Exercise HW2

Let $U \subset \mathbb{R}^n$ be an open subset.

(a) If $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable, and attains a local minimum at a point $x^0 \in U$, show that $Dg(x^0) = 0$. Hint: consider partial derivatives.

We consider a differentiable map $\Phi : U \to \mathbb{R}^n$ such that $D\Phi(x)$ is invertible for every $x \in U$. Let $a \in U$ and put $b := \Phi(a)$.

(b) Show that there exists a constant $C > 0$ such that $\|D\Phi(a)v\| \geq C\|v\|$ for all $v \in \mathbb{R}^n$.

(c) Show that there exists a $\delta > 0$ such that $\bar{B}(a;\delta) \subset U$ and

$$\|\Phi(x) - b\| \geq \frac{2}{3} C \|x - a\|$$

for all $x \in \bar{B}(a;\delta)$ (the bar indicates that the closed ball is taken).

For $y \in B(b; \frac{1}{3} C\delta)$ we consider the function

$$f_y : \bar{B}(a;\delta) \to \mathbb{R},\ x \mapsto \|\Phi(x) - y\|^2.$$

(d) Show that the function f_y attains a minimum value $m(y)$ at a point $x(y) \in \bar{B}(a;\delta)$.

(e) Show that $\sqrt{m(y)} < \frac{1}{3} C\delta$ and that $x(y) \in B(a;\delta)$.

(f) Show that $\Phi(x(y)) = y$.

(g) By using the previous items, prove that $\Phi(U)$ is open.