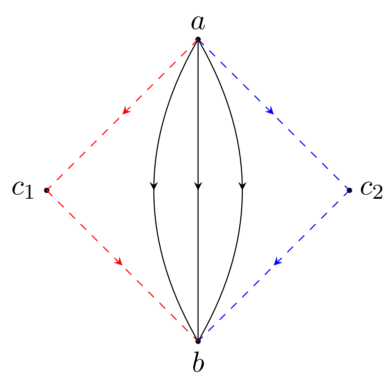


Figure 2. visualisation

To show that the boundary operator squares to zero, we will look at the following space.

Definition 3.8. *Let a and b be critical points of the Morse function f. Define*

$$\overline{\mathcal{L}}(a,b) = \bigcup_{c_i\in Cr(f)} \mathcal{L}(a,c_1)\times\mathcal{L}(c_1,c_2)\times ... \times\mathcal{L}(c_{m-1},b)$$

to be the space of broken trajectories. Elements in this space are called broken trajectories.

As the notation suggests, $\overline{\mathcal{L}}(a,b)$ is a compactification of $\mathcal{L}(a,b)$. In Figure 2 this would become the closed interval. To make sense of the construction of the topology of this space we need the following definition.

Definition 3.9. *A compactification of a space A is a pair (X,h) consisting of a compact Hausdorff space X and a homeomorphism h of A onto a dense subset of X.*

To define the topology we construct a subbasis. Then the topology generated by this subbasis is the smallest topology containing our subbasis.

DEFINITION 3.10. *Let $U \subset M$ be open. Then an open for our subbasis is*

$$W_U := \{\gamma \in \overline{\mathcal{L}}(a,b) \mid \gamma \cap U \neq \emptyset\}$$

and we define the subbasis to be

$$\mathcal{B} := \{W_U : \forall\, U \subset M \text{ open }\}.$$

The only requirement we have is that the subbasis covers our manifold. This is indeed the case since we can take $U = M$ and then $W_M = \overline{\mathcal{L}}(a,b)$. The use of the subbase is to check whether a map is continuous. To be more specific, given a map $g : X \to Y$ and $\mathcal{B}$ be a basis of the topology on Y, then g is continuous if $g^{-1}(U)$ is open in X for all $U \in \mathcal{B}$.

To get more intuition about the topology lets look at two examples.

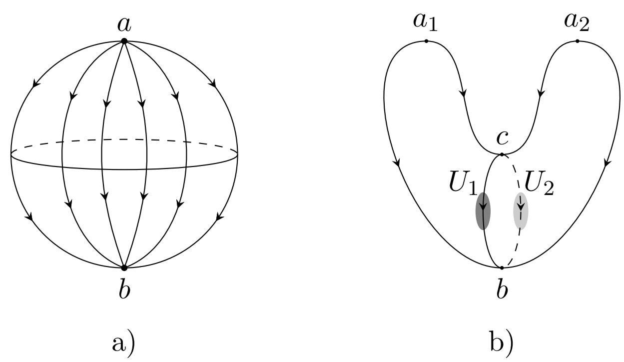


Figure 3. Examples

EXAMPLE 3.11. Take $M = S^2$ with the standard height function. The gradient lines are shown in Figure 3 a). There are two critical points, namely a with index 2 and b with index 0. Then

$$\overline{\mathcal{L}}(a,b) = \mathcal{L}(a,b),$$

which is isomorphic to the circle.

Maybe a more interesting example is the deformed two sphere.

EXAMPLE 3.12. There are four critical points a_1, a_2, b and c with indices $2, 2, 0$ and 1 respectively as shown in Figure 3 b). The space of broken trajectories looks as follows

$$\overline{\mathcal{L}}(a,b) = \mathcal{L}(a,b) \cup \mathcal{L}(a,c) \times \mathcal{L}(c,b)$$

As can be verified, $\mathcal{L}(a,b)$ is homeomorphic to an interval and $\mathcal{L}(a,c) \times \mathcal{L}(c,b)$ is homeomorphic to two points. For the two opens U_1 and U_2 we can make sure that the space of broken trajectories is homeomorphic to the closed interval and not the circle.

REMARK 3.13. Now we want the space of broken trajectories to be compact. This space is second countable, as can be seen from the fact that the manifold M is second countable. In a second countable space the condition that the space is compact is equivalent to saying that the space is sequentially compact.

THEOREM 3.14 (Sequentially compact). *The space of broken trajectories $\overline{\mathcal{L}}(a,b)$ is sequentially compact.*

To proof this theorem we need the following lemma.

LEMMA 3.15. *Let $x \in M$ not be a critical point and let (x_n) be a sequence converging to x. Let y_n and y be points lying on the same gradient trajectories as x_n and x respectively. If we then have $f(y_n) = f(y)$ then* $\lim_{n\to\infty} y_n = y$.

Let's now first prove Theorem 3.14.

PROOF OF THEOREM 3.14. Let (γ_n) be a sequence of paths in $\overline{\mathcal{L}}(a,b)$. To show sequentially compactness we need to show that this sequence has a converging subsequence.

First take a Morse neighborhood U of the critical point a and set $f(a) = \alpha$. Then there exists an ϵ such that the following definition makes sense:

$$U_a^u := W^u(a) \cap f^{-1}(f(a) - \epsilon) \text{ and } a_n^- := \gamma_n \cap U_a^u.$$

Note that U_a^u is homeomorphic to the sphere, so it is compact and hence sequentially compact. Therefore the sequence $\{a_n\}_{n\in\mathbb{N}}$, which lies in U_a^u, has a subsequence, for simplicity we denote this also as $\{a_n\}_{n\in\mathbb{N}}$, that converges to $a^- \in U_a^u$. Now look at the path χ_1 that goes through a^-. This has as end point a critical point x, by lemma 2.12. The flow corresponding to the vector field is a smooth function, so there exists an n big enough such that the paths γ_n also have to cross the Morse neighborhood U_x of x. Again there exists an $\epsilon_x > 0$ such that the following is well defined:

$$U_x^{\pm} = U \cap f^{-1}(f(x) \pm \epsilon).$$

Looking at the sequence of intersections $\{x_n^+\}_{\mathbb{N}}$, with

$$x_n^+ := U_x^+ \cap \gamma_n$$

converges to $x^+ = U_x^+ \cap \chi$ by lemma 3.15. If $x = b$ we are done and we have shown that the limit is in $\mathcal{L}(a,b)$. If $x \neq b$ then this means that γ_n will eventually leave the Morse neighborhood of x. Set

$$x_n^- := U_x^- \cap \gamma_n.$$

This sequence again converges to a point x^-. The point x^- lies on the unstable manifold of x. If this was not the case then there is a point $\bar{x}$ in U_x^+ that lies on the flow line of x_-. This would mean that $\bar{x}$ is the limit of the sequence x_n^+, but we already concluded that the limit lies on the stable manifold of x. Hence the point x_- lies on the unstable manifold of x. Now we can repeat this argument. Take a path χ_2 through x_-, this goes to an other critical point x_1, then again γ_n for big enough n will go into the Morse neighborhood of x_1 etcetera. This eventually gives a broken trajectory $\chi = [\chi_1, ..., \chi_n]$.

This will prove that for all U such that $U \cap \chi \neq \emptyset$ there exists an N for which γ_n lies in W_U for all $n > N$.

We still need to show the theorem in general. To this end, assume that $\gamma_n = ((\gamma_1)_n, ..., (\gamma_l)_n)$ is a converging sequence of broken trajectories. Now apply the above result to every sequence $(\gamma_i)_n$. □

We still need to prove lemma 3.15.

PROOF. There is a neighborhood U around $Cr(f)$ and an N such that the flow paths from x_n to y_n and the path from x to y are not contained in $M \setminus U$ for $n > N$. Since this is far enough away from the critical points, we define the following vector field on $M \setminus U$.

$$\tilde{X} = -\frac{X}{df \cdot X}.$$

Let ψ be the flow field belonging to this vector field. Then, similar as in the proof of theorem 2.23,

$$f(\psi_t(x)) = f(x) - t.$$

Because x_n and y_n lie on the same path we know that $\psi_t(x_n) = y_n$ for some $t \in \mathbb{R}$. By applying the function f on both sides we get that $t = f(x_n) - f(y_n)$. Then, since we are working with continuous functions,

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \psi_{f(x_n)-f(y_n)} x_n \\
&= \lim_{n\to\infty} \psi_{f(x_n)-f(y)} x_n \quad \text{since } f(y) = f(y_n) \\
&= \psi_{f(x)-f(y)} x \\
&= y,
\end{aligned}$$

which proves $\lim_{n\to\infty} y_n = y$. □

Combining remark 3.13 and theorem 3.14 shows that the space of broken trajectories is compact. But what we really want to show is that it is a compactification, meaning that the set $\mathcal{L}(a,b)$ lies dense in $\overline{\mathcal{L}}(a,b)$. This is crucial since we want to show the broken trajectories are in fact the boundary. The fact that it is a compactification is treated in the next section.

3. The boundary operator squares to zero

In this section we prove that the boundary operator squares to zero. For this we will show that the space of broken trajectories is a compactification of the space of trajectoriesl, in the case that the index of the critical points differ by two.

The following theorem expressed what we want of the compactification.

THEOREM 3.16 (Broken paths are the boundary). *Let M be a compact manifold with a Morse function f and X a gradient-like vector field which obeys the Morse-Smale condition. Given three critical point a, c and b with indices $k+1$, k and $k-1$ respectively. Let (γ_1, γ_2) be a broken trajectory connecting a, b and c. Then there is a continuous embedding ϕ from the interval $[0, \epsilon)$ onto a neighborhood of (γ_1, γ_2) in $\overline{\mathcal{L}}(a,b)$ that satisfies $\psi(0) = (\gamma_1, \gamma_2)$ and $\psi(s) \in \mathcal{L}(a,b)$ for $s \neq 0$.*

Before the proof of this theorem we will prove a few lemma's and propositions. The proof of the theorem is given on page 30.

Figure 4 visualises what we are going to do.

