Quantum Statistical Mechanics, L-series and Anabelian Geometry I: Partition Functions

Gunther Cornelissen and Matilde Marcolli

Abstract The zeta function of a number field can be interpreted as the partition function of an associated quantum statistical mechanical (QSM) system, built from abelian class field theory.
We introduce a general notion of isomorphism of QSM-systems and prove that it preserves (extremal) KMS equilibrium states.
We prove that two number fields with isomorphic quantum statistical mechanical systems are arithmetically equivalent, i.e., have the same zeta function. If one of the fields is normal over $\mathbb{Q}$, this implies that the fields are isomorphic. Thus, in this case, isomorphism of QSM-systems is the same as isomorphism of number fields, and the *noncommutative* space built from the *abelianized* Galois group can replace the *anabelian* absolute Galois group from the theorem of Neukirch and Uchida.

Introduction

The starting point for this study is the observation that the zeta function of a number field $\mathbb{K}$ can be realized as the partition function of a quantum statistical mechanical (QSM) system in the style of Bost and Connes (cf. [3] for $\mathbb{K} = \mathbb{Q}$). The QSM-systems for general number fields that we consider are those that were constructed by Ha and Paugam (see section 8 of [12], which is a specialization of their more

---

Gunther Cornelissen
Mathematisch Instituut, Universiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, Nederland, e-mail: g.cornelissen@uu.nl

Matilde Marcolli
Mathematics Department, Mail Code 253-37, Caltech, 1200 E. California Blvd. Pasadena, CA 91125, USA e-mail: matilde@caltech.edu

This paper is an updated version of part of [9]. We have split the original preprint into various parts, depending on the methods that are used in them. In the current part, these belong mainly to mathematical physics.

general class of QSM-systems associated to Shimura varieties), and further studied by Laca, Larsen and Neshveyev in [16]. This quantum statistical mechanical system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ consists of a C^*-algebra $A_{\mathbb{K}}$ (the noncommutative analogue of a topological space) with a time evolution $\sigma_{\mathbb{K}}$ (i.e., a continuous group homomorphism $\mathbb{R} \to \mathrm{Aut} A_{\mathbb{K}}$). The structure of the algebra is that of a semigroup crossed product

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}, \text{ with } X_{\mathbb{K}} := G^{\mathrm{ab}}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}},$$

where $\hat{\mathcal{O}}_{\mathbb{K}}$ is the ring of finite integral adeles and $J^+_{\mathbb{K}}$ is the semigroup of ideals, which acts on the space $X_{\mathbb{K}}$ by Artin reciprocity. The time evolution is only nontrivial on elements $\mu_{\mathfrak{n}} \in A_{\mathbb{K}}$ corresponding to ideals $\mathfrak{n} \in J^+_{\mathbb{K}}$, where it acts by multiplication with the norm $N(\mathfrak{n})^{it}$. For exact definitions, see Section 2.

We call two general QSM-systems *isomorphic* if there is a C^*-algebra isomorphism between the algebras that intertwines the time evolutions. In Section 1, we prove that such an isomorphism induces a homeomorphism between (extremal) KMS-equilibrium states of the systems at a given temperature.

Our main result for the QSM-systems of number fields is:

Theorem (= Theorem 4.1 below)**.** *If the QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ of two number fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, then $\mathbb{K}$ and $\mathbb{L}$ are arithmetically equivalent, i.e., they have the same Dedekind zeta function.*

