Extra exercises for the course Inleiding Topology, 2018

The following exercise provides background for Exercises 2.32 and 2.54.

Exercise E.2.1

The purpose of this exercise is to show that for $\mathbb{R}$ equipped the Euclidean topology has the following topological property (which is known as the connectedness of $\mathbb{R}$, see Section 4.1 of the lecture notes).

Let $A \subset \mathbb{R}$ be a non-empty open and closed subset. Then $A = \mathbb{R}$.

This result has been proven in Inleiding Analyse. The purpose of this exercise is to go through the proof again.

Let $A \subset \mathbb{R}$ be non-empty and both open and closed. Select $a \in A$. Consider the set $V := \{x > a \mid [a, x] \subset A\}$.

(a) Show that $V \neq \emptyset$.

(b) Show that V is not bounded from above. Hint: assuming that V is bounded from above, show that $\sup V \in A$ and derive a contradiction.

(c) Show that $A \supset [a, \infty)$.

(d) Show that $A = \mathbb{R}$.

Exercise E.3.1

In this exercise, we will show that every equivalence relation can be realized through the orbits of a group action. (This exercise has nothing to do with topology, but arose from a question by a student.)

Let X be a set, and R an equivalence relation of X. Let $P = X/R$ be the associated partition of X. X. We look at the group G of all bijections $X \to X$. The group operation if given by $fg = f \circ g$ for $f, g \in G$ and the neutral element is given by $e = \mathrm{id}_X$.

(a) Let G_0 be the subset of G consisting of all bijections $f : X \to X$ such that $f(C) = C$ for all $C \in X/R$. Show that G_0 is a subgroup of G.

(b) Let $x, y \in X$ belong to the same element $C \in X/R$. Show that there exists an $f \in G_0$ such that $f(x) = y$.

(c) Show that $X/R = X/G_0$.

(d) Show that for $x, y \in X$ we have $xRy \iff G_0x = G_0y$.

The following exercise is an extension of Exercise 5.11.

Exercise E.5.1

For $\Omega \subset \mathbb{R}^n$ open, we denote by $C^1(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ which are partially differentiable with continuous partial derivatives $\partial_j f : \Omega \to \mathbb{R}$, for $j = 1, \dots, n$. We note that $C^1(\Omega) \subset C(\Omega)$.

(a) Show that the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = (x-1)^2(x+1)^2$ for $|x| \leq 1$ and by $\varphi(x) = 0$ for $|x| > 1$ belongs to $C^1(\mathbb{R})$.

(b) Show that for every $a \in \mathbb{R}^n$ and any open neighborhood U of a in $\mathbb{R}^n$ there exists a function $g \in C^1(\mathbb{R}^n)$ with $g \geq 0$, $g(a) > 0$, and $\operatorname{supp} g \subset U$.

(c) Let $C \subset \mathbb{R}^n$ be closed and bounded, and let $U \subset \mathbb{R}^n$ be an open subset containing C. Show that there exists a function $\eta \in C^1(\mathbb{R}^n)$ such that $\eta \geq 0$, $\eta|_C > 0$ and $\operatorname{supp} \eta \subset U$. Hint: use compactness.

(d) Find a result in the lecture notes which guarantees that $C^1(X)$ is normal. In particular, in item (c) there exists a function η with the properties mentioned, and with $\eta = 1$ on C.

Exercise E.5.2

The purpose of this exercise is to give an application of partitions of unity which illustrates how to pass from local to global.

(a) Let $\{\lambda_1, \dots, \lambda_k\}$ be a subset of $[0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$. Show that for every interval $J \subset \mathbb{R}$ and every subset $\{r_1, \dots, r_k\} \subset J$ we have $\sum_{i=1}^k \lambda_i r_i \in J$.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and C a compact subset of $\mathbb{R}^n$. Show that for each $\varepsilon > 0$ there exists a finite cover $\mathcal{U} = \{U_0, U_1, \dots, U_k\}$ of $\mathbb{R}^n$, with $U_0 = \mathbb{R}^n \setminus C$, and real numbers $s_1, \dots, s_k$ such that

$$f(x) - \varepsilon < s_i < f(x) + \varepsilon$$

for each $1 \leq i \leq k$ and all $x \in U_i$.

(c) Show that for every $\varepsilon > 0$ there exists a C^1-function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon \qquad (\forall\, x \in C).$$

(d) Use paracompactness of $\mathbb{R}^n$ and the idea of the above argument to show that g can even be found such that $d_{\sup}(f, g) < \varepsilon$.

(e) Show that $C^\infty(\mathbb{R}^n)$ is dense in $C(\mathbb{R}^n)$ equipped with the topology of uniform convergence.

Exercise E.5.3

The purpose of this exercise is to show that $C^\infty(\mathbb{R}^n)$ is a normal collection in $C(\mathbb{R}^n)$.

Our basic tool is the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x) = e^{-1/x} \text{ for } x > 0, \quad \text{and } \psi(x) = 0 \text{ for } x \leq 0.$$

(a) Show that ψ is continuous.