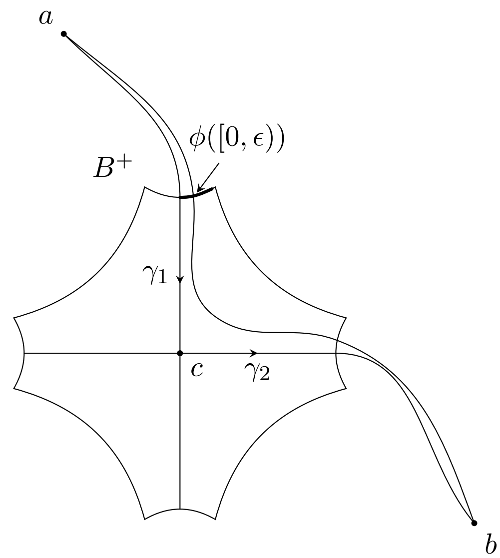


Figure 4. The embedding

The condition that the critical points differ by exactly one with each step is crucial. This ensures that the path space is one dimensional. Furthermore, if a sequence converges to (γ_1, γ_2), then this sequence eventually lies in the image of $\phi([0, \epsilon))$. In other words the broken trajectories can only be approached from one side. So, when looking at Figure 5 and c_1 and c_2 are broken trajectories, we cannot get something like Figure 5 a). But locally, near a broken trajectory, it looks like Figure 5 b).

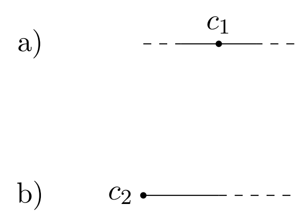


Figure 5. Locally it looks like b) and not a)

Throughout this remainder of the section we remain in the mind set of Theorem 3.16. So we always assume to have a Morse function f on the manifold M together with a transverse gradient-like vector field. For the critical point c set $f(c) = \alpha$. Then there exists a $\delta > 0$ such that $f^{-1}(\alpha \pm \delta)$ intersects the Morse neighborhood U and we can define

$$S^{\pm} := S^{\pm}_{c,\delta} = W^s(c) \cap f^{-1}(\alpha \pm \delta)$$
$$B^{\pm} := B^{\pm}_{c,\delta} = U \cap f^{-1}(\alpha \pm \delta)$$

The flow generates a map from $B^+ \setminus S^+$ to $B^- \setminus S^-$.

Lemma 3.17. *The embedding*

$$\Psi : B^+ \setminus S^+ \to B^- \setminus S^-$$

defined by the flow ψ of X is given by

$$\Psi(x_-, x_+) = (\frac{\|x_+\|}{\|x_-\|} x_-, \frac{\|x_-\|}{\|x_+\|} x_+).$$

Proof. Locally around the critical point c the vector field is the negative gradient of f. With the Morse Lemma 2.6 our function lcan be written locally as

$$f(x) = c - (x_1^2 + ... + x_k^2) + x_{k+1}^2 + ... + x_n^2.$$

Then the negative gradient equals

$$-\nabla f(x) = (2x_1, ..., 2x_k, -2x_{k+1}, ..., -2x_n).$$

Then the flow obeys

$$\frac{d\psi^s}{ds}(x) = -\nabla f(\psi^s(x))$$

which has as solution

$$\psi^s(x) = (e^{2s}x_-, e^{-2s}x_+).$$

When $(x_-, x_+) \in B^+ \setminus S^+$, i.e. $x_- \neq 0$ and $x_+ \neq 0$, then there exists and s_0 such that $\psi^{s_0}(x_-, x_+) = (\frac{\|x_+\|}{\|x_-\|}x_-, \frac{\|x_-\|}{\|x_+\|}x_+)$ and we define

$$\Psi(x_-, x_+) = (\frac{\|x_+\|}{\|x_-\|}x_-, \frac{\|x_-\|}{\|x_+\|}x_+).$$

When $(x_-, x_+) \in B^+ \setminus S^+$ then $\|x_+\|^2 - \|x_-\|^2 = \epsilon$ and we verify that

$$\begin{aligned}\Psi(x_-, x_+) &= c - (\frac{\|x_+\|^2}{\|x_-\|^2}(x_1^2 + .. + x_k^2)) + \frac{\|x_-\|^2}{\|x_+\|^2}(x_{k+1}^2 + ... + x_n^2) \\ &= c - \frac{\|x_+\|^2}{\|x_-\|^2}\|x_-\|^2 + \frac{\|x_-\|^2}{\|x_+\|^2}\|x_+\|^2 \\ &= c - \|x_+\|^2 + \|x_-\|^2 \\ &= c - (\|x_+\|^2 - \|x_-\|^2) = c - \epsilon.\end{aligned}$$

Hence $\Psi(x_-, x_+) \in B^- \setminus S^-$ and we have a well defined map

$$\Psi : B^+ \setminus S^+ \to B^- \setminus S^-.$$

□

To prove theorem 3.16 we also need the following propostition.

PROPOSITION 3.18. *Let $y \in S^+$ and $D \subset B^+$ be a k-dimensional disk that meets $W^s(c)$ transversally at y and satisfies $D \cap S^+ = \{y\}$. Then*

$$Q = \Psi(D \setminus \{y\}) \cup S^-$$

is a k-dimensional sub-manifold of M and its boundary is S^-, with Ψ from 3.17.

PROOF. Given $D \subset B^+$ a k-dimensional disc as described in the theorem. That is D meets $W^s(c)$ transversally at y and $D \cap S^+ = \{y\}$. The projection restricted to D:

$$\pi : D \to W^u(c);\ (x_-, x_+) \mapsto x_-$$

is a local diffeomorphism around y. So, locally there is an inverse, meaning that there exists a $\epsilon > 0$ such that

$$\pi^{-1} : D_\epsilon(:= \{(x_-, x_+) \in U \mid \|x\| < \epsilon, x_+ = 0\}) \to D;\ x_- \mapsto (x_-, h(x_-)),$$

where $h : D_\epsilon \to W^s(c)$ and $\|h\| = \sqrt{\delta + \|x_-\|^2}$. As a result, possibly after shrinking D, we can write

$$D = (x_-, h(x_-))$$

Now apply lemma 3.17 and look at $\Psi(D \setminus \{y\})$, which is given as

$$\Psi(x_-, h(x_-)) = (\frac{\sqrt{\delta + \|x_-\|^2}}{\|x_-\|}x_-, \frac{\|x_-\|}{\sqrt{\delta + \|x_-\|^2}}h(x_-)).$$

Writing $x_- \in D_\epsilon$ as $r \cdot x$ with $r \in (0, \epsilon)$ and $x \in S^{k-1}$ gives the function

$$F : (0, \delta) \times S^{k-1} \to \Psi(D \setminus \{y\});\ (r, x) \mapsto ((\sqrt{\delta + r^2})x, \frac{r}{\sqrt{\delta + r^2}} h(r, x)).$$

Now since $h(0)$ is well defined we can extend F onto $[0, \epsilon) \times S^{k-1}$ by setting $F(0, x) = (\sqrt{\delta} x, 0)$ and furthermore we have $F(0, x) \in S^-$, giving us the required map to say that $\Psi(D \setminus \{y\}) \cup S^-$ is a k-dimensional manifold with boundary S^-.

□

Now we have prepared enough to prove theorem 3.16.

Proof of Theorem 3.16. The broken trajectory (γ_1, γ_2) is given and denote

$$a_1 = \gamma_1 \cap f^{-1}(\alpha + \delta) \quad a_2 = \gamma_2 \cap f^{-1}(\alpha - \delta).$$

By the Morse-Smale condition we have

$$W^u(a) \pitchfork f^{-1}(\alpha + \delta)$$

now define the intersection

$$P := W^u(a) \cap f^{-1}(\alpha + \delta).$$

The submanifold P has dimension $n - 1 + k + 1 - n = k$ by lemma 2.16.

By the transversality condition

$$W^u(a) \pitchfork W^s(c)$$

we have

$$W^u(a) \cap f^{-1}(\alpha + \delta) \pitchfork W^s(c) \cap f^{-1}(\alpha + \delta)$$

inside the submanifold $f^{-1}(\alpha + \delta)$. In other words, inside the submanifold $f^{-1}(\alpha + \delta)$, P is transverse to S^+.

Since the dimension of S^+ equals $n - k - 1$ and the dimension of P equals k, the intersection $P \cap S^+$ is zero-dimensional in $f^{-1}(\alpha + \delta)$. Note that $a_1 \in P \cap S^+$.

When we set ,$D^k_\epsilon := \{x \in \mathbb{R}^k \mid \|x\| < \epsilon\}$ the k-dimensional open disk, then there exists a pointed map

$$\Phi : (D^k_\epsilon, 0) \to (P, a_1)$$

which is a local parametrization of P around a_1 such that $\operatorname{Im} \Phi \cap S^+_c = a_1$. Define D to be the image of Φ. By restricting the size of the open disk we may assume that $\operatorname{Im} \Phi \subset B^+$. With Lemma 3.17 we have the embedding

$$\Psi : D \setminus \{a_1\} \to B^- \setminus S^-.$$

We have verified that we are in the situation of Proposition 3.18. In figure 6 we visualise what is going on

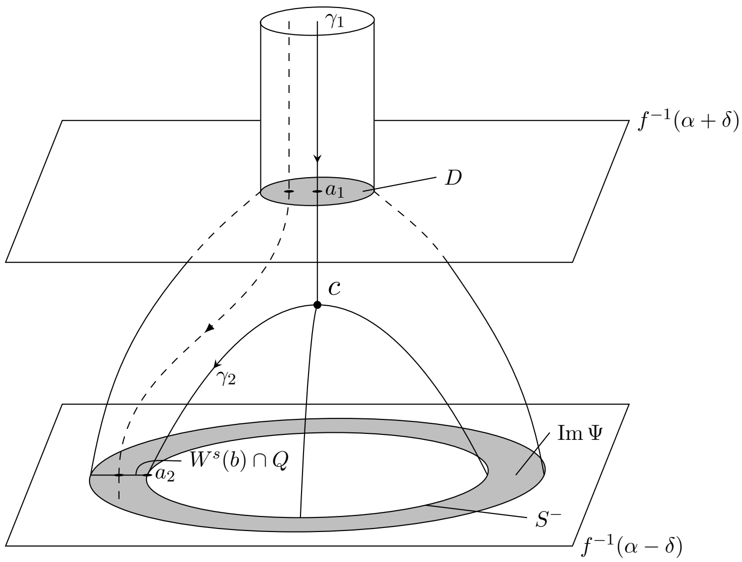


Figure 6. Bottle

Recall that

$$W^s(b) \pitchfork \left(W^u(c) \cap f^{-1}(\alpha-\delta)\right) = W^s(b) \pitchfork S^-.$$

and also

$$W^s(b) \pitchfork \left(W^u(a) \cap f^{-1}(\alpha-\delta)\right).$$

Now $\operatorname{Im}\Psi$ is an open in the intersection $W^u(a) \cap f^{-1}(\alpha-\delta)$. The submanifold $\operatorname{Im}\Psi$ has the same dimension as $W^u(a) \cap f^{-1}(\alpha-\delta)$ because

$$\begin{aligned}\dim\left(W^u(a) \cap f^{-1}(\alpha-\delta)\right) &= \dim(W^u(a)) + \dim(f^{-1}(\alpha-\delta)) - \dim(M)\\ &= k+1+n-1-n = k.\end{aligned}$$

Therefore we also have that $W^s(b) \pitchfork \operatorname{Im}\Psi$. Denoting $Q = \operatorname{Im}\Psi \cup S^-$ gives

$$\dim\left(W^s(b) \cap Q\right) = \dim(W^s(b)) + \dim(Q) - \dim(M) = n-(k-1)+k-n = 1,$$

and the boundary of $W^s(b) \cap Q$ equals $W^s(b) \cap S^- = \mathcal{L}(c,b)$.

