Exercise HW6

The purpose of this exercise is to give an application of partitions of unity which illustrates how to pass from local to global.

For $\Omega \subset \mathbb{R}^n$ open, we denote by $C^1(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ which are partially differentiable with continuous partial derivatives $\partial_j f : \Omega \to \mathbb{R}$, for $j = 1, \ldots, n$.

(a) Show that the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = (x-1)^2(x+1)^2$ for $|x| \leq 1$ and by $\varphi(x) = 0$ for $|x| > 1$ belongs to $C^1(\mathbb{R})$.

(b) Show that $C^1(\mathbb{R})$ is normal.

(c) Show that $C^1(\mathbb{R}^n)$ is normal.

(d) Let $\{\lambda_1, \ldots, \lambda_k\}$ be a subset of $[0,1]$ such that $\sum_{i=1}^k \lambda_i = 1$. Show that for every interval $J \subset \mathbb{R}$ and every subset $\{r_1, \ldots, r_k\} \subset J$ we have $\sum_{i=1}^k \lambda_i r_i \in J$.

(e) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and C a compact subset of $\mathbb{R}^n$. Show that for each $\varepsilon > 0$ there exists a finite cover $\mathscr{U} = \{U_0, U_1, \ldots, U_k\}$ of $\mathbb{R}^n$, with $U_0 = \mathbb{R}^n \setminus C$, and real numbers $s_1, \ldots, s_k$ such that

$$f(x) - \varepsilon < s_i < f(x) + \varepsilon$$

for each $1 \leq i \leq k$ and all $x \in U_i$.

(f) Show that for every $\varepsilon > 0$ there exists a C^1-function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon \qquad (\forall\, x \in C).$$