Extra exercises Analysis in several variables

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Exercise 1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subsets and let $\pi : U \to V$ be a C^k map which is a surjective submersion. By a local C^k section of π we mean a C^k-map $s : V_0 \to U$ with V_0 open in U, such that $\pi \circ s = \mathrm{id}_{V_0}$.

(a) Let $b \in V$ and $a \in U$ be such that $\pi(a) = b$. Show that there exists an open neighborhood V_0 of b in V and a local C^k section $s : V_0 \to U$ such that $s(b) = a$.

(b) Let $f : V \to \mathbb{R}^q$ be a map. Show that f is C^k if and only if $f \circ \pi$ is C^k.

Exercise 2. Let $A \in \mathrm{M}_n(\mathbb{R})$ be a symmetric matrix, i.e., $A_{ij} = A_{ji}$ for all $1 \leq i, j \leq n$.

(a) Show that

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

for all $x, y \in \mathbb{R}^n$.

(b) Consider the C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \langle Ax, x\rangle$. Determine the total derivative $Df(x) \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$, for every $x \in \mathbb{R}^n$. Determine the gradient $\mathrm{grad} f(x) \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$.

(c) Consider the unit sphere $S = S^{n-1}$ in $\mathbb{R}^n$ given by the equation $g(x) = 0$, where $g : \mathbb{R}^n \to \mathbb{R}$, $\|x\|^2 - 1$. Show that $f|_S$ attains a maximal value M at a suitable $x^0 \in S$.

(d) By using the multiplier method, show that $Ax^0 = Mx^0$. Thus, M is an eigenvalue for A.

(e) Formulate and prove a similar result with the minimal value m of $f|_S$.

(f) Show that all eigenvalues of A are contained in $[m, M]$.

Exercise 3.

(a) Let $\mathrm{S}_n(\mathbb{R})$ denote the set of symmetric $n \times n$-matrices. Show that this set is a linear subspace of $\mathrm{M}_n(\mathbb{R})$ which is linearly isomorphic to $\mathbb{R}^{n(n+1)/2}$.

(b) Let $\mathrm{O}(n)$ be the set of matrices $A \in M_n(\mathbb{R})$ such that $A^{\mathrm{T}}A = I$. We consider the map $g : \mathrm{M}_n(\mathbb{R}) \to \mathrm{S}_n(\mathbb{R}), X \mapsto X^{\mathrm{T}}X - I$. Show that the total derivative of g at $A \in \mathrm{M}_n(\mathbb{R})$ equals the linear map given by

$$Dg(A) : \mathrm{M}_n(\mathbb{R}) \to \mathrm{S}_n(\mathbb{R}), \quad H \mapsto A^{\mathrm{T}}H + H^{\mathrm{T}}A.$$

Hint: use the definition.

(c) Show that g is a submersion at $I \in \mathrm{O}(n)$.

(d) Show that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$ at the point I. Determine the tangent space $T_I\mathrm{O}_n(\mathbb{R})$.

(e) Show that for $B \in \mathrm{O}(n)$ the map $L_B : \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ given by $L_B(X) = BX$ is a linear automorphism of $\mathrm{M}_n(\mathbb{R})$, which preserves $\mathrm{O}(n)$.

(f) Show that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$. Determine the dimension of this submanifold. Determine the tangent space $T_A\mathrm{O}(n)$ for every $A \in \mathrm{O}(n)$.

Exercise 4.

(a) Show that the following result is a particular case of [DK2, Thm. 6.4.5]: Let $B = \prod_{j=1}^n [a_j, b_j]$ be a block in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ a continuous function. Then

$$\int_B f(x)\, dx = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \ldots dx_1.$$

(b) Write $B = [a_1, b_1] \times C$ with C a rectangle in $\mathbb{R}^{n-1}$. Inspect the proof of [DK2, Thm. 6.4.5] and show that the function

$$F : x_1 \mapsto \int_C f(x_1, y)\, dy$$

is continuous $[a_1, b_1] \to \mathbb{R}$.

Exercise 5. For the purpose of this exercise, by a semi-rectangle in $\mathbb{R}^n$ we shall mean a subset R for which there exists an n-dimensional rectangle $B \subset \mathbb{R}^n$ such that $\mathrm{int}(B) \subset R \subset B$.

(a) Argue that a semi-rectangle R is Jordan measurable, with volume given by

$$\mathrm{vol}_n(R) = \mathrm{vol}_n(\bar{R}).$$

(b) Let $B \subset \mathbb{R}^n$ be a rectangle and $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ a partition of B. Show that there exist semi-rectangles $R_1, \ldots, R_k$ with $R_j \subset B_j$ and

$$\sum_{j=1}^k 1_{R_j} = 1_B.$$

By a step function on $\mathbb{R}^n$ we mean a finite linear combination of functions of the form 1_R, with R a semi-rectangle in $\mathbb{R}^n$. The linear space of these step functions is denoted by $\Sigma(\mathbb{R}^n)$.

(c) Let $f : B \to \mathbb{R}$ be a bounded function. Show that there exist step functions $s_\pm$ with

$$s_- \leq f \leq s_+, \qquad \overline{S}(f, \mathcal{B}) = \int s_-(x)\, dx, \qquad \underline{S}(f, \mathcal{B}) = \int s_+(x)\, dx.$$

(d) We denote by $\Sigma_+(f)$ the set of step functions $s : \mathbb{R}^n \to \mathbb{R}$ with $f \leq s$. Show that

$$\overline{\int_B} f(x)\, dx = \inf_{s \in \Sigma_+(f)} \int s(x)\, dx.$$

Give a similar characterisation of the lower integral of f over B.