So now we can make a continuous paramertization $\phi : [0,\epsilon) \to W^s(b) \cap Q$ where $\phi(0) = a_2$. With this map ϕ we get the map

$$\Psi^{-1} \circ \phi : (0,\epsilon) \to (W^s(b) \cap (D \setminus \{a_1\})$$

by flowing back along the vector field. When $s \to 0$, then with lemma 3.15 it follows that $(\Psi^{-1} \circ \phi)(s) \to a_1$, because it has to go to the stable manifold of c and the only point in D that lies in the stable manifold is a_1.

As a result it is possible to extend $\Psi^{-1} \circ \phi$ to $[0,\epsilon)$ in a continuous way. Now as usual the flow is denoted by ψ and $[\psi_x(t)] \in \overline{\mathcal{L}}(a,b)$ is the class of the path $\psi_x(t)$

$$\varphi : [0,\epsilon) \to \overline{\mathcal{L}}(a,b); \quad s \mapsto \begin{cases} [\psi_{\phi(s)}(t)] & \text{if } s \in (0,\epsilon) \\ [\gamma_1,\gamma_2] & \text{if } s = 0. \end{cases}$$

This is the function desired and by construction it is continuous □

Lemma 3.19. *If* (γ'_n) *is a sequence converging to* (γ_1, γ_2) *then for big enough* n *it is contained in the image* φ.

Proof. Take (γ'_n) a sequence of trajectories converging to (γ_1, γ_2). Then for n big enough it has to enter through $D \setminus \{a_1\}$ and will therefore leave through $\operatorname{Im}\Psi \cap W^s(b)$ and hence lie in the image of φ □

In the previous section we have show that the space of broken trajectories is compact. Now in the special case, such that $\operatorname{Ind}(b) = \operatorname{Ind}(a) - 2$ we have that $\overline{\mathcal{L}}(a, b)$ is a compactification of $\mathcal{L}(a, b)$, in such a way that the boundary of $\overline{\mathcal{L}}(a, b)$ equals $\bigcup_{c \in Cr_k} \mathcal{L}(a, c) \times \mathcal{L}(c, b)$.

Theorem 3.20 (Classification theorem). *A one-dimensional connected manifold is homeomorphic to* S^1, $[0, 1]$, $[0, 1)$ *or* $(0, 1)$.

Proof. The proof can be found in [Mil65], although Milnor proofs the statement with diffeomorphisms □

Since $\overline{\mathcal{L}}(a, b)$ consists of finitely many compact connected components. Each of these components is homeomorphic to S^1 or $[0, 1]$.

Thus to come back to were we started. We wanted to proof the Morse homology is well defined, meaning the boundary operator δ should square to zero as states in theorem 3.2.

Proof of theorem 3.2. We already argued that $\delta^2 = 0$ if every coefficient, as in (2.1), equals zero. Now considering the coefficient

$$\sum_{c \in Cr_k} n(a, c)n(c, b) = \sum_{c \in Cr_k} |\mathcal{L}(a, c)| \times |\mathcal{L}(c, b)| + 2\mathbb{Z} = |\sum_{c \in Cr_k} \mathcal{L}(a, c) \times \mathcal{L}(c, b)| + 2\mathbb{Z}.$$

This is exactly the number of points in the boundary of $\overline{\mathcal{L}}(a, b)$ modulo two. Since $\overline{\mathcal{L}}(a, b)$ is a compact one-dimensional manifold we can apply theorem 3.20 and say that the number of boundary points is even. Therefore the boundary operator squares to zero. □

CHAPTER 4

Morse Homology Theorem

This chapter is devoted to the Morse homology theorem, which states that the Morse homology is isomorphic to the singular homology. To understand the isomorphism, we need some results from cellular homology. This is treated in the first section.

1. Intermezzo: Cellular Homology

For the Morse homology theorem we need to know what the cellular homology is and also a certain expression for the boundary map for the cellular homology.

Let $f : M \to N$ be a continuous function. The composition of the continuous maps $\sigma : \Delta^k \to M$ and f, $\sigma \circ f$, induces a homomorphism $f_* : H_k(M) \to H_k(N)$. This, together with the fact that the n-th homology group of the n-dimensional sphere is the free group generated by one generator i.e. $H_n(S^n) = \mathbb{Z}_2$. Let the generator of $\mathbb{Z}_2$ be denotes as 1.This gives us that the following definition of degree is well defined.

Definition 4.1. *Given a continuous map $f : S^n \to S^n$ we have a homomorphism $f_* : \mathbb{Z}_2 \to \mathbb{Z}_2$. Then the degree of f, $\deg(f)$, is $f_*(1)$.*

This definition of degree has a few nice properties

- If f is the identity map then $\deg(f) = 1$.
- Given two maps
$$S^n \xrightarrow{f} S^n \xrightarrow{g} S^n$$
then $\deg(g \circ f) = \deg(g)\deg(f)$. In particular this gives us that if f is a homeomorphism then $\deg(f) = 1$. Indeed take $g = f^{-1}$ then $1 = \deg(Id) = \deg(f \circ f^{-1}) = \deg(f)\deg(f^{-1})$.

The Cellular homology is defined for a CW-complex X. The complexes are defined as the relative homology groups $H_n(X^{(n)}, X^{(n-1)})$. This definition comes from the following exact sequence of the relative homology

$$\cdots \to H_{n+1}(A) \to H_{n+1}(X) \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to \cdots$$

and the following lemma from [Hat10, Lemma 2.34, p. 137]

Lemma 4.2. *For a CW-complex the following statements are true:*

(1) $H_k(X^{(n)}, X^{(n-1)})$ equals 0 when $k \neq n$, and for $k = n$ it is a free abelian group with a basis corresponding with the n-cells.
(2) $H_k(X^{(n)})$ equals 0 for $k > n$.
(3) $i_* : H_k(X^{(n)}) \to H_k(X)$ is an isomorphism for $k < n$.

Using this theorem we can construct the following diagram.

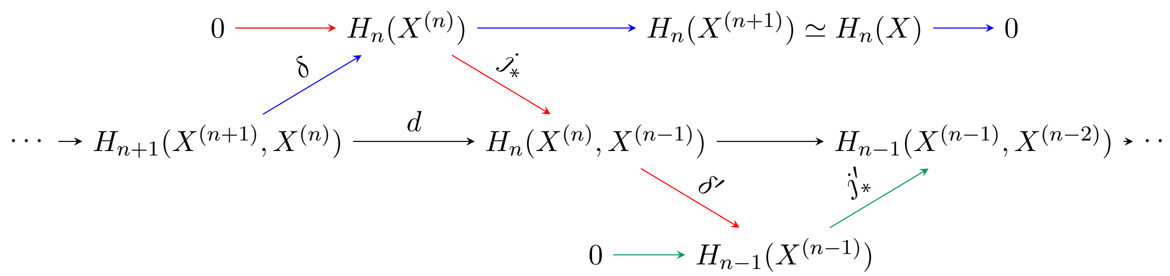


Figure 1. Cellular Diagram

So we have a chain complex of relative homologies and we can define the homology corresponding to this chain complex. This is called the cellular homology:

$$H_n^{cw}(X) := \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$$

THEOREM 4.3. *For a CW-complex its cellular homology is isomorphic to its singular homology.*

PROOF. From the diagram in figure 1 we conclude that

$$H_n(X) \simeq H_n(X^{(n)})/\operatorname{Im}(\delta).$$

Then, since j_* is injective, we have $\operatorname{Im}(\delta) \simeq \operatorname{Im}(j_*\delta) = \operatorname{Im}(d)$. Similarly we have

$$H_n(X^{(n)}) \simeq \operatorname{Im} j_* = \operatorname{Ker} \delta'$$

Then injectivity of j'_* implies $\operatorname{Ker} \delta' = \operatorname{Ker} d$ and hence $H_n(X) \simeq \operatorname{Ker} d/\operatorname{Im} d$. □

When we have a CW-complex, then, looking at the $(n-1)$-skeleton $X^{(n-1)}$, we can collapse the $(n-2)$-skeleton, $X^{(n-2)}$. The resulting space $X^{(n-1)}/X^{(n-2)}$ is a bouquet of $n-1$ spheres, a sphere for every $(n-1)$-cell attached. Label the spheres β_i. Define the map $\psi_{\beta_{i_0}} : X^{(n-1)}/X^{(n-2)} \to S^{n-1}$ as the quotient map collapsing all the spheres except β_{i_0}.

For each n-cell α we have an the attaching map $\phi_\alpha : S^{n-1} \to X^{(n-1)}$. This map can be composed with ψ_β (after collapsing the $(n-2)$-skeleton) to give

$$\psi_\beta \circ \phi_\alpha : \alpha = S^{n-1} \to S^{n-1} = \beta.$$

Then define $d_{\alpha\beta}$ as the modulo 2 degree of $\psi_\beta \circ \phi_\alpha$.

THEOREM 4.4. *With the previous notation the boundary map d in the chain complex of the cellular homology can be represented as*

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}.$$

2. Morse Homology Theorem

In this section we prove the Morse Homology theorem. We start intuitively, and give a sketch of the proof. After this we formally prove the theorem.

We denote the k-th singular homology group of a space X as $H_k(X)$ and the relative homology of $A \subset X$ as $H_k(X, A)$. This is also treated in the section 5 of chapter 2.

Theorem 4.5 (Morse Homology Theorem). *Given a smooth compact manifold M equipped with a Morse function and gradient-like vector field. Then*

$$HM_k(M) \simeq H_k(M)$$

First there will be a intuitive explanation of this theorem and later on we will prove the theorem formally.

Remark 4.6. This theorem can be improved upon, since for a smooth compact manifold there always is a Morse function and a Smale vector field.