Using some other known consequences of arithmetical equivalence, we get the following ([19], Theorem 1): if number fields $\mathbb{K}$ and $\mathbb{L}$ have isomorphic QSM-systems, then, for any rational prime p, there is a bijection between the prime ideals $\mathfrak{p}$ of $\mathbb{K}$ above p and the prime ideals $\mathfrak{q}$ of $\mathbb{L}$ above p that preserves the inertia degrees: $f(\mathfrak{p} \,|\, \mathbb{K}) = f(\mathfrak{q} \,|\, \mathbb{L})$. Furthermore, the number fields have the same degree over $\mathbb{Q}$, the same discriminant, normal closure, isomorphic unit groups, and the same number of real and complex embeddings. However, it does not follow from arithmetical equivalence that $\mathbb{K}$ and $\mathbb{L}$ have the same class group (or even class number), cf. [10]. In general, arithmetic equivalence does not imply that $\mathbb{K}$ and $\mathbb{L}$ are isomorphic, as was shown by Gaßmann ([11], cf. also Perlis [19], or [14]). An example is provided by $\mathbb{K} = \mathbb{Q}(\sqrt[8]{3})$ and $\mathbb{L} = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$ ([19], [15]). However, the implication is true *if* $\mathbb{K}$ and $\mathbb{L}$ are Galois over $\mathbb{Q}$ (Theorem of Bauer [1] [2], nowadays a corollary of Chebotarev's density theorem, see, e.g., Neukirch [18] 13.9), so we find:

Corollary. *If the QSM-systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ of two number fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic and the extension $\mathbb{K}/\mathbb{Q}$ is normal, then $\mathbb{K}$ and $\mathbb{L}$ are isomorphic as fields.* □

This corollary is somewhat reminiscent of the anabelian theorem of Neukirch and Uchida ([17], [20]), which says that number fields with isomorphic absolute Galois groups are isomorphic (Neukirch [17] proved this if one of the fields is normal over $\mathbb{Q}$, just as in our corollary). It is interesting to notice that the QSM-system involves the abelianized Galois group and the adeles, but not the absolute Galois group. In this sense, it is "not anabelian"; but of course, it is "noncommutative" (in noncommutative topology, the crossed product construction is an analog of taking quotients). The emerging philosophy seems to be that one can substitute the consideration of the "anabelian" absolute Galois group (with its difficult representation theory studied in the Langlands programme) by the dynamics of the action of Frobeniusses in the abelianized Galois group, with its "easy" representation theory given by class field theory.

One may wonder whether QSM-system isomorphism in general implies field isomorphism. In [8], this is proven for global function fields. We will discuss the number field case in the remaining instalments of this work.

The structure of this paper is as follows: first, we introduce isomorphism of QSM-systems. We deduce some basic properties, such as preservation of (extremal) KMS-states. Then we recall the QSM-system of a number field, and we prove our main theorem. In the final section, we make explicit the matching of KMS states for number fields.

1 Isomorphism of QSM systems

We recall some definitions and refer to [4], [6], and Chapter 3 of [7] for more information and for some physics background. After that, we introduce isomorphism of QSM-systems, and prove it preserves KMS-states.

1.1 Definition. A *quantum statistical mechanical system* (QSM-system) (A,σ) is a (unital) C^*-algebra A together with a so-called *time evolution* σ, which is a continuous group homomorphism $\sigma : \mathbb{R} \to \text{Aut } A : t \mapsto \sigma_t$. A *state* on A is a continuous positive unital linear functional $\omega : A \to \mathbb{C}$. We say ω is a *KMS*$_\beta$ *state* for some $\beta \in \mathbb{R}_{>0}$ if for all $a,b \in A$, there exists a function $F_{a,b}$, holomorphic in the strip $0 < \Im z < \beta$ and bounded continuous on its boundary, such that

$$F_{a,b}(t) = \omega(a\sigma_t(b)) \text{ and } F_{a,b}(t+i\beta) = \omega(\sigma_t(b)a) \quad (\forall t \in \mathbb{R}).$$

Equivalently, ω is a σ-invariant state with $\omega(ab) = \omega(b\sigma_{i\beta}(a))$ for a,b in a dense set of σ-analytic elements. The set $\text{KMS}_\beta(A,\sigma)$ of KMS_β states is topologized as a subspace of the convex set of states, a weak* closed subset of the unit ball in the operator norm of bounded linear functionals on the algebra. A KMS_β state is called *extremal* if it is an extremal point in the (compact convex) set of KMS_β states for the weak (i.e., pointwise convergence) topology.