Exercise 6. We define $\Sigma(\mathbb{R}^n)$ as above.

(a) Let B be a rectangle, and $S \subset \partial B$. Show that 1_S is a step function.

(b) Let $\mathcal{B}$ be a partition of B. Let $s : \mathbb{R}^n \to \mathbb{R}$ be a function such that

(1) $s(\mathbb{R}^n)$ has finitely many values;

(2) for all $B' \in \mathcal{B}$, s is constant on $\mathrm{int}(B')$;

(3) $s = 0$ outside B.

(c) Let $s \in \Sigma(\mathbb{R}^n)$ vanish outside the rectangle B. Show that there exists a partition $\mathcal{B}$ of B such that the above condition (2) is fulifilled.

(d) If s, t are step functions, show that both $\min(s, t)$ and $\max(s, t)$ are step functions.

Exercise 7. Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathcal{J}(U)$ the collection of compact subsets of U which are Jordan measurable.

(a) If $f : U \to \mathbb{R}$ is absolutely Riemann integrable, show that there exists a unique real number $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a $K_0 \in \mathcal{J}(U)$ such that for all $K \in \mathcal{J}(U)$ with $K \supset K_0$ we have

$$\left| \int_K f(x)\, dx - I \right| < \epsilon.$$

(b) Show that $I = \int_U f(x)\, dx$.

Exercise 8.

(a) Determine the collection of all $s \in \mathbb{R}$ for which the integral

$$\int_{\mathbb{R}^2} (1 + \|x\|)^{-s} \, dx$$

is absolutely convergent. Hint: use polar coordinates. Prove the correctness of your answer.

(b) Answer the same question for

$$\int_{\mathbb{R}^3} (1 + \|x\|)^{-s} \, dx.$$

(c) Wat is your guess for the similar integral over $\mathbb{R}^n$, for $n \geq 4$. ? We will return to this question at a later stage.

Exercise 9. We consider the cone $C : x_2^2 + x_3^2 = mx_1^2$, with $m > 0$ a constant.

(a) Show that $M := C \setminus \{0\}$ is a C^∞ submanifold of dimension 2 of $\mathbb{R}^3$.

(b) Determine the area of the subset M_h of M given by

$$M_h = \{x \in C \mid 0 < x_1 < h\}, \qquad (h > 0).$$

Exercise 10. We consider a C^1-function $f : (a, b) \to (0, \infty)$. Consider the graph G of f in $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Let S be the surface arising from G by rotating G about the x_1-axis over all angles from $[0, 2\pi]$.

(a) Guess a formula for the area of S. Explain the heuristics.

(b) Show that S is a C^1-submanifold of $\mathbb{R}^3$.

(c) Prove that the conjectured formula is correct.

Exercise 11. Let $v_1, \ldots, v_{n-1}$ be $n - 1$ vectors in $\mathbb{R}^n$.

(a) Show that $\xi : v \mapsto \det(v, v_1, v_2, \ldots, v_{n-1})$ defines a map in $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$.

(b) Show that there exists a unique vector $v \in \mathbb{R}^n$ such that

$$\xi(u) = \langle u, v \rangle \qquad (\forall u \in \mathbb{R}^n).$$

The uniquely determined element v of (c) is called the exterior product of $v_1, \ldots, v_{n-1}$ and denoted by $v_1 \times \cdots v_{n-1}$.

(c) Show that the map $(v_1, \dots, v_{n-1}) \mapsto v_1 \times \cdots v_{n-1}$ is alternating multilinear $(\mathbb{R}^n)^{\times(n-1)} \to \mathbb{R}^n$. Furthermore, show that $v_1 \times \cdots \times v_{n-1} \perp v_j$ for every $j = 1, \dots, n-1$. Finally, show that $v_1 \times \cdots \times v_{n-1} = 0$ if and only if $v_1, \dots, v_{n-1}$ are linearly independent.

(d) Show that for $n = 3$, the above corresponds to the usual exterior product.

We consider an injective linear map $L : \mathbb{R}^{n-1} \to \mathbb{R}^n$. Let $V = L(\mathbb{R}^n)$ and let d_V be the Euclidean density on V. Let $\mathbf{n}$ be a unit vector in $V^\perp$ such that $\det(\mathbf{n} \mid L) > 0$.

(e) Show that for every $v \in \mathbb{R}^n$ we have

$$L^*(\langle v, \mathbf{n}\rangle\, d_V) = \det(v \mid L) \qquad v \in \mathbb{R}^n$$

Hint: first show this for $v = \mathbf{n}$.

(f) Show that

$$L^*(d_V) = \|Le_1 \times \cdots \times Le_n\| \cdot d_{\mathbb{R}^{n-1}}.$$

(g) Show that

$$\|Le_1 \times \cdots \times Le_n\| = \sqrt{\det(L^{\mathrm{T}}L)}.$$

Let now $U \subset \mathbb{R}^{n-1}$ be open and $\varphi : U \to \mathbb{R}^n$ an embedding onto a codimension 1 submanifold M of $\mathbb{R}^n$. Let $\mathbf{n} : M \to \mathbb{R}^n$ be defined by $\mathbf{n}(x) \perp T_x M$ and

$$\det(\mathbf{n}(\varphi(y)), D_1\varphi(y), \dots D_{n-1}\varphi(y)) > 0, \qquad (y \in U).$$

(h) Show that for every vector field $v : M \to \mathbb{R}^n$ we have

$$\varphi^*(\langle v, \mathbf{n}\rangle\; dx) = \det(v(\varphi(y)) \mid D\varphi(y))\; d_{\mathbb{R}^{n-1}}.$$