The main idea of the Morse Homology theorem is showing that the diagram

$$\begin{array}{ccc} C_k & \xrightarrow{\quad\delta\quad} & C_{k-1} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ H_k(M^{(k)}, M^{(k-1)}) & \xrightarrow[d]{} & H_{k-1}(M^{(k-1)}, M^{(k-2)}) \end{array}$$

Figure 2. Commutative diagram

commutes. So in fact we prove an isomorphism between the chain complex of the Morse homology and the cellular chain complex. From the commutativity the isomorphism of the Morse homology and the cellular homology follows immediately. Then, since for a CW-complex the singular homology coincides with the cellular homology, the theorem is proven by using theorem 4.3.

Note we use the function f to construct a homotopy between the manifold M, and a cellular complex intheorem 2.26. This already suggests that the Morse homology is independent of the Morse function f in the same way that the singular homology does not depend on the cellular decomposition of the manifold.

Recall that $H_k(M^{(k)}, M^{(k-1)})$ is a free abelian group with a generator for every k cell. The complex C_k was defined as the formal sum of critical points with index k. Since the critical point has index k it has a unstable manifold of dimension k. This unstable manifold corresponds to a k-cell. Then the map F sends this critical point of index k to the corresponding k-cell.

We need to correctly define the $F : C_\star \to H_\star(M^{(\star)}, M^{(\star-1)})$. Intuitively we send the critical point to its unstable manifold. Theorem 2.26 implies that this function is an isomorphism. There is only a little subtlety that the unstable manifold is homeomorphic to an open disc, while we need to send it to a closed n-cell. Therefore we need to define a boundary of the unstable manifold. Then the commutativity of the diagram will follow from the way the boundary is attached.

Let us define the closure as follows

Definition 4.7. *The closure of the unstable manifold $W^u(c)$ or the closed unstable manifold is:*

$$\overline{W}^u(c) := W^u(c) \cup \left(\bigcup \overline{\mathcal{L}}(c,d) \times W^u(d)\right)$$

The idea behind this definition is as follows. Given a path that originates from c and has as limit another critical point d. This limit point should then be contained in the closure of your unstable disk. Furthermore, there are paths that get really close to d but will eventually leave the Morse neighborhood. In this way these paths can get arbitrarily close to the unstable manifold of d. Therefore the unstable manifold is contained in the closure. For every such path from c to d we want to add the unstable manifold again. The idea behind this is that we want something that is homeomorphic to the circle. For example if we take the open interval $(0,1)$ then the closure should be $[0,1]$ and not S^1.

At the moment the definition of the closure only makes sense as a set. A topology will be defined next.

Definition 4.8. *Let U, V be opens in the manifold M. The following sets*

$$\mathcal{W}_U(V) := \{(\gamma, x) \in \overline{W}^u(c) : x \in V \quad and\ \gamma \cap U \neq \emptyset\}$$

define a subbasis for the topology on $\overline{W}^u(c)$, making it a topological space.

This is a well defined subbasis since the only thing we need to check is whether the space is covered by our subbasis. We defined the closure of the unstable manifold and stated that it is the closure of the unstable manifold and homeomorphic to the disk. The following theorem allows us to do so.

Theorem 4.9. *Given a critical point c of index k, then the closed unstable manifold $\overline{W}^u(c)$ is homeomorphic to the closed disk i.e. there exists a homeomorphism,*

$$\tilde{\varphi}_c : D^k \to \overline{W}^u(c).$$

This in its turn defines the homeomorphism

$$\varphi_c : S^{n-1} \to \partial\overline{W}^u(c)(= \bigcup_d \overline{\mathcal{L}}(c,d) \times W^u(d))$$

With this homeomorphism we can prove the Morse Homology theorem

Proof of Theorem 4.5. The strategy is to show that there exists an isomorphism of chain-complexes F such that the diagram from figure 2 is commutative. If we show this we also show that the Morse homology is isomorphic to the cellular homology.Then since our manifold is a CW-complex, this equals the singular homology. With the homeomorphism from Theorem 4.9 we can define the isomorphism F as

$$F : C_k \to H_k(M^{(k)}, M^{(k-1)}); \quad c \mapsto [i \circ \tilde{\varphi}_c].$$

Recall that $H_k(M^{(k)}, M^{(k-1)})$ is the free abelian group with a generator for every k-cell, the isomorphism F sends the critical point to the closure of its unstable manifold which is a k-cell.

Since we want to see how the boundary operator of the cellular homology acts on such a cell, we will use the same procedure as in the homology case to reduce the problem by looking at the degree of functions.

Consider the map

$$q : M^{(k-1)} \to M^{(k-1)}/M^{(k-2)} \simeq \bigvee S^{k-1}$$

where the wedge is taken over the amount of $k-1$ cells. Collapsing all the spheres except one, we get the map $c : \bigvee S^{k-1} \to S^{k-1}$.

Composing all these maps gives

$$S^{k-1} \xrightarrow{\varphi} \partial\overline{W}^u(c) \xrightarrow{i} M^{(k-1)} \xrightarrow{c\circ q} S^{k-1}$$

Denote the composition as $\mu = \varphi \circ i \circ c \circ q$. We can compute the degree of this composition. This is the same degree returning in the formula for the boundary operator, theorem 4.4.

To see that the diagram commutes we need a relation between $d_{\alpha\beta}$ and $n(\alpha, \beta)$. Since i works as the inclusion what we do is we glue the $W^u(b)$ space $|\mathcal{L}(a, b)|$ times on itself. Which translates to the fact that $\mu_*(1) = |\mathcal{L}(a, b)| = n(a, b)$ where $\mu_* : H_n(S^n) \to H_n(S^n)$. If we now will look how how the boundary operator of the cellular homology works on the unstable manifold then

$$\begin{aligned} dF(c) &= d[i \circ \tilde{\varphi}_c] \\ &= \sum_\beta d_{\alpha\beta}[i \circ \overline{W}^u(\beta)] \\ &= \sum_\beta n(\alpha, \beta)[i \circ \overline{W}^u(\beta)] \\ &= \sum_\beta n(\alpha, \beta) F(\beta) \\ &= F(\delta c) \end{aligned}$$

We conclude that the diagram commutes and we get the isomorphism of the complexes and hence the Morse homology is isomorphic to the singular homology. □

Remark 4.10. This isomorphism still holds if we look at Morse and singular homology with integer coefficients see for example [BH04].

A question that arises is how much does this depend on the homeomorphism from Theorem 4.9. So, assume that a different homeomorphism $\psi : S^{n-1} \to \partial\overline{W}^u(c)$ exists. Then we can look at the composition.

$$S^{n-1} \overset{\varphi\circ\psi^{-1}}{\longrightarrow} S^{n-1} \overset{\psi}{\longrightarrow} \partial\overline{W}^u(c) \overset{c\circ q\circ i}{\to} S^{n-1}.$$

For simplicity we denote $\mu = c \circ q \circ i$ and recall that for a homeomorphism we have that the degree equals 1, then we get that

$$\begin{aligned} \deg(\mu \circ \varphi) &= \deg(\mu \circ \psi \circ \psi^{-1} \circ \varphi) \\ &= \deg(\mu \circ \psi) \deg(\psi^{-1} \circ \varphi) \\ &= \deg(\mu \circ \psi) \end{aligned}$$

So a different homeomorphism gives us still the isomorphism.

3. Closure of the Unstable Manifold

To finish the proof of the Morse homology theorem we need to prove theorem 4.9, which states that the closed unstable manifold is homeomorphic to the closed disc.

To show this homeomorphism we will define

$$\overline{W}^u(c, A) : \{(\gamma, x) \in \overline{W}^u(c) : f(x) \geq A\}$$

and show that it will remain homeomorphic when we change A. Then when we take A sufficiently close to $f(c)$ then we have immediately that it is homeomorphic to the closed disk, by regarding the Morse lemma 2.6 for example. Furthermore, when A is smaller than the minimum of f then $\overline{W}^u(c, A) = \overline{W}^u(c)$.

The difficult part of showing that is remains homeomorphic is when A crosses a critical value. But let us first explain the case where there are only regular values.

THEOREM 4.11. *Let c a critical point and let $[A, B]$ be an interval without a critical value. Then we have a homeomorphism*

$$\overline{W}^u(c, A) \simeq \overline{W}^u(c, B)$$

PROOF. Similarly as in 2.23, there exists a function $\rho : M \to \mathbb{R}$ such that

$$\rho(x) = -\frac{1}{df(x) \cdot X(x)}$$

on $f^{-1}[A, B]$ and which vanishes outside a compact neighborhood of $f^{-1}[A, B]$. Then we can define a different vector field

$$Y(x) = \rho(x)X(x)$$

and denote ψ as the flow belonging to the vector field. Since ψ is defined on a compact subset it is complete.

Then the homeomorphism is given by

$$\Psi : \overline{W}^u(c, B) \to \overline{W}^u(c, A); \quad (\gamma, x) \mapsto (\gamma, \psi_{B-A}(x)).$$

This has as inverse the map constructed using the negative flow. □

THEOREM 4.12. *Given a critical point c and let d be a critical point with $f(d) = \beta$. There exists a homeomorphism*

$$\overline{W}^u(c, \beta + \varepsilon) \simeq \overline{W}^u(c, \beta - \varepsilon).$$

Intuitively, given a critical point d and a trajectory $\gamma \in \mathcal{L}(c, d)$, there exists a disc D of dimension $\mathrm{Ind}(d)$, such that $D \subset W^u(c) \cap f^{-1}(\beta + \epsilon)$ and $D \cap W^s(d) = \gamma \cap f^{-1}(\beta + \epsilon)$. This disc can be pushed down through adapted flow lines, see figure 3. This adapted flow should agree with the regular flow on boundary.

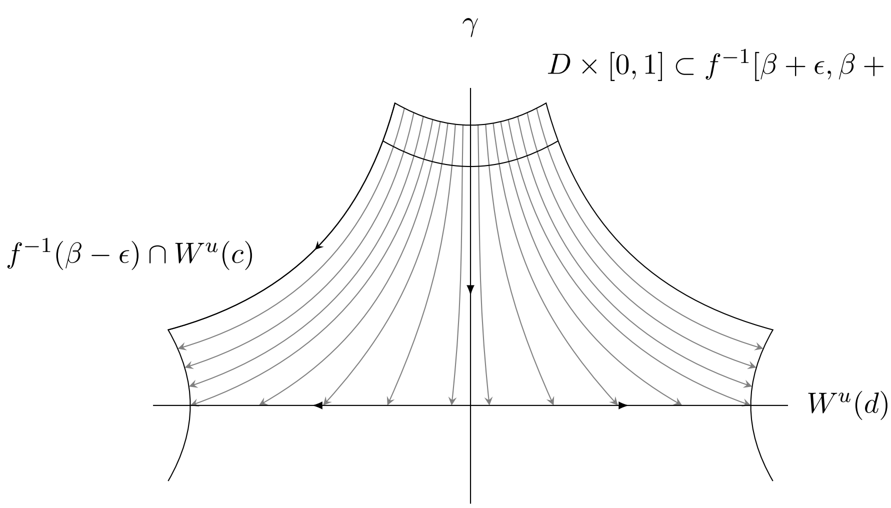


Figure 3. Near a critical point

PROOF. Let us denote the Morse neighborhood as U and assume for simplicity that $\beta = 0$. Outside the Morse neighborhood there are no critical points and thus we can use the previous theorem and use the flow. This is a homeomorphism ψ that is the identity on $\overline{W}^u(c, \epsilon + \delta)$. We may assume that δ and ϵ are sufficiently small such that $f^{-1}(\epsilon + \delta) \cap U \neq \emptyset$.