1.2 Remark (Physical origins)**.** This notion of QSM-system is one of the possible physical theories of quantum statistical mechanics; one should think of A as the algebra of observables, represented on some Hilbert space $\mathscr{H}$ with orthonormal basis $\{\Psi_i\}$; the time evolution, in the given representation, is generated by a Hamiltonian H by

$$\sigma_t(a) = e^{itH}ae^{-itH}, \tag{1}$$

and (mixed) states of the system are combinations

$$a \mapsto \sum \lambda_i \langle \Psi_i | a \Psi_i \rangle$$

which will mostly be of the form $a \mapsto \mathrm{trace}(\rho a)$ for some density matrix ρ. A typical equilibrium state (here, this means stable by time evolution) is a Gibbs state

$$a \mapsto \mathrm{trace}(ae^{-\beta H}) / \mathrm{trace}(e^{-\beta H})$$

at temperature $1/\beta$, where we have normalized by the *partition function* $\mathrm{trace}(e^{-\beta H})$. The KMS-condition (named after Kubo, Martin and Schwinger) is a correct generalization of the notion of equilibrium state to more general situations, for example when the trace class condition $\mathrm{trace}(e^{-\beta H}) < \infty$, needed to define Gibbs states, no longer holds (cf. Haag, Hugenholtz and Winnink [13]).

We now introduce the following equivalence relation for QSM-systems:

1.3 Definition. An *isomorphism* of two QSM-systems (A, σ) and (B, τ) is a C^*-algebra isomorphism $\varphi : A \xrightarrow{\sim} B$ that intertwines time evolutions, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow[\sim]{\varphi} & B \\ \sigma \downarrow & & \downarrow \tau \\ A & \xrightarrow[\sim]{\varphi} & B \end{array}$$

1.4 Proposition. *Let $\varphi : (A, \sigma) \xrightarrow{\sim} (B, \tau)$ denote an isomorphism of QSM systems. Then for any $\beta > 0$,*

(i) *pullback*

$$\varphi^* : \mathrm{KMS}_\beta(B, \tau) \to \mathrm{KMS}_\beta(A, \sigma) : \omega \mapsto \omega \circ \varphi$$

is a homeomorphism between the spaces of KMS_β *states on B and A;*

(ii) φ^* *induces a homeomorphism between extremal* KMS_β *states on B and A.*

Proof. The map φ obviously induces a bijection between states on B and states on A.

For (i), let $F_{a,b}$ be the holomorphic function that implements the KMS_β-condition for the state ω on (B, τ) at $a, b \in B$, so

$$F_{a,b}(t) = \omega(a\tau_t(b)) \text{ and } F_{a,b}(t + i\beta) = \omega(\tau_t(b)a).$$

The following direct computation then shows that the function $F_{\varphi(c),\varphi(d)}$ implements the KMS_β-condition for the state $\varphi^*\omega$ on (A, σ) at $c, d \in A$:

$$(\omega \circ \varphi)(c\sigma_t(d)) = \omega(\varphi(c)\tau_t(\varphi(d)) = F_{\varphi(c),\varphi(d)}(t),$$

and similarly at $t + i\beta$. Also, note that pullback is continuous, since C^*-algebra isomorphism is compatible with the topology on the set of KMS-states.

For (ii), if a KMS_β state ω on B is not extremal, then the GNS-representation π_ω of ω is not factorial. As in Prop 3.8 of [6], there exists a positive linear functional, which is dominated by ω, namely $\omega_1 \leq \omega$, and which extends from B to the von Neumann algebra given by the weak closure $\mathscr{M}_\omega$ of B in the GNS representation. The functional ω_1 is of the form $\omega_1(b) = \omega(hb)$ for some positive element h in the center of the von Neumann algebra $\mathscr{M}_\omega$. Consider then the pullbacks

