Inleiding Topologie (January 24, 2016)

Note: All the questions are worth 0.5 points (adding up to a total of 11 points). Please motivate all your answers. For instance, in Exercise 1, point c) do not just answer with "yes" or "no", but provide a proof. Also, in d) of the same exercise, please do not just write down the final result, but also explain how you found it. The most difficult questions are 4-d) and 3-c).

Exercise 1. For $\mathbb{R}$ we consider the family $\mathcal{S}$ of subsets consisting of all the intervals of type $(a, b]$ with $a < b \leq 0$ and all intervals of type $[c, d)$ with $0 \leq c < d$. Let also $\mathcal{B}$ be the family which is $\mathcal{S}$ together with the subset $\{0\}$ of $\mathbb{R}$. We endow $\mathbb{R}$ with the smallest topology containing $\mathcal{S}$- denote it by $\mathcal{T}$.

a) Show that $\mathcal{S}$ is not a topology basis and that $\mathcal{B}$ is the smallest topology basis containing $\mathcal{S}$.

b) Show that $(\mathbb{R}, \mathcal{T})$ is Hausdorff.

c) Is the sequence $x_n = \frac{1}{n}$ convergent in $(\mathbb{R}, \mathcal{T})$?

d) Compute the closure and the interior of

$$A = (-3, -2] \cup (-1, 0) \cup (0, 1) \cup (2, 3)$$

in the topological space $(\mathbb{R}, \mathcal{T})$

e) Show that any open in the Euclidean topology $\mathcal{T}_{Eucl}$ is open also in $(\mathbb{R}, \mathcal{T})$.

f) Show that $[0, 1]$, with the topology induced from $(\mathbb{R}, \mathcal{T})$, is not compact.

g) Show that $(\mathbb{R}, \mathcal{T})$ is not connected.

h) Describe all the connected subsets of $(\mathbb{R}, \mathcal{T})$.

i) Show that any continuous function $f : (\mathbb{R}, \mathcal{T}_{Eucl}) \to (\mathbb{R}, \mathcal{T})$ is constant.

Exercise 2. Consider

$$X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \text{ or } x = 0\},$$

$$Y = \{(x, y) \in \mathbb{R}^2 : xy = 0\},$$

both endowed with the Euclidean topology. Show that:

a) Y can be embedded in X,

b) X cannot be embedded in Y,

c) that the one-point compactifications of X and Y are homeomorphic.

In this exercise you do not have to provide explicit formulas for the embedding and the homeomorphism from a) and c), respectively. But please use pictures to describe those functions and please provide all the arguments to support your claims (e.g., if you draw a picture for the one-point compactification of X, please explain why the picture you draw is indeed the one-point compactification).

Exercise 3. Show that:

a) If X is a compact Hausdorff space and A is closed in X, then the space X/A obtained from X by collapsing A to a point is compact and Hausdorff.

b) If $U \subset \mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ (with the Euclidean topology), considering $X = \overline{U}$ (the closure of U in $\mathbb{R}^n$) and $A = \partial(U)$ (the boundary of U in $\mathbb{R}^n$) show:

   b1) X is compact and A is closed in X.

   b2) the one-point compactification of U is homeomorphic to X/A.

c) If $U \subset \mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ then its one-point compactification can be embedded in $\mathbb{R}^{n+1}$.

Exercise 4. Here we work over the field $\mathbb{R}$ of real numbers. Let A be an arbitrary algebra, assume that A is commutative (i.e. $a_0a_1 = a_1a_0$ for any $a_0, a_1 \in A$) and denote by $\mathbf{1_A}$ the unit element of A. Fix an element $\alpha \in A$. Consider

$$B = A \times A = \{(a_0, a_1) : a_o, a_1 \in A\}$$

with the operations

$$(a_0, a_1) {\color{red}+} (a_0^{'}, a_1^{'}) = (a_0 + a_0^{'}, a_1 + a_1^{'}),$$
$$\lambda {\color{red}\cdot} (a_0, a_1) = (\lambda a_0, \lambda a_1),$$
$$(a_0, a_1) {\color{red}\cdot} (a_0^{'}, a_1^{'}) = (a_0 a_0^{'} + \alpha a_1 a_1^{'}, a_0 a_1^{'} + a_0^{'} a_1).$$

a) Show that B, together with the previous operations (written in red), is an algebra (don't forget the unit element of B!).

b) Assume that $\chi : B \to \mathbb{R}$ is a character and consider

$$\overline{\chi} : A \to \mathbb{R}, \quad \overline{\chi}(a) = \chi(a, 0),$$
$$r := \chi(0, 1) \in \mathbb{R}.$$

Show that:

   b1) $\overline{\chi}$ is a character on A.

   b2) $\chi(a_0, a_1) = \overline{\chi}(a_0) + r \cdot \overline{\chi}(a_1)$ for any $(a_0, a_1) \in B$.

   b3) $\overline{\chi}(\alpha) = r^2$.

c) conversely, for any character $\overline{\chi}$ on A and any $r \in \mathbb{R}$ satisfying the equation from b3), show that χ defined by the formula from b2) is a character on B.

d) When $A := \mathbb{R}[X_0, X_1]$ is the algebra of polynomials in two variables and $\alpha := 1 - (X_0)^2 - (X_1)^2$, show that the topological spectrum of B is *homeomorphic* to S^2.