In the first case we assume that $k = \text{Ind}(d) = \text{Ind}(c) - 1$. Then let $l \in Ł(c, d)$ and set $l \cap f^{-1}(\epsilon) = a$. Note that we use the notation l also as the points in the manifold that lie on that trajectory, from context it is clear what is meant and we will not introduce new notation. There exists a k-dimensional disk such that $D \pitchfork W^s(d)$ and $D \cap W^s(d) = \{a\}$. Furthermore there exists an embedding from $i : D_1 := \{x \in \mathbb{R}^k : \|x\| \leq 1\} \to D$. The image of the restriction $i_{D_{1/2}}$ is denoted as D'. The set $f^{-1}[\epsilon, \epsilon + \delta]$ does not contain any critical values and thus we can create a 'normalized' flow. Using the normalised flow we can can embed the cilinder $D \times [0, \delta]$ in $f^{-1}[\epsilon, \epsilon + \delta]$.

What we want is to create a homeomorphism from $D \times [0, \delta]$ to

$$\tilde{D} := \{x \in f^{-1}[-\epsilon, \epsilon + \delta] : \gamma_x \cap D \times [0, \delta] \neq \emptyset\} \cup W^u(d, -\epsilon).$$

This can be done by taking the flow of the boundary and gradually 'pulling' it to to the (un)stable manifold as shown in the figure 3.

The function $\Gamma : D \times [0, \delta] \to \tilde{D}$ created this way has the following properties:

- $\Gamma(x) = \psi(x)$ for $x \in \partial D \times [0, \delta]$
- $\Gamma(x) = x$ for $x \in D \times \{0\}$
- $\Gamma(x) \in W^u(d, -\epsilon)$ for $x \in D' \times \{\delta\}$.

We can create the function $\Psi : \overline{W}^u(c, \epsilon) \to \overline{W}^u(c, -\epsilon)$ as follows

$$\Psi(x) = \begin{cases} (\gamma, \Gamma(x)) & \text{if } x \in D' \times \{\delta\} \\ (\Gamma(x)) & \text{if } x \in D \times [0, \delta] \setminus D_{1/2} \times \{\delta\} \\ \psi(x) & \text{otherwise.} \end{cases}$$

By construction this map is a homeomorphism and glues well with the flow outside the neighborhood.

Now consider the case that $k = \text{Ind}(d) \neq 0$ and let d' be an arbitrary critical point such that there is a path from d' to d. Therefore $Ł(d', d)$ is a manifold of dimension $\text{Ind}(d') - \text{Ind}(d) - 1$. Hence, by transversality, there exists a tubular k-dimensional disk neighborhood around $W^s(d') \cap (W^u(c) \cap f^{-1}(\epsilon + \delta))$ in $W^u(d')$ then for every $l \in Ł(d', d)$ we can construct a continuous map in the similar way as in the first case. We define Γ_l on $D_l \times [0, \delta]$ as in the first case, where D_l is the disk fibre.

Then we have the map $\Psi : \overline{W}^u(c, \epsilon) \to \overline{W}^u(c, -\epsilon)$, note that if we have a $(\gamma, x) \in \overline{W}^u(c, \epsilon)$ such that x is the Morse neighborhood of d then it is contained in a fibre of some path l_x. Then Ψ is defined as follows

$$(\gamma, x) \mapsto \begin{cases} (\gamma, \Gamma_l(x)) \\ ([\gamma, l_x], \Gamma_l(x)). \end{cases}$$

Again this map is continuous by construction.

For the final case $\text{Ind}(d) = 0$ then we can then we construct the homeomorphism as follows

$$\Psi : \overline{W}^u(c, \epsilon) \to \overline{W}^u(c)$$

by letting l_x be the path through x.

$$(\gamma, x) \mapsto \begin{cases} (\gamma, \Gamma(x)) & \text{if } f(x) > \epsilon \\ ([\gamma, l_x], \Gamma(x) & \text{otherwise.} \end{cases}$$

□

CHAPTER 5

Novikov Homology

In the previous chapters we used a Morse function, i.e. a function with non-degenerate critical points and a gradient-like vector field obeying the Morse-Smale condition. The conditions of the Morse functions can be translated to conditions on its differential. This differential is an exact one-form, hence also closed. One may wonder if we could reproduce what we have done for closed one-forms that are not exact. First we start with a small overview of cohomology to introduce notations. Then we start to translate the theory and definitions we had for a Morse function to Novikov theory. In section 4 we explain the Novikov inequalities, these are similar to the Morse inequalities. Eventually we also define the Novikov complex, this results in a homology that is isomorphic to the homology with a certain local system.

1. Intermezzo: Cohomology

This is a small overview of singular cohomology and the statement that it is isomorphic to the de Rham cohomology.

The idea behind singular cohomology is that it is the dualization of singular homology. What we mean with dualization is the following, given three groups A, B, G and a homomorphism $a : A \to B$ then the dual of a is defined as

$$\begin{aligned} a^* : \hom(B, G) &\to \hom(A, G) \\ \beta &\mapsto \beta \circ a \end{aligned}$$

Let X be a topological space together with the singular complex $C_k(X)$ and its boundary operator δ, the complex is changed by replacing $C_k(X)$ with $C^k(X) = \hom(C_k(X), G)$. We will consider the case when $G = \mathbb{R}$. For the singular homology we have that the boundary operator $\delta : C_k(X) \to C_{k-1}(X)$ is given by

$$\delta(c) = \sum (-1)^i \sigma_{[\hat{i}]}.$$

The dualisation of δ is

$$\delta^* : C^{k-1}(X) \to C^k(X).$$

If $f \in C^{k-1}$ so $f : C_{k-1} \to \mathbb{R}$, then $\delta^*(f) = f \circ \delta : C_k \to \mathbb{R}$. and hence

$$\delta^*(f)(\sigma^k) = \sum_{i=0}^{k} (-1)^i f(\sigma^k \mid_{[\hat{i}]}).$$

Since $(\delta^*)^2 f = f \circ \delta^2$, the boundary operator δ^* squares to zero. Therefore we have a chain complex

$$C^0(X) \xrightarrow{\delta_1^*} C^1(X) \xrightarrow{\delta_2^*} \ldots$$

Define the singular cohomology

$$H^k(X;\mathbb{R}) := \frac{\mathrm{Ker}(\delta^*_{k+1})}{\mathrm{Im}(\delta^*_k)}.$$

Because $\mathbb{R}$ is also a field the universal coefficient theorem [JFD01, Cor 2.31] says that

$$H^k(X;\mathbb{R}) \to \hom(H_k(X;\mathbb{Z}),\mathbb{R})$$

is an isomorphism.

On the other hand the de Rham cohomology is defined using the following chain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

Here $\Omega^k(M)$ is the space of k-forms. The exterior derivative d has the property that $d^2 = 0$ and we can define a cohomology that we call the de Rham cohomology by

$$H^k_{dR} := \frac{\mathrm{Ker}\, d : \Omega^k \to \Omega^{k+1}}{\mathrm{Im}\, d : \Omega^{k-1} \to \Omega^k}.$$

The de Rham cohomology is isomorphic to the singular cohomology [Lee13]. This isomorphism uses $H^\infty_k(M)$, the smooth singular homology which is defined similarly as singular homology (section 5 in chapter 2), only the maps from the simplex

$$\sigma^k : \Delta^k \to M$$

are required to be smooth. The singular homology is isomorphic to the smooth singular homology. We can integrate forms and the theorem of de Rham states that

$$\hom(H^\infty_k(M),\mathbb{R}) \simeq H^k(M;\mathbb{R})$$

is an isomorphism. The map is called the de Rham isomorphism and it is given by

$$\varphi : \omega \mapsto \left([c] \mapsto \int_c \omega\right)$$

Definition 5.1. *A form is called integral if $\varphi(\omega) \in (H_k(M) \to \mathbb{Z})$, meaning that $\int_\gamma \omega \in Z$.*

2. Novikov Theory

In this section we will generalise the conditions imposed on a Morse function to a closed one-form. We will explain why circle-valued functions are an important class of closed one-forms.

Locally a one-form ω can be written as

$$\omega = \sum_{i=1}^n a_i(x) dx_i,$$

where $a_i(x)$ is a smooth function. Then the requirement that the form is closed is equivalent to saying that

$$\frac{\partial a_j}{\partial x_i} = \frac{\partial a_i}{\partial x_j} \qquad \forall i,j$$

With Poincare's lemma we know that for any simply connected open the closed one form is exact, meaning that for any simply connected open U there exists a smooth function f on U such that

$$\omega\mid_U = df\mid_U .$$

When this happens we have that $a_i(x) = \frac{\partial f}{\partial x_i}(x)$.

When we have two sections $s_1, s_2 : M \to TM$ we say that they are transversal if the submanifolds $s_1(M)$ and $s_2(M)$ are transversal.

Recall that x is a critical point of a Morse function f if $df(x) = 0$. A critical point is called non-degenerate if the Hessian is non-singular. The Hessian can be given by the matrix,

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

Note that we can interpret the differential of a Morse function as a map from our manifold M to the tangent bundle TM, i.e.

$$df : M \to TM; \quad (x_1, ..., x_n) \mapsto (x_1, ..., x_n, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$$

The zero-section can be given as

$$s : M \to TM; \quad (x_1, ..., x_n) \mapsto (x_1, ..., x_n, 0, ..., 0)$$

We will show that $df(M) \pitchfork s(M)$. Take a point $x \in df(M) \cap s(M)$. Then this means that $\frac{\partial f}{\partial x_i} = 0$ for all $i \in \{1, ..., n\}$ and hence we are looking at critical point. Calculating the tangent maps at a critical point gives

$$T_x s : T_x M \to T_{s(x)}(TM); \quad (\partial x_1, ..., \partial x_n) \mapsto (\partial x_1, ..., \partial x_n, 0, ..., 0),$$

$$T_x df : T_x M \to T_{df(x)}(TM); \quad (\partial x_1, ..., \partial x_n) \mapsto \begin{pmatrix} Id_n \\ H \end{pmatrix} \begin{pmatrix} \partial x_1 \\ \vdots \\ \partial x_n \end{pmatrix}.$$

Now note that $df(M) \pitchfork s(M)$ if and only if H is non-singular. We say that df is called to be transversal to the zero section.