$$\varphi^*(\omega)(a) = \omega(\varphi(a))$$

and

$$\varphi^*(\omega_1)(a) = \omega_1(\varphi(a)) = \omega(h\varphi(a))$$

for $a \in A$. The continuous linear functional $\varphi^*(\omega_1)$ has norm $\|\varphi^*(\omega_1)\| \leq 1$. In fact, since we are dealing with unital algebras, $\|\varphi^*(\omega_1)\| = \varphi^*(\omega_1)(1) = \omega(h)$. The linear functional $\omega_2(b) = \omega((1-h)b)$ also satisfies the positivity property $\omega_2(b^*b) \geq 0$, since $\omega_1 \leq \omega$. The decomposition

$$\varphi^*(\omega) = \lambda\eta_1 + (1-\lambda)\eta_2,$$

with $\lambda = \omega(h)$,

$$\eta_1 = \varphi^*(\omega_1)/\omega(h) \text{ and } \eta_2 = \varphi^*(\omega_2)/\omega(1-h)$$

shows that the state $\varphi^*(\omega)$ is not extremal. Notice that η_1 and η_2 are both KMS states. To see this, it suffices to check that the state $\omega_1(b)/\omega(h)$ is KMS. In fact, one has for all analytic elements $a, b \in B$:

$$\omega_1(ab) = \omega(hab) = \omega(ahb) = \omega(hb\tau_{i\beta}(a)).$$

This proves the proposition. □

2 A QSM-system for number fields

Bost and Connes ([3]) introduced a QSM-system for the field of rational numbers. More general QSM-systems associated to arbitrary number fields were constructed by Ha and Paugam in [12] as a special case of their more general class of systems for Shimura varieties, which in turn generalize the GL(2)-system of [6]. We recall here briefly the construction of the systems for number fields in an equivalent formulation (cf. also [16]).

2.1. We denote by $J_{\mathbb{K}}^+$ the semigroup of integral ideals, with the norm function

$$N : J_{\mathbb{K}}^+ \to \mathbb{Z} : \mathfrak{n} \mapsto N(\mathfrak{n}) = N_{\mathbb{Q}}^{\mathbb{K}}(\mathfrak{n}) = N_{\mathbb{K}}(\mathfrak{n}).$$

Denote by $G_{\mathbb{K}}^{\mathrm{ab}}$ the Galois group of the maximal abelian extension of $\mathbb{K}$. The Artin reciprocity map is denoted by

$$\vartheta_{\mathbb{K}} : \mathbf{A}_{\mathbb{K}}^{*} \to G_{\mathbb{K}}^{\mathrm{ab}}.$$

By abuse of notation, we will also write $\vartheta_{\mathbb{K}}(\mathfrak{n})$ for the image under this map of an ideal $\mathfrak{n}$, which is seen as an idele by choosing a non-canonical section s of

$$\mathbf{A}_{\mathbb{K},f}^{*} \twoheadrightarrow J_{\mathbb{K}} \quad (\text{section } s) \qquad : \qquad (x_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})} .$$

The abuse lies in the fact that the image depends on this choice of section (thus, up to a unit in the finite ideles), but it is canonically defined in (every quotient of) the Galois group $G_{\mathbb{K},\mathfrak{n}}^{\mathrm{ab}}$ of the maximal abelian extension unramified at prime divisors of $\mathfrak{n}$: on every finite quotient of this, it is the “Frobenius element” of $\mathfrak{n}$. The notation $\vartheta_{\mathbb{K}}(\mathfrak{n})$ will only occur in situations where this ambiguity plays no role.

We consider the fibered product

$$X_{\mathbb{K}} := G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_{\mathbb{K}}^{*}} \hat{\mathscr{O}}_{\mathbb{K}},$$

(where $\hat{\mathscr{O}}_{\mathbb{K}}$ is the ring of finite integral adeles), where the balancing is defined for $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ and $\rho \in \hat{\mathscr{O}}_{\mathbb{K}}$ by the equivalence

$$(\gamma,\rho) \sim (\vartheta_{\mathbb{K}}(u^{-1})\cdot\gamma, u\rho) \text{ for all } u \in \hat{\mathscr{O}}_{\mathbb{K}}^{*}.$$

2.2 Definition. The *QSM-system* $(A_{\mathbb{K}},\sigma_{\mathbb{K}})$ *associated to a number field* $\mathbb{K}$ is defined as the semigroup crossed product algebra

