Exercise HW1

(a) Let $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^N$ be a bilinear map, i.e., $\beta(\,\cdot\,, v) \in \mathrm{Lin}(\mathbb{R}^p, \mathbb{R}^N)$ and $\beta(u, \,\cdot\,) \in \mathrm{Lin}(\mathbb{R}^q, \mathbb{R}^N)$ for all $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Show that there exists a constant $C > 0$ such that

$$\|\beta(u,v)\| \leq C\|u\|\|v\|$$

for all $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$.

(b) Let $\mu : \mathrm{Mat}(n,\mathbb{R}) \times \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ be a bilinear map, let $U \subset \mathrm{Mat}(n,\mathbb{R})$ be open, and let $A \in U$. Let $f : U \to \mathrm{Mat}(n,\mathbb{R})$ be a map which is (totally) differentiable at A. Show that the map

$$M : U \to \mathrm{Mat}(n,\mathbb{R}),\ X \mapsto \mu(f(X), X)$$

is (totally) differentiable at A, with derivative given by

$$DM(A)(H) = \mu(Df(A)(H), A) + \mu(f(A), H), \quad (H \in \mathrm{Mat}(n,\mathbb{R})).$$

Hint: consider $M(A+H) - M(A)$, use the definition of (total) differentiability and apply (a) to obtain certain necessary estimates.

We consider $\mathrm{GL}(n,\mathbb{R}) := \{X \in \mathrm{Mat}(n,\mathbb{R}) \mid \det(X) \neq 0\}$. This is set open in $\mathrm{Mat}(n,\mathbb{R})$ since it is the preimage under the continuous function $\det : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ of the open subset $\mathbb{R} \setminus \{0\}$ of $\mathbb{R}$.

(c) Show that the map $F : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}),\ X \mapsto X^{-1}$ is C^1.

(d) For each $A \in \mathrm{GL}(n,\mathbb{R})$ show that the map F is (totally) differentiable at A, with derivative $DF(A) : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ given by

$$DF(A)H = -A^{-1}HA^{-1}.$$

Hint: use the equality $F(X)X = I$ and apply (b).