Retake Inleiding Topologie, WISB243 2018-04-16, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ in the usual way.
- You are free to write the solutions either in English, or in Dutch.

Succes !

11 pt total **Exercise 1.** Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ such that

(1) $[0,1) \in \mathscr{T}$;

(2) the map $V : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ is continuous with respect to $\mathscr{T}$;

(3) for each $c \in \mathbb{R}$ the map $T_c : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + c$ is continuous with respect to $\mathscr{T}$.

2 pt (a) Show that for every $U \in \mathscr{T}$ and $a \in \mathbb{R}$ the set $U + a = \{x + a \mid x \in U\}$ belongs to $\mathscr{T}$.

3 pt (b) Show that the collection $\mathscr{B} := \{[a, a + 2^{-n}) \mid a \in \mathbb{R},\ n \in \mathbb{N}\}$ is a subset of $\mathscr{T}$.

3 pt (c) Show that $\mathscr{B}$ is a topology basis.

3 pt (d) Show that the topology generated by $\mathscr{B}$ equals $\mathscr{T}$.

11 pt total **Exercise 2.** Let $J = [-1, 1]$, let $S := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1, -1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1, z_2, t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t-1), \quad (0 \le s,t \le 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \;\longrightarrow\; X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cilinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any motivation.)

11 pt total **Exercise 3.** We consider a (non-empty) topological space Y, a set Z and a map $f : Y \to Z$. The map f is said to be locally constant if for every $y \in Y$ there exists an open neighborhood U of y in Y such that the restriction $f|_U$ is constant. Equivalently, this means that $f(U)$ consists of a single element.

3 pt (a) Show that f is locally constant if and only if every fiber $f^{-1}(\{z\})$, for $z \in Z$, is open in Y.

3 pt (b) Show that $f : Y \to Z$ is locally constant if and only if f is continuous for the discrete topology on Z.

3 pt (c) If Y is connected and $f : Y \to Z$ locally constant, show that f is constant.

2 pt (d) If Y is not connected, show that there exists a set Z and a locally constant function $f : Y \to Z$ which is not constant.

11 pt total **Exercise 4.** Let X be a Hausdorff topological space and $f : X \to (0,\infty)$ a function such that that for every $a \in X$ there exists an open neighborhood U of a and a constant $m > 0$ such that $f(x) \ge m$ for all $x \in U$.

3 pt (a) If X is compact, show that there exists a constant $m_X > 0$ such that $f(x) \ge m_X$ for all $x \in X$.

We now assume that X is paracompact.

3 pt (b) Show that there exists a locally finite covering $\{V_i \mid i \in I\}$ of X such that for every $i \in I$ there exists a constant $m_i > 0$ such that $f(x) \ge m_i$ for all $x \in V_i$.

5 pt (c) Show that there exists a continuous function $\mu : X \to (0,\infty)$ such that $f(x) \ge \mu(x)$ for all $x \in X$.