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+} = C(G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_{\mathbb{K}}^{*}} \hat{\mathscr{O}}_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+}, \qquad (2)$$

where the crossed product structure is given by $\mathfrak{n} \in J_{\mathbb{K}}^{+}$ acting on $f \in C(X_{\mathbb{K}})$ as

$$(\mathfrak{n}, f) \mapsto \rho_{\mathfrak{n}}(f)(\gamma,\rho) = f(\vartheta_{\mathbb{K}}(\mathfrak{n})\gamma, s(\mathfrak{n})^{-1}\rho)e_{\mathfrak{n}},$$

with $e_{\mathfrak{n}} = \mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^{*}$ the projector onto the space of $[(\gamma,\rho)]$ where $s(\mathfrak{n})^{-1}\rho \in \hat{\mathscr{O}}_{\mathbb{K}}$. Here $\mu_{\mathfrak{n}}$ is the isometry that implements the action of $J_{\mathbb{K}}^{+}$. Note that, because of the balancing over the finite idelic units $\hat{\mathscr{O}}_{\mathbb{K}}^{*}$, the dependence of $\vartheta_{\mathbb{K}}(\mathfrak{n})$ on s is again of no influence. The action has a partial inverse defined by

$$\sigma_{\mathfrak{n}}(f)(x) = f(\mathfrak{n} * x)$$

where we have defined the action $\mathfrak{n} * x$ of an ideal $\mathfrak{n} \in J_{\mathbb{K}}^{+}$ on an element $x \in X_{\mathbb{K}}$ as

$$\mathfrak{n} * [(\gamma,\rho)] = [(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}\gamma, s(\mathfrak{n})\rho)].$$

Then the following defining relations hold in the semigroup crossed product algebra:

$$\mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^* = e_{\mathfrak{n}};\ \mu_{\mathfrak{n}}^*\mu_{\mathfrak{n}} = 1;\ \rho_{\mathfrak{n}}(f) = \mu_{\mathfrak{n}} f \mu_{\mathfrak{n}}^*;$$

$$\sigma_{\mathfrak{n}}(f) = \mu_{\mathfrak{n}}^* f \mu_{\mathfrak{n}};\ \sigma_{\mathfrak{n}}(\rho_{\mathfrak{n}}(f)) = f;\ \rho_{\mathfrak{n}}(\sigma_{\mathfrak{n}}(f)) = f e_{\mathfrak{n}}.$$

Finally, the time evolution is given by

$$\begin{cases} \sigma_{\mathbb{K},t}(f) = f, & \forall f \in C(G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathscr{O}}_{\mathbb{K}}^*} \hat{\mathscr{O}}_{\mathbb{K}}); \\ \sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it}\,\mu_{\mathfrak{n}}, & \forall \mathfrak{n} \in J_{\mathbb{K}}^+. \end{cases} \tag{3}$$

where $\mu_{\mathfrak{n}}$ are the isometries that implement the semigroup action of $J_{\mathbb{K}}^+$.

3 Hilbert space representation, partition function, KMS-states

3.1. Let us abbreviate $\mathrm{KMS}_\beta(\mathbb{K}) := \mathrm{KMS}_\beta(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$. A complete classification of the KMS states for the systems $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ was obtained in [16], Thm. 2.1. In particular, in the low temperature range $\beta > 1$, the extremal KMS_β states are parameterized by elements $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$, and are in Gibbs form, given by

$$\omega_{\beta,\gamma}(f) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} L_{\mathbb{K}}(\gamma, f, \beta),\ \text{where } L_{\mathbb{K}}(\gamma, f, \beta) := \sum_{\mathfrak{n} \in J_{\mathbb{K}}^+} \frac{f(\mathfrak{n} * \gamma)}{N(\mathfrak{n})^\beta} \tag{4}$$

is a *generalized L-series* associated to $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ and $f \in A_{\mathbb{K}}$.