DEFINITION 5.2. *Given ω a closed one-form and $x \in M$,*

- *We say that x is a critical point of ω if $\omega(x) = 0$.*
- *A critical point x of ω is called non-degenerate if ω is transverse to the zero section.*

REMARK 5.3. Locally around a critical point x write $\omega = df$. then saying x is a non-degenerate point for f coincides with saying x is non-degenerate for ω.

The index of a non-degenerate critical point is defined in the same way as the index for a Morse function by writing it out locally.

DEFINITION 5.4. *Given a closed one-form ω. If all its critical points are non-degenerate then ω is called a Morse form.*

EXAMPLE 5.5. Given a function f which has a non-degenerate critical point x_0, then with 2.6 there exists a coordinate chart such that

$$f(x) = f(x_0) - (x_1^2 + ... + x_k^2) + (x_{k+1}^2 + ... + x_n^2).$$

Taking the exterior derivative on both sides yields

$$df(x) = -(2x_1 dx_1 + ... + 2x_k dx_k) + (2x_{k+1} dx_{k+1} + ... + 2x_n dx_n).$$

Since locally $\omega = df$, we have a local expression for our Morse form.

EXAMPLE 5.6. On the circle we can look at the angular form $\frac{1}{2\pi}(xdy - ydx)$. This form is not exact however, it is closed. Writing $(x, y) = (\cos(2\pi\theta), \sin(2\pi\theta))$ gives that $\frac{1}{2\pi}(xdy - ydx) = d\theta$. Note that for γ, the generator of $\pi_1(S^1)$ one obtains

$$\frac{1}{2\pi}\int_\gamma xdy - ydx = \int_0^1 d\theta = 1$$

Let $f : M \to S^1$ be a function the form $f^*(d\theta)$ is a closed one-form on M. Furthermore this closed form is integral since the fundamental group of the circle is freely generated by one element.

This also works the other way around. If we have an integral one from $\omega \in H^1_{dR}(M)$ on a path connected manifold M, then there exists a function $f : M \to S^1$ such that $f^*(d\theta) = \omega$. Fix a base point x_0. Then the function is given by

$$f : M \to S^1 : x \mapsto \exp(2\pi i \int_{x_0}^x \omega).$$

This function is well defined since for two different paths from x_0 to x, say γ_1 and γ_2, the fact that ω is integral gives us

$$\int_{\gamma_1 * \gamma_2^{-1}} \omega \in \mathbb{Z},$$

and hence

$$\exp(2\pi i \int_{\gamma_1} \omega) = \exp(2\pi i \int_{\gamma_1 * \gamma_2^{-1} * \gamma_2} \omega) = \exp(2\pi i \int_{\gamma_2} \omega).$$

We also have $f^*(d\theta) = \omega$, as desired. It can be verified that the non-degenerate critical points of f coincide with the non-degenerate critical points of ω.

Circle-calued Morse functions are a special case of Novikov theory, these are studied extensively in [Paj06].

3. Intermezzo: Homology with local coefficients

This section will treat a generalisation of singular homology called homology with local coefficients. First we will define a local system and homology with local coefficients. Then we will show how we get a local system from a ring homomorphism. This will be necessary to understand the Novikov inequalities which we will state in the next section.

We immediately start with the definition of a local system.

DEFINITION 5.7. *Given a ring R. A local system of R-modules $\mathcal{L}$ over a topological space X is defined as a function which assigns to any point $x \in X$ a left R-module $\mathcal{L}(x)$ and to any continous path $\gamma : [0, 1] \to X$ an R-homomorphism $\gamma_* : \mathcal{L}(\gamma(1)) \to \mathcal{L}(\gamma(0))$, such that the following conditions are satisfied:*

- *if any two path are homotopic relative end points then the induces R-homomorphisms are identical.*
- *if the path is the constant path then the induced homomorphism is the identity map.*
- *the induced homomorphism of the concatenation of two paths is the composition of the homomorphisms.*

Recall the definition of singular homology from section 5 of chapter 2. So for the singular homology we have continuous functions

$$\sigma^k : \Delta^k (= [v_0, ..., v_k]) \to M$$

and a chain complex $C_k(M)$ consisting of finite linear combinations of such maps. With this notation we can define a chain complex.

Definition 5.8. *Define the set $C_k(M;\mathcal{L})$ as the set of all function*

$$c : C_k(M) \to \mathcal{L}; \quad \sigma \mapsto \mathcal{L}(\sigma(e_0))$$

which are almost everywhere zero, meaning that $|\{\sigma \in C_i(M) : c(\sigma) \neq 0\}|$ is finite.

Remark 5.9. We can write every $c \in C_k(M;\mathcal{L})$ as a finite sum

$$c := \sum_i s_i \cdot \sigma_i^k,$$

and this set has a R-module structure.

Given a $c = s \cdot \sigma \in C_k(M;\mathcal{L})$ then with this σ we construct a path

$$\gamma : [0,1] \to M; \quad t \mapsto \sigma((1-t)v_1 + tv_0)$$

and since we work with a local system we get an induced R-homomorphism $\gamma_* : \mathcal{L}(\sigma(v_0)) \to \mathcal{L}(\sigma(v_1))$. With this homomorphism we can define the boundary operator

$$\delta(c) = \gamma_*(s)\sigma|_{[\hat{0}]} + \sum_{i=1}^{q} (-1)^q s \cdot \sigma_{[\hat{i}]}$$

and we can extend it linearly so that it operates on $C_k(M;\mathcal{L})$

Remark 5.10. Notice that the definition of the boundary operator looks a lot like the one in singular homology. The only difference is that we need the property of the local system to make sure that we get an element out of $\mathcal{L}(\sigma(v_0))$.

Eventually we want to define homology so we need to show that $\delta^2 = 0$.

Proof. Let $c = s \cdot \sigma$. We denote the path $\sigma((1-t)v_i + tv_j)$ as γ_j^i. This is a path from $\sigma(v_i)$ to $\sigma(v_j)$. Further, denote

$$[\hat{i},\hat{j}] = \{x \in \Delta^q : x_i = 0 \text{ and } x_j = 0\}$$

With this definition we have the following rules

$$(\sigma|_{[\hat{i}]})|_{[\hat{j}]} = \sigma|_{[\hat{j},\hat{i}]} \qquad \text{when } j < i,$$
$$(\sigma|_{[\hat{i}]})|_{[\hat{j}]} = \sigma_{[\hat{i},\widehat{j+1}]} \qquad \text{when } j \geq i.$$

Then

$$\delta(\delta(c)) = \delta((\gamma_0^1)_*(s)\sigma|_{[\hat{0}]}) + \sum_{i=1}^{q} (-1)^i \delta(s \cdot \sigma_{[\hat{i}]}). \tag{3.1}$$

For the first part notice that the paths $\gamma_0^1 * \gamma_1^2$ and γ_0^2 are homotopic by a homotopy on the simplex. So by the definition of a local system the induced homomorphisms are the same. Hence $(\gamma_1^2)_*(\gamma_0^1)_* = (\gamma_1^2 * \gamma_0^1)_* = (\gamma_0^2)_*$, which gives

$$\begin{aligned}\delta((\gamma_0^1)_*(s)\sigma|_{[\hat{0}]}) &= (\gamma_1^2)_*((\gamma_0^1)_*(s)\sigma|_{[\hat{0},\hat{1}]}) + \sum_{j=2}^{q} (-1)^{j+1} (\gamma_0^1)_*(s) \cdot \sigma_{[\hat{0},\hat{j}]} \\ &= (\gamma_0^2)_* \cdot \sigma|_{[\hat{0},\hat{1}]} + \sum_{j=2}^{q} (-1)^{j+1} (\gamma_0^1)_*(s) \cdot \sigma_{[\hat{0},\hat{j}]}\end{aligned}$$

For the part $\sum_{i=1}^{q}(-1)^i\delta(s\cdot\sigma_{[\hat{\imath}]})$ we look at two cases $i=1$ and $i>1$. For the first case $i=1$ we see that

$$\delta(s\cdot\sigma_{[\hat{1}]}) = (\gamma_0^2)_*(s)\sigma_{[\hat{0},\hat{1}]} + \sum_{j=2}^{q}(-1)^{j+1}s\cdot\sigma|_{[\hat{1},\hat{\jmath}]}.$$

For the case that $i>1$ we have

$$\begin{aligned}\delta(s\cdot\sigma_{[\hat{\imath}]}) &= (\gamma_0^1)_*(s)\sigma_{[\hat{0},\hat{\imath}]} + \sum_{j=1}^{q-1}(-1)^j s\cdot(\sigma|_{[\hat{\imath}]})|_{[\hat{\jmath}]}\\ &= (\gamma_0^1)_*(s)\sigma_{[\hat{0},\hat{\imath}]} + \sum_{1\le j<i}(-1)^j s\cdot(\sigma|_{[\hat{\jmath},\hat{\imath}]}) + \sum_{j\ge i}^{q-1}(-1)^j s\cdot(\sigma|_{[\hat{\imath},\widehat{j+1}]})\end{aligned}$$

Now note that we can rearrange the index ordering and then simply change the names of the indices. This yields

$$\begin{aligned}\sum_{i=2}^{q}\sum_{j\ge i}^{q-1}(-1)^{i+j}s\cdot(\sigma|_{[\hat{\imath},\widehat{j+1}]}) &= \sum_{j=2}^{q-1}\sum_{2\le i\le j}(-1)^{i+j}s\cdot(\sigma|_{[\hat{\imath},\widehat{j+1}]})\\ &= \sum_{i=2}^{q-1}\sum_{2\le j\le i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{\jmath},\widehat{i+1}]})\end{aligned} \tag{3.2}$$