It is an exercise of basic analysis to show that $\psi \in C^\infty(\mathbb{R})$. You may use this result without proof.

(b) Show that there exists a function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\varphi(x) = 0$ for $|x| \geq 1$.

(c) Show that for every $a \in \mathbb{R}^n$ and every open neighborhood U of a in $\mathbb{R}^n$ there exists a function $g \in C^\infty(\mathbb{R}^n)$ with $g \geq 0$, $g(a) > 0$ and $\operatorname{supp} g \subset U$.

(d) Show that $C^\infty(\mathbb{R}^n)$ is a normal collection in $C(\mathbb{R}^n)$.

Exercise E.5.4

Let X be a topological space. If $\{S_i \mid i \in I\}$ is a locally finite collection of subsets of X, show that

(a) $\{\overline{S}_i\}_{i \in I}$ is locally finite;

(b) the closure of $\cup_{i \in I} S_i$ is given by

$$\overline{\cup_{i \in I} S_i} = \cup_{I \in I} \overline{S_i}.$$

Exercise E.5.5

Let X be a second countable locally compact Hausdorff space.

(a) Suppose that $\{S_i\}_{i \in I}$ is a family of subsets of X, indexed by an index set I. Show that the following conditions are equivalent.

  (i) The collection $\{S_i\}_{i \in I}$ is locally finite.

  (ii) For every compact subset $C \subset X$ the collection $I_C := \{i \in I \mid S_i \cap C \neq \emptyset\}$ is finite.

(b) If $\{S_i\}_{i \in I}$ is locally finite, show that the collection of $i \in I$ with $S_i \neq \emptyset$ is at most countable.

(c) Let $\{\eta_i\}_{i \in I}$ be a partition of unity on X. Show that the collection of $i \in I$ with $\eta_i \neq 0$ is at most countable.

Exercise E.5.6

Let X be a topological space, and $\mathcal{A}$ a subset of $C(X)$ which contains the zero function and is closed under locally finite sums. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X and $\{\psi_\alpha\}_{\alpha\in\mathfrak{a}}$ an $\mathcal{A}$-partition of unity such that for every $\alpha \in \mathfrak{a}$ there exists a $g(\alpha) \in I$ such that $\operatorname{supp}\psi_\alpha \subset U_{g(\alpha)}$.

(a) For each $i \in I$ show that

$$\eta_i := \sum_{\alpha\in g^{-1}(i)} \psi_\alpha$$

is a well-defined function $X \to \mathbb{R}$ which belongs to $\mathcal{A}$.

(b) Show that for every $i \in I$ we have

$$\operatorname{supp}\eta_i \subset U_i.$$

Exercise E.5.7

Let X be a paracompact Hausdorff space, and let $\mathcal{A} \subset C(X)$ be a subset which is normal and closed under taking locally finite sums and quotients. In addition assume that $\mathcal{A}$ is closed under scalar multiplication by $\mathbb{R}$. Thus, $\mathcal{A}$ is a linear subspace of $C(X)$.

Show that for every $f \in C(X)$ and every $\varepsilon > 0$ there exists a function $\varphi \in \mathcal{A}$ such that $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in X$.

Hint: first show that there exists an open covering $\{U_i\}_{i\in I}$ such that for every $i \in I$ there exists $\lambda_i \in \mathbb{R}$ such that $|f(x) - \lambda_i| < \varepsilon$ for all $x \in U_i$.

Then show that there exists a locally finite collection $\{\eta_i\}_{i\in I}$ of functions from $\mathcal{A}$ such that

$$|f - \sum_i \lambda_i\eta_i| < \varepsilon$$

on X.

Exercise E.7.1

Let X be a set.

(a) Let $d_\circ$ be a metric on X with associated topology $\mathcal{T}_\circ$. Show that $d_{\circ\circ} = \min(1, d_\circ)$ is a metric on X. Show that the associated topology $\mathcal{T}_{\circ\circ}$ equals $\mathcal{T}_\circ$.

We now assume that for each $j \geq 1$ a metric $d_j : X \times X \to [0, \infty)$ is given. Let $\mathcal{T}_j$ be the associated topology.

(b) Define $d : X \times X \to [0, \infty)$

$$d(x, y) = \sup\{d_j(x, y) \mid j \geq 1\}.$$

Show that d is a metric on X. Show that the associated topology $\mathcal{T}$ contains $\mathcal{T}_j$ for every $j \geq 1$.

We now assume in addition that $d_j \geq 1/j$ on X; this may be easily arranged without changing topologies, by replacing d_j with $\min(1/j, d_j)$.

(c) Show that $\mathcal{T}$ is the smallest topology containing all $\mathcal{T}_j$, for $j \geq 1$.

(d) Show that the space $X = C(\mathbb{R})$ equipped with the topology $\mathcal{T}_{\mathrm{cpt}}$ of uniform convergence on compact sets is metrizable.