Exercises Geometric Analysis

Exercise 1

We assume that (M, g) is an oriented Riemannian manifold, and that Ω is the associated positively oriented volume form on M. Thus, for $x \in M$ and $v_1, \ldots, v_m$ a positively oriented orthonormal basis of T_xM we have $\Omega_x(v_1, \ldots, v_m) = 1$. We define the bilinear pairing $\langle \cdot\,, \cdot \rangle :\ C^\infty(M) \times C_c^\infty(M) \to \mathbb{C}$ by

$$\langle f, g \rangle = \int_M f g \, \Omega$$

(a) Show that this pairing is non-degerate in the sense that $\langle f, g\rangle = 0$ for all $g \in C_c^\infty(M)$ implies $f = 0$.

Show that the Laplace-Beltrami operator is the unique second order differential operator Δ on $C^\infty(M)$ such that the following properties are fulfilled:

(b) The principal symbol is given by $\sigma^2_\Delta(x, \xi) = -g_x(\xi, \xi)$ for $x \in M$, $\xi \in T_xM$.

(c) For all $f, g \in C_c^\infty(M)$ we have $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$.

(d) The operator Δ annihilates the constant function 1_M.

Exercise 2 Let V be a linear space. Let $\mathcal{P}$ be a collection of seminorms on V. We equip V with the translation invariant topology generated by the neighborhoods

$$B_p(0; r) := \{x \in V \mid p(x) < r\}.$$

(a) Show that the following assertions are equivalent:

  (1) V is a Hausdorff space;

  (2) $\{0\}$ is closed;

  (3) if $x \in V$ and $p(x) = 0$ for all $p \in \mathcal{P}$, then $x = 0$.

(b) If $p_1, \ldots, p_n \in \mathcal{P}$ show that $p := \max_j p_j$ is a seminorm on V.

We assume that V is Hausdorff and define $\widetilde{\mathcal{P}}$ to be the collection of seminorms on V of the form $p = \max_{1 \le j \le k} p_j$, for $k \in \mathbb{Z}_+$ and $p_j \in \mathcal{P}$.

(c) Show that the sets $B_p(0; r)$ for $p \in \widetilde{\mathcal{P}}$ and $r > 0$ form a basis of neighborhoods of 0.

We note that $\widetilde{P}$ is a fundamental system of seminorms on $\mathcal{P}$, i.e.,

(i) If $x \in V$ and $p(x) = 0$ for all $p \in \widetilde{\mathcal{P}}$, then $x = 0$.

(ii) For all $p_1, p_2 \in \widetilde{\mathcal{P}}$ there exists $q \in \widetilde{\mathcal{P}}$ such that $p_1, p_2 \le q$.

(d) Let s be a seminorm on V. Show that the following conditions are equivalent:

(1) s is continuous;

(2) $B_s(0;1)$ is a neighborhood of 0 in V;

(3) there exists a $p \in \widetilde{\mathcal{P}}$ and a constant $C > 0$ such that $s \leq Cp$.

We denote by $\mathcal{P}_V$ the collection of all continuous seminorms on V.

(e) Let $T : V \to W$ be a linear map of locally convex spaces. Show that the following conditions are equivalent:

(1) T is continuous

(2) for every $q \in \mathcal{P}_W$ there exists a $p \in \mathcal{P}_V$ such that $q \circ T \leq p$

the latter estimate should be read as: $\forall x \in V : \quad q(T(x)) \leq p(x)$.

Exercise 3 The purpose of this exercise is to have a better understanding of the locally convex topology on $C_c^\infty(\mathbb{R}^n)$.

(a) Show that there exists a sequence K_j of compact subsets of $\mathbb{R}^n$ such that the collection $\{K_j \mid j \in \mathbb{N}\}$ is locally finite and has union $\mathbb{R}^n$.

(b) For each sequence $c = (c_j)$ in $]\,0, \infty\,[$ show that

$$s_c : f \mapsto \sup_{j \geq 0} \,(c_j \cdot \sup_{K_j} |f|)$$

defines a seminorm on $C_c(\mathbb{R}^n)$. The collection of these seminorms is denoted by $\mathcal{S}$.

(c) Show that the following assertions are equivalent for a sequence (f_j) in $C_c(\mathbb{R}^n)$.

(1) there exists a compact subset $K \subset \mathbb{R}^n$ such that $\operatorname{supp} f_j \subset K$ for all j and f_j converges uniformly to zero.

(2) $s(f_j) \to 0$ for every seminorm $s \in \mathcal{S}$.

(d) Let $u : C_c(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional. Show that the following assertions are equivalent:

(1) u is continuous relative to (the topology defined by) $\mathcal{S}$.

(2) for every compact subset $K \subset \mathbb{R}^n$ the restriction of u to $C_K(\mathbb{R}^n)$ (equipped with the sup-norm) is continuous.

(e) Define a collection $\mathcal{P}$ of seminorms on $C_c^\infty(\mathbb{R}^n)$ such that the following conditions are equivalent for every linear functional u on $C_c^\infty(\mathbb{R}^n)$.

(1) u is continuous relative to (the topology defined by) $\mathcal{P}$.

(2) for every compact subset $K \subset \mathbb{R}^n$ the restriction $u|_{C_K^\infty(\mathbb{R}^n)}$ is continuous.