3.2. Associated to any element $\gamma \in G_{\mathbb{K}}^{\mathrm{ab}}$ is a natural representation π_γ of the algebra $A_{\mathbb{K}}$ on the Hilbert space $\ell^2(J_{\mathbb{K}}^+)$. Namely, let $\varepsilon_{\mathfrak{m}}$ denote the canonical basis of $\ell^2(J_{\mathbb{K}}^+)$. Then the action on $\ell^2(J_{\mathbb{K}}^+)$ of an element $f_{\mathfrak{n}}\mu_{\mathfrak{n}} \in A_{\mathbb{K}}$ with $\mathfrak{n} \in J_{\mathbb{K}}^+$ and $f_{\mathfrak{n}} \in C(X_{\mathbb{K}})$ is given by

$$\pi_\gamma(f_{\mathfrak{n}}\mu_{\mathfrak{n}})\,\varepsilon_{\mathfrak{m}} = f_{\mathfrak{n}}(\mathfrak{n}\mathfrak{m} * \gamma)\,\varepsilon_{\mathfrak{n}\mathfrak{m}}.$$

In this picture, the time evolution is implemented (in the sense of formula (1)) by a Hamiltonian

$$H_{\sigma_{\mathbb{K}}}\varepsilon_{\mathfrak{n}} = \log N(\mathfrak{n})\,\varepsilon_{\mathfrak{n}}. \tag{5}$$

3.3. In this representation,

$$\mathrm{trace}(\pi_\gamma(f) e^{-\beta H_{\sigma_{\mathbb{K}}}}) = \sum_{\mathfrak{n} \in J_{\mathbb{K}}^+} \frac{f(\mathfrak{n} * \gamma)}{N(\mathfrak{n})^\beta}.$$

Setting $f = 1$, the Dedekind zeta function

$$\zeta_{\mathbb{K}}(\beta) = \sum_{\mathfrak{n} \in J_{\mathbb{K}}^+} N(\mathfrak{n})^{-\beta}$$

appears as the partition function

$$\zeta_{\mathbb{K}}(\beta) = \mathrm{trace}(e^{-\beta H_{\sigma_{\mathbb{K}}}})$$

of the system (convergent for $\beta > 1$).

4 Hamiltonians and arithmetic equivalence

We show that the existence of an isomorphism of the quantum statistical mechanical systems implies arithmetic equivalence; this is basically because the zeta functions of $\mathbb{K}$ and $\mathbb{L}$ are the partition functions of the respective systems. Some care has to be taken since the systems are not represented on the same Hilbert space.

4.1 Theorem. *Let $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ be an isomorphism of QSM-systems of number fields $\mathbb{K}$ and $\mathbb{L}$. Then $\mathbb{K}$ and $\mathbb{L}$ are arithmetically equivalent, i.e., they have the same Dedekind zeta function.*

Proof. The isomorphism $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ induces an identification of the sets of extremal KMS-states of the two systems, via pullback $\varphi^* : \mathrm{KMS}_\beta(\mathbb{L}) \to \mathrm{KMS}_\beta(\mathbb{K})$.

Consider the GNS representations associated to regular low temperature KMS states $\omega = \omega_\beta$ and $\varphi^*(\omega)$. We denote the respective Hilbert spaces by $\mathscr{H}_\omega$ and $\mathscr{H}_{\varphi^*\omega}$. As in Lemma 4.3 of [5], we observe that the factor $\mathscr{M}_\omega$ obtained as the weak closure of $A_{\mathbb{L}}$ in the GNS representation is of type I_∞, since we are only considering the low temperature KMS states that are of Gibbs form. Thus, the space $\mathscr{H}_\omega$ decomposes as

$$\mathscr{H}_\omega = \mathscr{H}(\omega) \otimes \mathscr{H}',$$

with an irreducible representation π_ω of $A_{\mathbb{L}}$ on $\mathscr{H}(\omega)$ and

$$\mathscr{M}_\omega = \{T \otimes 1 \,|\, T \in \mathscr{B}(\mathscr{H}(\omega))\}$$

($\mathscr{B}$ indicates the set of bounded operators). Moreover, we have

$$\langle (T \otimes 1) 1_\omega, 1_\omega \rangle = \mathrm{Tr}(T\rho)$$

for a density matrix ρ (positive, of trace class, of unit trace).