Then substituting it all into (3.1) gives

$$\begin{aligned}\delta^2(c) = (\gamma_0^2)_*\cdot\sigma|_{[\hat{0},\hat{1}]} + \sum_{j=2}^{q}(-1)^{j+1}(\gamma_0^1)_*(s)\cdot\sigma_{[\hat{0},\hat{\jmath}]} - (\gamma_0^2)_*(s)\sigma_{[\hat{0},\hat{1}]} - \sum_{j=2}^{q}(-1)^{j+1}s\cdot\sigma|_{[\hat{1},\hat{\jmath}]}\\ + \sum_{i=2}^{q}(-1)^i\left((\gamma_0^1)_*(s)\sigma_{[\hat{0},\hat{\imath}]} + \sum_{1\le j<i}(-1)^j s\cdot(\sigma|_{[\hat{\jmath},\hat{\imath}]}) + \sum_{j\ge i}^{q-1}(-1)^j s\cdot(\sigma|_{[\hat{\imath},\widehat{j+1}]})\right)\end{aligned}$$

We immediately see that in the above equation the blue parts cancel each other out. Also we expand the second row as

$$\begin{aligned}= -\sum_{j=2}^{q}(-1)^j(\gamma_0^1)_*(s)\cdot\sigma_{[\hat{0},\hat{\jmath}]} - \sum_{j=2}^{q}(-1)^{j+1}s\cdot\sigma|_{[\hat{1},\hat{\jmath}]} + \sum_{i=2}^{q}(-1)^i(\gamma_0^1)_*(s)\sigma_{[\hat{0},\hat{\imath}]}\\ + \underline{\sum_{i=2}^{q}\sum_{1\le j<i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{\jmath},\hat{\imath}]})} + \sum_{i=2}^{q}\sum_{j\ge i}^{q-1}(-1)^{i+j}s\cdot(\sigma|_{[\hat{\imath},\widehat{j+1}]})\end{aligned}$$

Now the red parts cancel out and we break the underlined sum into two parts. The first is the case were $j=1$ and the second is the rest. Since we do this we can start at $i=3$ in the second sum. For the other sum on the second row we change the indices as we have done in (3.2)/ This gives

$$= -\sum_{j=2}^{q}(-1)^{j+1}s\cdot\sigma|_{[\hat{1},\hat{j}]} + \sum_{i=2}^{q}(-1)^{i+1}s\cdot(\sigma|_{[\hat{1},\hat{i}]})$$

$$+\sum_{i=3}^{q}\sum_{2\leq j<i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{j},\hat{i}]}) + \underline{\sum_{i=2}^{q-1}\sum_{2\leq j\leq i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{j},\widehat{i+1}]})}$$

Now we see the green part fall away. For the underlined part we shift the index of i.

$$= \sum_{i=2}^{q}\sum_{2\leq j<i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{j},\hat{i}]}) - \sum_{i=3}^{q}\sum_{2\leq j<i}(-1)^{i+j}s\cdot(\sigma|_{[\hat{j},\hat{i}]}) = 0$$

Now the last part also cancels and we see that indeed the boundary operator δ squares to zero. □

So, we have a chain complex of R-modules with a boundary operator that squares to zero.

Definition 5.11. *The k-th homology modules of M with local coefficients in $\mathcal{L}$ is given by*

$$H_k(M;\mathcal{L}) := \frac{\operatorname{Ker}\delta : C_k(M;\mathcal{L}) \to C_{k-1}(M;\mathcal{L})}{\operatorname{Im}\delta : C_{k+1}(M;\mathcal{L}) \to C_k(M;\mathcal{L})}$$

Each $H_k(M;\mathcal{L})$ is again an R-module.

Let M be a manifold which has a local system of R-modules. Pick $x \in M$ so we have $\mathcal{L}(x)$ which is a R-module and on this module there is an action of our fundamental group $\pi := \pi_1(M,x)$. Any loop γ with $\gamma(0) = \gamma(1) = x$ determines an automorphism $\gamma_* : \mathcal{L}(x) \to \mathcal{L}(x)$, which only depends on the homotopy class. With this action the module $\mathcal{L}(x)$ turns into a left module over the group ring $R[\pi]$. We have shown that a local system of R-modules induces the $R[\pi]$-module $\mathcal{L}(x)$. But the other way around we have the same thing, i.e. an $R[\pi]$-module induces a local system of R-modules.

Theorem 5.12. *Let M be pathconnected. Let L be the R-module on which there is an action of the fundamental group $\pi_1(M,x_0)$. Then there exists a local system of R-modules $\mathcal{L}$ such that $\mathcal{L}(x) = L$.*

Proof. We begin by denoting the local system as $\mathcal{L}(x) = L$. For every element $x \in M$ we choose a path $\gamma_x \in \pi_1(M;x_0,x)$, with the assumption that γ_{x_0} is the constant path. Now when we have a path $\eta \in \pi_1(M;x_1,x_2)$ then $\gamma_{x_1} * \eta * (\gamma_{x_2})^{-1}$ is an element of $\pi_1(M,x_0)$. Then we also have an R-homomorphism

$$\eta_* : \mathcal{L}(x_2)(=L) \to \mathcal{L}(x_1)(=L); \quad a \mapsto (\gamma_{x_1} * \eta * (\gamma_{x_2})^{-1})\cdot a$$

We need to check whether all the conditions of a local system are satisfied.

- When we have a different path η' that is homotopic relative to the endpoints to η then $\gamma_{x_1} * \eta * (\gamma_{x_2})^{-1}$ is also homotopic to $\gamma_{x_1} * \eta' * (\gamma_{x_2})^{-1}$, which gives the same automorphism and hence the same R-homomorphism.
- If $\eta = c_x$ is the constant path at a point x, then

$$\gamma_x * c_x * (\gamma_x)^{-1} \sim \gamma_x(\gamma_x)^{-1} \sim \gamma_{x_0}$$

  and this induces the identity automorphism.

- Note that we have an action of the fundamental group, meaning that if $\xi_1, \xi_2 \in \pi_1(X, x_0)$ then
$$(\xi_1 * \xi_2) \cdot x = \xi_1 \cdot (\xi_2 \cdot x)$$
So, given two paths $\eta_1 \in \pi_1(M; x_1, x_2)$ and $\eta_2 \in \pi_1(M; x_2, x_3)$, we see that
$$\begin{aligned}(\eta_1 * \eta_2)_*(x) &= (\gamma_{x_1} * \eta_1 * \eta_2 * (\gamma_{x_3})^{-1}) \cdot x \\ &= (\gamma_{x_1} * \eta_1 * (\gamma_{x_2})^{-1} * \gamma_{x_2} * \eta_2 * (\gamma_{x_3})^{-1}) \cdot x \\ &= (\gamma_{x_1} * \eta_1 * (\gamma_{x_2})^{-1}) \cdot ((\gamma_{x_2} * \eta_2 * (\gamma_{x_3})^{-1}) \cdot x) \\ &= (\eta_1)_* \cdot (\eta_2)_*(x)\end{aligned}$$
Which checks the last part.

So, we have constructed a local system of R-modules. □

We have created a local system of R-modules. This local system in turn gives back the $R[\pi]$-module L. Now it could be possible that there are more local systems which determine the same $R[\pi]$-module. But we will show that these are equivalent up to natural isomorphisms.

Definition 5.13. *Given two local systems of R-modules $\mathcal{L}$ and $\mathcal{L}'$. We say that*
$$\Phi : \mathcal{L} \to \mathcal{L}'$$
is a natural isomorphism if the diagram below commutes for every path η. Note that we denote η'_* for the homomorphism in the local system $\mathcal{L}'$.

$$\begin{array}{ccc} \mathcal{L}(x) & \xrightarrow{\eta_*} & \mathcal{L}(y) \\ \Phi(x) \downarrow & & \downarrow \Phi(y) \\ \mathcal{L}'(x) & \xrightarrow[\eta'_*]{} & \mathcal{L}'(y) \end{array}$$

Theorem 5.14. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two local systems and a operator homomorphism $\phi : \mathcal{L}(x_0) \to \mathcal{L}'(x_0)$, meaning that $\gamma'_* \circ \phi = \phi \circ \gamma_*$ for every $\gamma \in \pi_1(M, x_0)$. Then there is a unique natural isomorphism $\Phi : G \to G'$ with $\Phi(x_0) = \phi$.*

Proof. We again pick for every $x \in M$ a $\gamma_x \in \pi_1(M; x_0, x)$ and define
$$\Phi(x) : \mathcal{L}(x) \to \mathcal{L}'(x); \quad \alpha \mapsto (\gamma'_x)_*^{-1} \circ \phi \circ (\gamma_x)_*(\alpha)$$
We need to show that we this map is an natural isomorphism and that it is unique. Let $\eta \in \pi_1(M; y, x)$ and set $\rho = \gamma_y * \eta * \gamma_x^{-1}$. This is an element of $\pi_1(M, x_0)$ and so we have $\rho'_* \circ \phi = \phi \circ \rho_*$ by assumption. To show that it is a natural isomorphism we need to show that the diagram commutes.
$$\begin{aligned}\Phi(y) \circ \eta_* &= (\gamma'_y)_*^{-1} \circ \phi \circ (\gamma_y)_* \circ \eta_* \\ &= (\gamma'_y)_*^{-1} \circ \phi \circ \rho_* \circ (\gamma_x)_* \\ &= (\gamma'_y)_*^{-1} \circ \rho'_* \circ \phi \circ (\gamma_x)_* \\ &= \eta'_* \circ (\gamma'_x)_* \circ \phi \circ (\gamma_x)_* = \eta'_* \circ \Phi(x)\end{aligned}$$
thus it is an natural isomorphism. To show uniqueness, assume we have another natural isomorphism Ψ for which $\Psi(x_0) = \phi$. Then by definition we have the following commuting diagram

$$\begin{array}{ccc} \mathcal{L}(x) & \xrightarrow{(\gamma_x)_*} & \mathcal{L}(x_0) \\ \Psi(x)\Big\downarrow & & \Big\downarrow \Psi(x_0)=\phi \\ \mathcal{L}'(x) & \xrightarrow[(\gamma'_x)_*]{} & \mathcal{L}'(x_0) \end{array}$$

But this immediately says $\Psi(x) = (\gamma_x)_*^{-1} \circ \phi \circ (\gamma_x)_* = \Phi(x)$ □

Theorems 5.12 and 5.14 combined tells us that there is a one-to-one correspondence between local systems of R-modules and an R-module with an action of the fundamental group.

4. Novikov Inequalities and Homology

For the Novikov inequalities we need to understand the Novikov Betti and torsion numbers. These come from a generalisation of singular homology called homology with local coefficients explained in the previous section. In the beginning of this section we will see how a ring-homomorphism implies a local system, and how the closed one form can imply such a ring homomorphism. The local system obtained this way is used for the Novikov BBettietti and torsion numbers.

