Extra notes for the course Inleiding Topology, 2017

Notes on quotients

Let X be a set. Then by a partition of X we mean a collection P of non-empty subsets of X (thus, $P \subset \mathscr{P}(X)$), such that

(a) for all $S,T \in P$, if $S \neq T$ then $S \cap T = \emptyset$;

(b) X equals the union $\cup P$ of the sets from P.

If R is a relation on X, then for given $x,y \in X$ we shall sometimes write xRy in place of $(x,y) \in R$. Then R is said to be an equivalence relation if for all $x,y,z \in X$,

(a) xRx (reflexivity);

(b) $xRy \Rightarrow yRx$ (symmetry);

(c) $xRy \wedge yRz \Rightarrow xRz$ (transitivity).

Let R be an equivalence relation on X. Then for $x \in X$ we define the equivalence class of x by

$$R(x) = \{y \in X \mid xRy\}.$$

It is readily seen that the equivalence classes form a partition of $R(x)$. This partition, the collection of equivalence classes, is denoted by X/R, and called the (abstract) quotient of X by R. The surjective map $\pi : X \mapsto X/R$, $x \mapsto R(x)$ is called the quotient map.

Conversely, if P is a partition of X, then the relation R_P defined by

$$(x,y) \in R_P \iff (\exists S \in P): \ \{x,y\} \subset S,$$

is an equivalence relation. Its classes are precisely the elements of P. Thus, $P = X/R_P$.

Quotients appear naturally in the context of surjective maps. Let $f : X \to Y$ be a map between sets. For $y \in Y$ we define

$$f^{-1}(y) := f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}.$$

This subset of X is called the fiber of y for the map f. Clearly, f is surjective if and only if all fibers are non-empty. From now on we assume $f : X \to Y$ to be surjective. Then it is readily seen that the relation R_f on X defined by

$$(x,y) \in R_f \iff f(x) = f(y)$$

is an equivalence relation. Its equivalence classes are precisely the fibers of f. Indeed, for $y \in Y$ and $x \in f^{-1}(y)$ we have $R(x) = f^{-1}(y)$. In the second lecture we discussed the following result and its proof.

Lemma 1. *Let $f : X \to Y$ be a surjective map of sets, and let $R = R_f$ be the associated equivalence relation on X defined as above. Then there exists a unique map $\bar{f} : X/R \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \bar{f} & \\ X/R & & \end{array}$$

The map $\bar{f}$ is bijective.

Remark. Commutativity of the above diagram by definition means that $f = \bar{f} \circ \pi$. In general, commutativity of a diagram of maps means that all compositions of arrows are equal as soon as they have the same domain and target.

Proof. Let $\bar{f}$ be any map $X/R \to Y$ such that the diagram commutes. Let $\xi \in X/R$ and $x \in \xi$. Then $\pi(x) = \xi$, hence $\bar{f}(\xi) = \bar{f}(\pi(x)) = \bar{f} \circ \pi(x) = f(x)$. This shows that there is only one choice for the values of $\bar{f}$. Hence the map is uniquely determined.

We will now prove existence of $\bar{f}$. Let $\xi \in X/R$. Then ξ is an equivalence class for R. If $x, y \in \xi$ then xRy hence $f(x) = f(y)$ by definition of $R = R_f$. It follows that f has a common value on the equivalence class ξ. We define $\bar{f}(\xi) \in Y$ to be this common value. Now, for every $x \in X$ we have $\bar{f}(\pi(x)) = \bar{f}(R(x)) = f(x)$. Hence $\bar{f} \circ \pi = f$, so the diagram commutes for this $\bar{f}$. This establishes existence.

Finally, we will show that $\bar{f}$ is bijective. First, if $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Put $\xi = \pi(x)$. Then $\bar{f}(\xi) = \bar{f} \circ \pi(x) = f(x) = y$ and we see that $\bar{f}$ is surjective.

For injectivity, let $\xi_1, \xi_2 \in X$ and assume $\bar{f}(\xi_1) = \bar{f}(\xi_2)$. Select $x_1, x_2 \in X$ such that $\pi(x_j) = \xi_j$ for $j = 1, 2$. Then $\bar{f}(\xi_j) = f(x_j)$, so $f(x_1) = f(x_2)$. By definition of R it follows that $x_1 R x_2$ hence $R(x_1) = R(x_2)$, hence $\xi_1 = \pi(x_1) = R(x_1) = R(x_2) = \xi_2$. Injectivity follows. □

Quotient topology

Let $(X, \mathscr{T})$ be a topological space, R an equivalence relation on X and $\pi : X \to X/R$ the quotient map. We define

$$\mathscr{T}_{X/R} := \{V \subset X/R \ : \ \pi^{-1}(V) \in \mathscr{T}\}.$$

Claim: this set is a topology on $\mathscr{T}_{X/R}$.

Proof. First of all, $\pi^{-1}(X/R) = X$ and since $X \in \mathscr{T}$ we see that $X/R \in \mathscr{T}_{X/R}$. On the other hand, $\pi^{-1}(\emptyset) = \emptyset \in \mathscr{T}$ and we see that $\emptyset \in \mathscr{T}_{X/R}$. It follows that both $\emptyset$ and X/R belong to $\mathscr{T}_{X/R}$.

If $U, V \in \mathscr{T}_{X/R}$, then $\pi^{(}U)$ and $\pi^{-1}(V)$ belong to $\mathscr{T}$ so that also

$$\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) \in \mathscr{T}$$

and we find that $U \cap V \in \mathscr{T}_{X/R}$.

Finally, let $\{U_i\}_{i\in I}$ be any family of sets from $\mathscr{T}$. Then

$$\pi^{-1}(\cup_{i\in I} U_i) = \cup_{i\in I} \pi^{-1}(U_i)$$

is a union of the sets $\pi^{-1}(U_i) \in \mathscr{T}$, hence belongs to $\mathscr{T}$. Therefore, the union $\cup_{i\in I} U_i$ belongs to $\mathscr{T}_{X/R}$. □

It follows from the above that the quotient X/R of a topological space X by an equivalence relation R carries a natural topology, which we call the quotient topology. We note that the natural map $\pi : X \to X/R$ is continuous for $\mathscr{T}$ and $\mathscr{T}_{X/R}$. Furthermore, any topology $\mathscr{T}'$ on X/R for which π is continuous must be a subset of $\mathscr{T}_{X/R}$. Thus, the quotient topology $\mathscr{T}_{X/R}$ is the largest topology on X/R such that $\pi : X \to X/R$ is continuous relative to $\mathscr{T}$ and $\mathscr{T}_{X/R}$.

In the above we have used a few rules concerning preimages, intersections and unions. These rules, already proven in the first analysis course, are so important that we recall them explicitly. In the following we assume that $f : X \to Y$ is a map between sets. For a subset $A \subset Y$ the preimage of A under f is the subset of X defined by

$$f^{-1}(A) := \{x \in X \mid f(x) \in A\}.$$

We note that

(a) $f^{-1}(\emptyset) = \emptyset, \quad f^{-1}(Y) = X \quad \text{and} \quad f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$

Let $\{A_i\}_{i\in I}$ be a family of subsets A_i of Y, parametrized by an index set I. Then the following assertions are valid.

(a) $f^{-1}(\cap_{i\in I} A_i) = \cap_{i\in I} f^{-1}(A_i)$;

(b) $f^{-1}(\cup_{i\in I} A_i) = \cup_{i\in I} f^{-1}(A_i)$.

Phrased concisely, the map on power sets

$$f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X),\, A \mapsto f^{-1}(A)$$

preserves all complements, intersections and unions.