Lie derivatives, tensors and forms

Erik van den Ban
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Linear maps and tensors

The purpose of these notes is to give conceptual proofs of a number of results on Lie derivatives of tensor fields and differential forms. We start with some remarks on the effect of linear maps on tensors. In what follows, $U, V, W$ will be finite dimensional real vector spaces.

A linear map $A: V \rightarrow W$ induces a linear map $A^*: \otimes^k V \rightarrow \otimes^k W$.

Indeed, let $a: V^k \rightarrow \otimes^k W$ be the multilinear map given by $a(v_1, \ldots, v_k) \mapsto Av_1 \otimes \cdots \otimes Av_k$.

Then $A^*$ is just the map $\tilde{a}$ from the universal property of the tensor product. In particular, on elementary tensors $A^*$ is given by

$$A^*(v_1 \otimes \cdots \otimes v_k) = Av_1 \otimes \cdots \otimes Av_k.$$

On the other hand, $A$ also induces the adjoint linear map $A^* : W^* \rightarrow V^*$, $\eta \mapsto \eta \circ A$. This map in turn induces a linear map

$$A^* = \otimes^k (A^*): \otimes^k W^* \rightarrow \otimes^k V^*.$$

Note that the assignment $A \mapsto A^*$ preserves the direction of arrows, whereas $A \mapsto A^*$ reverses directions. Therefore, it is in general impossible to define a natural induced map between the spaces of mixed tensors of type $(r, s)$, $V_{r,s}$ and $W_{r,s}$. Here we have used the notation of Warner’s book.

We recall that there exists a natural isomorphism

$$V_{0,s} = \otimes^s V^* \simeq M_s(V),$$

where $M_s(V)$ denotes the space of $k$-multilinear forms on $V$. Via this isomorphism, a tensor of the form $\xi_1 \otimes \cdots \otimes \xi_s$ corresponds with the multilinear form

$$(v_1, \ldots, v_s) \mapsto \xi_1(v_1) \cdots \xi_s(v_s).$$

From this we see that the induced map $A^* : V_{0,s} \rightarrow W_{s,0}$ corresponds with the map $M_s(W) \rightarrow M_s(V)$, $\mu \mapsto \mu \circ A^*$, which is the genuine pull-back of multi-linear forms by $A$.

As in Warner’s book, let $C(V)$ denote the (graded) tensor algebra $\oplus_{k \geq 0} V_{k,0}$, where $V_{0,0} := \mathbb{R}$. Then a linear map $A: V \rightarrow W$ induces a linear map $A_* : C(V) \rightarrow C(W)$ which is readily seen to be a homomorphism of graded algebras. Also, it is clear that $A_*$ maps $I(V)$ into $I(W)$, hence induces an algebra homomorphism $A_* : \wedge V \rightarrow \wedge W$.  

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Similarly, the adjoint $A^* : W^* \to V^*$ induces an algebra homomorphism

$$(A^*)_*: \wedge W^* \to \wedge V^*,$$

which we briefly denote by $A^*$. Again, this is a homomorphism of graded algebras.

We recall that we introduced a particular linear isomorphism

$$\wedge^k V^* \cong A_k(V).$$

Under this isomorphism, an element $\xi_1 \wedge \cdots \wedge \xi_k \in \wedge^k V$ corresponds with the alternating $k$-form given by

$$(v_1, \ldots, v_k) \mapsto \sum_{\sigma \in S_k} \text{sgn}(\sigma)\xi_{\sigma 1}(v_1) \cdots \xi_{\sigma k}(v_k) = \det(\xi_i(v_j)).$$

Identifying the elements of $\wedge^k V^*$ with alternating $k$-forms in this fashion, we see that $A^*$ becomes the ordinary pull-back map

$$A_k(W) \to A_k(V), \mu \mapsto \mu \circ A^k.$$

As $A_k(V) \subset M_k(V)$, we may use the above isomorphisms to identify $\wedge^k V^*$ with a subspace of $V_{0,k}$. From (1) we then see that

$$\xi_1 \wedge \cdots \wedge \xi_k = \sum_{\sigma \in S_k} \text{sgn}(\sigma)\xi_{\sigma 1} \otimes \cdots \otimes \xi_{\sigma k}.$$

Finally, if $A$ is a linear isomorphism of $V$ onto $W$, we do have an induced map

$$A_* := \otimes^r A \otimes \otimes^s A^{-1} : V_{r,s} \to W_{r,s},$$

In this situation we agree to also write

$$A^* := A_*^{-1} = \otimes^r A^{-1} \otimes \otimes^s A^* : W_{r,s} \to V_{r,s}.$$  