We know that the low temperature extremal KMS states for the system $(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ are of Gibbs form and given by the explicit expression in equation (4) for some $\gamma \in G_{\mathbb{L}}^{\mathrm{ab}}$; and similarly for the system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$. Thus, we can identify $\mathscr{H}(\omega)$ with $\ell^2(J_{\mathbb{L}}^+)$ and the density ρ correspondingly with

$$\rho = e^{-\beta H_{\sigma_{\mathbb{L}}}} / \mathrm{Tr}(e^{-\beta H_{\sigma_{\mathbb{L}}}});$$

this is the representation considered in Section 3.2. As in Lemma 4.3 of [5], the evolution group e^{itH_ω} generated by the Hamiltonian H_ω that implements the time evolution $\sigma_{\mathbb{L}}$ in the GNS representation on $\mathscr{H}_\omega$ agrees with $e^{itH_{\sigma_{\mathbb{L}}}}$ on the factor $\mathscr{M}_\omega$. We find

$$e^{itH_\omega}\pi_\omega(f)e^{-itH_\omega} = \pi_\omega(\sigma_{\mathbb{L}}(f)) = e^{itH_{\sigma_{\mathbb{L}}}}\pi_\omega(f)e^{-itH_{\sigma_{\mathbb{L}}}}.$$

As observed in §4.2 of [5], this gives us that the Hamiltonians differ by a constant:

$$H_\omega = H_{\sigma_{\mathbb{L}}} + \log\lambda_1 \text{ for some } \lambda_1 \in \mathbb{R}_+^*. \tag{6}$$

The argument for the GNS representation for $\pi_{\varphi^*(\omega)}$ is similar and it gives an identification of the Hamiltonians

$$H_{\varphi^*(\omega)} = H_{\sigma_{\mathbb{K}}} + \log\lambda_2 \text{ for some } \lambda_2 \in \mathbb{R}_+^*. \tag{7}$$

The algebra isomorphism φ induces a unitary equivalence Φ of the Hilbert spaces of the GNS representations of the corresponding states, and the Hamiltonians that implement the time evolution in these representations are therefore related by

$$H_{\varphi^*(\omega)} = \Phi H_\omega \Phi^*. \tag{8}$$

In particular the Hamiltonians $H_{\varphi^*(\omega)}$ and H_ω then have the same spectrum.

By combining (6), (7) and (8), we find that

$$H_{\sigma_{\mathbb{K}}} = \Phi H_{\sigma_{\mathbb{L}}}\Phi^* + \log\lambda$$

for a unitary operator Φ and a $\lambda \in \mathbb{R}_+^*$. This gives at the level of zeta functions

$$\zeta_{\mathbb{L}}(\beta) = \lambda^{-\beta}\zeta_{\mathbb{K}}(\beta) \tag{9}$$

for sufficiently large real β, hence for all β by analytic continuation. Now consider the left hand side and right hand side as classical Dirichlet series of the form

$$\sum_{n\geq 1}\frac{a_n}{n^\beta} \quad \text{and} \quad \sum_{n\geq 1}\frac{b_n}{(\lambda n)^\beta},$$

respectively. Observe that $a_1 = b_1 = 1$. Taking the limit as $\beta \to +\infty$ in (9), we find

$$a_1 = \lim_{\beta\to+\infty} b_1\lambda^{-\beta},$$

from which we conclude that $\lambda = 1$. Thus, we obtain $\zeta_{\mathbb{K}}(\beta) = \zeta_{\mathbb{L}}(\beta)$, which gives arithmetic equivalence of the number fields. □

5 Matching of generalized *L*-series

Since the zeta functions are equal, the matching of extremal KMS$_\beta$ states as in 3.1 implies a matching of generalized *L*-series, as follows:

5.1 Corollary. *Let $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \xrightarrow{\sim} (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ be an isomorphism of QSM-systems of number fields $\mathbb{K}$ and $\mathbb{L}$. There exists a homeomorphism $\psi\colon G_{\mathbb{L}}^{\mathrm{ab}} \xrightarrow{\sim} G_{\mathbb{K}}^{\mathrm{ab}}$ such that we have an identification of generalized L-series*

$$L_{\mathbb{L}}(\gamma, f, \beta) = L_{\mathbb{K}}(\psi(\gamma), \varphi^{-1}(f), \beta)$$

for all $f \in A_{\mathbb{L}}$ and all $\gamma \in G_{\mathbb{L}}^{\mathrm{ab}}$. □

References

1. Bauer, M.: Über einen Satz von Kronecker. Arch. der Math. u. Phys. (3) **6** 218–219 (1903)
2. Bauer, M.: Über zusammengesetzte Körper. Arch. der Math. u. Phys. (3) **6** 221–222 (1903)
3. Bost, J.-B., Connes, A.: Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. Selecta Math. (N.S.) **1** 411–457 (1995)
4. Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics 2, second ed. Texts and Monographs in Physics, Springer-Verlag, Berlin (1997)
5. Connes, A., Consani, C., Marcolli, M.: Noncommutative geometry and motives: the thermodynamics of endomotives. Adv. Math. **214** 761–831 (2007)
6. Connes, A., Marcolli, M.: From physics to number theory via noncommutative geometry, I: Quantum statistical mechanics of $\mathbb{Q}$-lattices. In: Cartier, P.E., Julia, B., Moussa, P., Vanhove, P. (eds.) Frontiers in number theory, physics, and geometry I, pp. 269–347. Springer, Berlin (2006)
7. Connes, A., Marcolli, M.: Noncommutative geometry, quantum fields and motives. American Mathematical Society Colloquium Publications, vol. 55, American Mathematical Society, Providence, RI (2008)
8. Cornelissen, G.: Curves, dynamical systems, and weighted point counting. Proc. Natl. Acad. Sci. USA **110** (no. 24) 9669–9673 (2013)
9. Cornelissen, G., Marcolli, M.: Quantum statistical mechanics, *L*-series and anabelian geometry. Preprint, arXiv:1009.0736 (2010)
10. de Smit, B., Perlis, R.: Zeta functions do not determine class numbers. Bull. Amer. Math. Soc. (N.S.) **31** 213–215 (1994)
11. Gaßmann, F.: Bemerkungen zur Vorstehenden Arbeit von Hurwitz: Über Beziehungen zwischen den Primidealen eines algebraischen Körpers und den Substitutionen seiner Gruppen.. Math. Z. **25** 661(665)–675 (1926)
12. Ha, E., Paugam, F.: Bost-Connes-Marcolli systems for Shimura varieties. I. Definitions and formal analytic properties. IMRP Int. Math. Res. Pap. **5** 237–286 (2005)
13. Haag, R., Hugenholtz, N. M., Winnink, M.: On the equilibrium states in quantum statistical mechanics. Comm. Math. Phys. **5** 215–236 (1967).
14. Klingen, N.: Arithmetical similarities. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York (1998)
15. Komatsu, K.: On the adele rings of algebraic number fields. Kōdai Math. Sem. Rep. **28** 78–84 (1976)
16. Laca, M., Larsen N.S., Neshveyev, S.: On Bost-Connes types systems for number fields. J. Number Theory **129** 325–338 (2009)
17. Neukirch, J.: Kennzeichnung der *p*-adischen und der endlichen algebraischen Zahlkörper. Invent. Math. **6** 296–314 (1969)
18. Neukirch, J.: Algebraic number theory. Grundlehren der Mathematischen Wissenschaften, vol. 322, Springer-Verlag, Berlin (1999)
19. Perlis, R.: On the equation $\zeta_K(s) = \zeta_{K'}(s)$. J. Number Theory **9** 342–360 (1977)
20. Uchida, K.: Isomorphisms of Galois groups. J. Math. Soc. Japan **28** 617–620 (1976)