Exercise HW3

In this exercise we assume that $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are C^k submanifolds of dimensions d and e, respectively. In addition, we assume that $f : M \to N$ is a C^k map, that $a \in M$ and $b = f(a) \in N$. Here we do not assume that f is defined on an neighborhood of M in $\mathbb{R}^m$.

(a) Let I be an open interval in $\mathbb{R}$ and $\gamma : I \to M$ a C^1 curve (just differentiable would be enough). Show that $f \circ \gamma : I \to N$ is a C^1-curve.

(b) (U, κ) be a chart of M with $a \in U$. Assume $0 \in I$ and let $\gamma_1, \gamma_2 : I \to M$ be two C^1-curves such that $\gamma_j(0) = a$, for $j = 1, 2$. Show that $\kappa \circ \gamma_j$ define C^1 curves with domain a suitable open interval containing 0. Furthermore, show that

$$\gamma_1'(0) = \gamma_2'(0) \iff (\kappa \circ \gamma_1)'(0) = (\kappa \circ \gamma_2)'(0)$$

(c) Show that there exists a unique map $L : T_aM \to T_bN$ such that

$$L(\gamma'(0)) = (f \circ \gamma)'(0)$$

for every C^1-curve $\gamma : I \to M$ with $\gamma(0) = a$.

(d) Let (V, λ) be a chart of N with $b \in V$ and (U, κ) a chart of M with $a \in U$ and $f(U) \subset V$. Argue that the following diagram makes sense and commutes:

$$\begin{array}{ccc} T_aM & \xrightarrow{L} & T_bN \\ {\scriptstyle D(\kappa^{-1})(a')} \uparrow & & \uparrow {\scriptstyle D(\lambda^{-1})(b')} \\ \mathbb{R}^d & \xrightarrow{L'} & \mathbb{R}^e \end{array}$$

Here we have written $a' = \kappa(a)$, $b' = \lambda(b)$ and $L' = D(\lambda \circ f \circ \kappa^{-1})(a')$.

(e) Show that the map L in (c) is linear. We will denote it by T_af and call it the tangent map of f at a.

(f) Let $g : N \to R$ be a C^k map from N to a C^k submanifold R of $\mathbb{R}^r$. Show that $g \circ f : M \to R$ is C^k and that

$$T_a(g \circ f) = T_bg \circ T_af.$$

This formula is known as the chain rule for tangent maps.