Given a ring R and a ring homomorphism $\phi : \mathbb{Z}[\pi] \to R$, we can see R as a left $\mathbb{Z}[\pi]$-module:

$$g \cdot r = r\phi(g)^{-1} \quad g \in \pi \quad r \in R$$

This $\mathbb{Z}[\pi]$-module structure commutes with the standard R-module structure we always have on a ring. It follows that R has a $R[\pi]$ module structure. With what we have seen in the previous section such a module structure implies a local system of R-modules. If the local system is the result of a ring homomorphism ϕ we denote the local system as $\mathcal{L}_\phi$.

The ring we use in the Novikov homology is called the Novikov ring.

Definition 5.15. *The Novikov ring is defined as:*

$$\boldsymbol{Nov} := \{\sum_{i=0}^{\infty} a_i t^{\gamma_i} : a_i \in \mathbb{Z}, \gamma_i \in \mathbb{R} \text{ and } \gamma_i \to -\infty\}$$

We need to clarify what addition and multiplication is on this ring to see it is truly a ring we define

$$\sum_{i=0}^{\infty} a_i t^{\gamma_i} + \sum_{i=0}^{\infty} b_j t^{\gamma_j} = \sum_{i,j=0}^{\infty} a_i t^{\gamma_i} + b_j t^{\gamma_j}$$
$$(\sum_{i=0}^{\infty} a_i t^{\gamma_i})(\sum_{i=0}^{\infty} b_j t^{\gamma_j}) = \sum_{k \in \mathbb{R}} (\sum_{\gamma_i + \gamma_j = k} a_i b_j) t^k$$

The Novikov ring is closed under the above definitions of multiplication and addition. Furthermore one can prove that this ring is a principal ideal domain and is torsion free. [Far04, Lemma 1.10 and 1.12]

Let ξ a class of closed 1-forms, this gives a ring-homomorphism

$$\phi_\xi : \mathbb{Z}[\pi] \to \mathbf{Nov}$$

The homomorphism sends a generator

$$g \mapsto t^{\langle \xi, g \rangle}; \quad \langle \xi, g \rangle := \int_g \xi$$

and then extending it linearly. This ring homomorphism gives a local system which we denote $\mathcal{L}_\xi$.

DEFINITION 5.16. *For a closed one-form ξ, we define the Novikov homology as*

$$H_k(M; \mathcal{L}_\xi)$$

*these homology groups again have a **Nov**-module structure*

With the structure theorem for finitely generated modules over a principal ideal domain (see for example [Hun74, Thm. 6.12. p. 225]), these homology groups can be decomposed in a free part and a torsion part.

Now that we have defined the Novikov homology we can define the Novikov Betti numbers and the Novikov torsion numbers.

DEFINITION 5.17. *Let ξ be a class of closed one-forms, the k-th Novikov Betti numbers $b_k(\xi)$ is the rank of the free group of $H_k(M; \mathcal{L}_\xi)$. The Novikov torsion number $q_k(\xi)$ is the minimum number of generators for the torsion part of $H_k(M; \mathcal{L}_\xi)$.*

REMARK 5.18. If we take the zero homology class $0 = \xi \in H^1(M)$, i.e. the class of exact forms, then

$$\int_g df = 0$$

Therefore we have that the ring homomorphism ϕ_ξ has as its image $\mathbb{Z}$. Hence for every path the induced isomorphism has to be the trivial isomorphism. Therefore the homology with local coefficients is, in this case, the singular homology with coefficients in **Nov**. This implies that the Novikov Betti number coincides with the regular Betti number, i.e. $b_i(\xi) = b_i(M)$. For more details see [Far04, p.17].

Now we have explained enough to state and understand the Novikov inequalities

THEOREM 5.19. *Let ω be a Morse form on a manifold M. Denote by $\xi = [\omega] \in H^1(M; \mathbb{R})$ the homology class of ω. Then for any j,*

$$\sum_{i=0}^{j} (-1)^i c_{j-i}(\omega) \geq q_j(\xi) + \sum_{i=0}^{j} (-1)^i b_{j-i}(\xi)$$

5. Novikov Complex

Now we look at a circle valued Morse function and we define the Novikov complex that corresponds to this function. We will follow a similar approach as in [Ran01] with influences from [Far04] and [Paj06].

So we have a closed one-form $\omega \in H^1(M; \mathbb{Z})$. Then, with example 5.6 we get a corresponding circle valued function f such that $\omega = f^*(d\theta)$ and the non-degenerate critical points of ω coincide with the Morse type critical points of f.

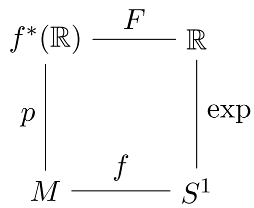


Figure 1. Lift

Looking at the exponent map $\exp : \mathbb{R} \to S^1$ we can pull back the M to the infinite cycle cover $\tilde{M} := f^*(M)$. Define $F : \tilde{M} \to \mathbb{R}$ to be the lift of the function f.

So, we have rewritten the problem to real valued Morse function

$$F : \tilde{M} \to \mathbb{R}.$$

The problem now is the fact that $\tilde{M}$ is not compact therefore we can not use regular Morse theory. Compactness was very important to state that the vector field is complete, and that there existed a gradient-like vector field. Furthermore there could be an infinite number of critical points for non-compact manifolds.

The idea is to lift the properties we want from M. Since M is compact it has a gradient-like vector field we can lift this vector field v to a gradient-like vector field $\tilde{v}$ on $\tilde{M}$. Note the v is transversal if and only if $\tilde{v}$ is transversal [Paj06, p. 337].

Look at the covering translations Π. Since $\tilde{M}$ is an infinite cycle cover of M these coverings are generated by one generator denoted as z. Take an arbitrary lift of the critical points of f, denote the lift of c as $\tilde{c}$. In this way every critical point d of F can be written as $z^j\tilde{p(d)}$. Define the Novikov chain complex, N, with as k-th complex

$$N_k = N_k(f, v) := \{\sum n_i\tilde{c}_i : n_i \in \mathbf{Nov}, c \in Cr(f)\},$$

together with the boundary operator

$$\begin{aligned} \delta : N_k &\to N_{k-1} \\ \tilde{c} &\mapsto \sum_{q\in c_i(f)} \sum_{j\in\mathbb{Z}} n(\tilde{c}, z^j\tilde{q})z^j\tilde{q} \end{aligned}$$

To see that the boundary operator is well defined we need to check that $\sum_{j\in\mathbb{Z}} n(\tilde{c}, z^j\tilde{q})z^j$ is an element of the Novikov ring. First of all, note that there exists a j_0 such that $F(\tilde{c}) < F(z^{j_0}\tilde{q})$ and thus $n(\tilde{c}, z^j\tilde{q}) = 0$ because of the gradient like vector field. So there is a upper bound for the j and since the $j \in \mathbb{Z}$ it follows that $j \to -\infty$. Hence the coefficient is contained in the Novikov ring.

We also need that the boundary operator squares to zero. To this end, observe that

$$\begin{aligned} \delta^2(c) &= \delta(\sum_{q\in c_i(f)} \sum_{j\in(Z)} n(\tilde{c}, z^j\tilde{q})z^j q) \\ &= \sum_{q\in c_i(f)} \sum_{j\in(Z)} n(\tilde{c}, z^j\tilde{q}) \left(\sum_{p\in c_{i-1}(f)} \sum_{i\in\mathbb{Z}} n(z^j\tilde{q}, z^i\tilde{p})z^i \right) \tilde{p} \end{aligned}$$

Then saying that the boundary operator squares to zero is equivalent to saying that

$$\sum_{j\in\mathbb{Z}}\sum_{q\in c_i} n(\tilde{c}, z^j\tilde{q})n(z^j\tilde{q}, z^i\tilde{p}) = 0$$

for all $z^i\tilde{p}$. Take regular values α and β such that $\alpha > F(\tilde{c}) > F(z^i\tilde{p}) > \beta$. Consider $F^{-1}[\beta, \alpha]$, this is a cobordism and for this we can use the Morse homology for cobordisms and then the boundary operators are equivalent. Since the boundary operator squares to zero in the Morse homology then it also square to zero in the Novikov chain. Hence we can define the homology belonging to the Novikov complex

$$H_k(N(f,v)) := \frac{\operatorname{Ker}\delta^{nov} : N_k \to N_{k-1}}{\operatorname{Im}\delta^{nov} : N_{k+1} \to N_k}.$$

Theorem 5.20. *[Ran01] Given a closed one-form $\xi \in H^1(M, \mathbb{Z})$, we have a circle valued function $f : M \to S^1$. The Novikov complex is **Nov**-module chain equivalent to $C(M; \textbf{Nov})$ and we have the isomorphisms*

$$H_k(N(f,v)) \simeq H_k^{nov}(M; \mathcal{L}_\xi)$$

In fact Ranicki states that

$$H_k(N(f,v) \simeq H_k(R \otimes_{\mathbb{Z}[\pi]} C(\tilde{M})).$$

But from [Whi78] or [Far04] we know that the equivariant homology is equivalent to the homology with local coefficients, i.e.

$$H_k(R \otimes_{\mathbb{Z}[\pi]} C(\tilde{M})) \simeq H_k(M; \text{Ł}_\xi)$$

A similar construction can be done for general Morse forms ω on the manifold M. To do so we consider the universal cover $\bar{M}$ of the manifold together with the projection

$$p : \bar{M} \to M.$$

Then the pullback $p_*(\omega)$ is a closed one-form on $\bar{M}$. The universal cover is simply connected, by definition, thus with poincare's lemma we know that the closed one-form is exact. Therefore there is an real valued function

$$f : \bar{M} \to \mathbb{R}$$

such that $df = p_*(\omega)$, this is again a Morse function, and the non-degenerate ciritcal points of f coincide with the non-degenerate critical points of ω. It is not necessarily true that the covering translations are generated by one element, in fact the group of covering translations is isomorphic to the fundamental group. Denote the group of covering translations Π. Let $g \in \Pi$. Take a take a path γ_g from x to $g \cdot x$, then we can integrate the closed one form over the closed path $p(\gamma_g)$. Note that this integral does not depend on the path taken.

For every critical point $c_i \in Cr(\omega)$, pick a lift $\tilde{c}_i$. Define the Novikov complex in the same way.

$$N_k = N_k(f,v) := \{\sum n_i\tilde{c}_i : n_i \in \textbf{Nov}, c \in Cr(\omega)\}$$

For $c \in Cr_k(\omega)$ we set

$$\delta(c) = \sum_{b\in Cr_{k-1}}\sum_{\gamma\in\Pi} n(c, \gamma\cdot b)t^{\langle\xi,\gamma\rangle}b$$

this we extend linearly. Farber shows that this is again well defined. [Far04]

Therefore we can again define the homology belonging to the Novikov complex

$$H_k(N(\omega, v))$$

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