For later purposes, we need the notion of contraction of tensors. First of all, the natural pairing $V \times V^* \to \mathbb{R}$ corresponds to a linear map $V \otimes V^* \to \mathbb{R}$, called contraction, and denoted by $\mathcal{C}$. Note that

$$\mathcal{C}(v \otimes \xi) = \xi(v).$$

If $A : V \to W$ is a linear isomorphism, then

$$\mathcal{C}_V \circ A^* = \mathcal{C}_W.$$  

This is readily seen from

$$\mathcal{C}_V \circ A^*(w \otimes \eta) = \mathcal{C}_V(A^{-1}w \otimes A^* \eta) = A^* \eta(A^{-1}w) = \eta(AA^{-1}w) = \mathcal{C}_W(w \otimes \eta).$$

More generally, if $r, s \geq 1$ and $1 \leq i \leq r$ and $1 \leq j \leq s$, we may define a contraction $C_{i,j} : V_{r,s} \to V_{r-1,s-1}$ on the $i$-th contravariant slot and the $j$-th covariant slot. More precisely, $C_{i,j}$ is the linear map induced by the multi-linear map

$$(v_1, \ldots, v_r, \xi_1 \ldots \xi_s) \mapsto$$

...
\[ C(v_i \otimes \xi_j)v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_r \otimes \xi_1 \otimes \cdots \otimes \hat{\xi}_j \otimes \cdots \otimes \xi_s. \]

The above formula is now readily seen to generalize to
\[ A^* \circ C_{W,i,j} = C_{V,i,j} \circ A^*. \]

Note that \( A^* = I \) on \( V_{0,0} = \mathbb{R} \).

**Maps and tensor fields**

We will now describe the effect of maps on tensor fields. Let \( \varphi : M \to N \) be a smooth map between manifolds. Then for every \( m \in M \) we have an induced linear map \( d_m \varphi : T_m M \to T_{\varphi(m)} N \) which in turn induces an algebra homomorphism
\[ (d_m \varphi)^* : \wedge^s T_{\varphi(m)}^* N \to \wedge^s T_m^* M. \]

Accordingly, we have the induced map
\[ \varphi^* : E(N) \to E(M) \]
given by
\[ (\varphi^* \omega)_m = (d_m \varphi)^* \omega_{\varphi(m)}, \quad (\omega \in E(N)). \]

As the wedge product of forms is defined pointwise, the map \( \varphi^* \) thus defined is a homomorphism of graded algebras.

Note that \( E_0(M) \simeq \Gamma^\infty(\mathbb{R}) = C^\infty(M) \). Moreover, for \( f \in C^\infty(M) \) the function \( \varphi^* f \) is given by the usual pull-back \( f \circ \varphi \).

Similar remarks can be made for the spaces of covariant tensor fields: a smooth map \( \varphi : M \to N \) naturally induces a linear map
\[ \varphi^* : \Gamma T_{0,s} N \to \Gamma T_{0,s} M. \]

By the identifications of the previous sections, we have a natural embedding of vector bundles
\[ \wedge^k T^* M \hookrightarrow T_{0,k} M, \]
and accordingly an embedding
\[ E_k(M) \hookrightarrow \Gamma T_{0,k} M, \]
identifying \( k \)-differential forms with alternating tensor fields. The embedding is compatible with the definitions of \( \varphi^* \) given above.

**Lemma 1**  Let \( \varphi : M \to N \) be a smooth map of manifolds. Then for every differential form \( \omega \in E(N) \),
\[ \varphi^*(d\omega) = d\varphi^* \omega. \]

**Proof:** We will first check the formula for \( \omega = f \in C^\infty(M) \). Then, for every \( m \in M \),
\[ d(\varphi^* f)_m = d_m(f \circ \varphi) = d_{\varphi(m)}f \circ d_m \varphi. \]

by the chain rule. The latter expression equals
\[ (df)_{\varphi(m)} \circ d_m \varphi = (d_m \varphi)^*(df)_{\varphi(m)} = (\varphi^* df)_m. \]

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The formula for $f$ follows.

In general, let $U$ be a coordinate patch in $N$ and observe that the restriction of $\varphi$ to $\varphi^{-1}(U)$ is a smooth map $\varphi^{-1}(U) \rightarrow U$. It suffices to prove the formula for a $k$-form $\omega \in E_k(U)$. By using local coordinates, we see that in such a patch $\omega \in E_k(U)$ can be expressed as a sum of $k$-forms of type

$$\lambda = f dg^1 \wedge \cdots \wedge dg^k,$$

with $f, g^j \in C^\infty(U)$. Hence, it suffices to prove the formula for such a $k$-form $\lambda$.

As $d$ is an anti-derivation, and $d^2 = 0$,

$$d\lambda = df \wedge dg^1 \wedge \cdots \wedge dg^k$$

so that

$$\varphi^*(d\lambda) = \varphi^*(df) \wedge \varphi^*(dg^1) \wedge \cdots \wedge \varphi^*(dg^k)$$

$$= d\varphi^*f \wedge d\varphi^*g^1 \wedge \cdots \wedge d\varphi^*g^k$$

$$= d[(\varphi^*f)\varphi^*g^1 \wedge \cdots \wedge \varphi^*g^k]$$

$$= d[\varphi^*\lambda].$$

Here we have been using that $\varphi^*$ is an algebra homomorphism $E(U) \rightarrow E(\varphi^{-1}(U))$.  

