Exercise HW5

Let $n \geq 1$. In this exercise we will investigate absolute integrability of the function

$$f_s : \mathbb{R}^n \to \mathbb{R}, \ \ x \mapsto (1 + \|x\|)^s,$$

for $s \in \mathbb{R}$.

(a) For $r > 0$ show that the map $\Psi : x \mapsto rx$ is a diffeomorphism from $\mathbb{R}^n$ onto itself. Use substitution of variables to show that for any compact Jordan measurable set $K \subset \mathbb{R}^n$ we have

$$\mathrm{vol}_n(rK) = r^n \mathrm{vol}_n(K).$$

In particular, this implies that $\mathrm{vol}_n(\overline{B}(0;r)) = r^n V_n$, with $V_n = \mathrm{vol}_n(\overline{B}(0;1))$.

(b) If $s \geq 0$, show that the function f_s is not absolutely Riemann integrable over $\mathbb{R}^n$.

From now on, we assume that $s < 0$. For every integer $k \geq 0$ we define the set $S(k) := \bar{B}(0;k+1) \setminus B(0;k)$.

(c) Show that

$$\int_{S(k)} (1 + \|x\|)^s \, dx \leq V_n (1+k)^s [(1+k)^n - k^n].$$

(d) Show that, for $k \geq 1$,

$$\int_{S(k)} (1 + \|x\|)^s \, dx \leq n V_n \int_k^{k+1} t^s t^{n-1} \, dt.$$

(e) Show that for $s < -n$ the function f_s is absolutely integrable over $\mathbb{R}^n$.

Conversely, by a similar method it can be shown that for $s \geq -n$ the function f_s is not absolutely integrable over $\mathbb{R}^n$. We do not ask you to prove this, but if you wish, you are welcome to do so.