**Lie derivatives**

If $\varphi$ is a (local) diffeomorphism $M \rightarrow N$, we may define a pull-back map $\varphi^* : \Gamma r,s N \rightarrow \Gamma r,s M$ on mixed tensor fields as follows. For $T \in \Gamma r,s N$ we define

$$\varphi^*(T)_m := (d_m\varphi)^*T_{\varphi(m)}.$$ 

This definition facilitates the definition of Lie derivative of tensors with respect to a given smooth vector field.

Let $X \in X(M)$ be a smooth vector field. Then for every $m \in M$ we denote by $t \mapsto \varphi^t_X(m)$ the (maximal) integral curve for $X$ with initial point $m$. The domain of this integral curve is an open interval $I_{X,m}$ containing 0. Let $T \in \Gamma r,s M$. Then we define the Lie derivative of $T$ with respect to $X$ by

$$(\mathcal{L}_XT)_m := \frac{d}{dt}
\bigg|_{t=0}
[\varphi^{t}_X]^*T_m.$$ 

Here we note that $\varphi^{t}_X$ is a diffeomorphism from a neighborhood of $m$ onto a neighborhood of $\varphi^{t}_X(m)$. Accordingly, the expression

$$[\varphi^{t}_X]^*T_m$$ 

is a well-defined element of $(T_m M)_{r,s}$ which depends smoothly on $t$ (in a neighborhood of 0). Accordingly, $(\mathcal{L}_XT)_m$ defines a tensor in $(T_m M)_{r,s}$. Moreover, by smoothness of the flow of the vector field $X$ it follows that the section $\mathcal{L}_XT$ of the tensor bundle $T_{r,s}M$ thus defined is smooth. In other words, we have defined a linear map

$$\mathcal{L}_X : \Gamma r,s M \rightarrow \Gamma r,s M,$$

called the Lie derivative. In a similar way it is seen that the Lie derivative defines a linear map $\mathcal{L}_X : E_k(M) \rightarrow E_k(M)$. 

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Lemma 2  Let $f \in C^\infty(M)$. Then $\mathcal{L}_X f = Xf$.

Proof:  By definition and application of the chain rule,
\[
(\mathcal{L}_X f)(m) = \frac{d}{dt} \bigg|_{t=0} (\varphi_X^t)^* f(m) \\
= \frac{d}{dt} \bigg|_{t=0} f(\varphi_X^t(m)) \\
= d_m f \frac{d}{dt} \varphi_X^t(m)\bigg|_{t=0} \\
= d_m f X_m = (Xf)_m.
\]

\[\square\]

We have the following Leibniz rule with respect to tensor products

Lemma 3  Let $S \in \Gamma_{r,s}M$ and $T \in \Gamma_{u,v}M$. Then
\[
\mathcal{L}_X (S \otimes T) = \mathcal{L}_X S \otimes T + S \otimes \mathcal{L}_X T.
\]

Proof: We note that
\[
(\varphi_X^t)^* (S \otimes T)_m = ((\varphi_X^t)^* S)_m \otimes ((\varphi_X^t)^* T)_m.
\]
Now differentiate at $t = 0$ and apply the lemma below.

Lemma 4  Let $I$ be an open interval containing 0 and let $f : I^n \to M$ be a smooth map. Then
\[
\frac{d}{dt} \bigg|_{t=0} f(t, t, \ldots, t) = \\
= \frac{d}{dt} \bigg|_{t=0} f(t, 0, \ldots, 0) + \frac{d}{dt} \bigg|_{t=0} f(0, t, \ldots, 0) + \frac{d}{dt} \bigg|_{t=0} f(0, 0, \ldots, t).
\]

Proof: We will prove this for $n = 2$. The general case is proved similarly.
Consider the diagonal map $\delta : I \to I^2, t \mapsto (t, t)$. Then
\[
\frac{d}{dt} \bigg|_{t=0} f(t, t) = \frac{d}{dt} \bigg|_{t=0} f \circ \delta(t) \\
= d_m f \cdot \delta'(0) = d_m f \cdot (1, 1) \\
= d_m f \cdot (1, 0) + d_m f \cdot (0, 1) \\
= \frac{d}{dt} \bigg|_{t=0} f(t, 0) + \frac{d}{dt} \bigg|_{t=0} f(0, t).
\]

\[\square\]
Taking the Lie derivative commutes with contractions. More precisely, if \( r \geq 1, s \geq 1, \) and \( 1 \leq i \leq r, 1 \leq j \leq s, \) we may define a contraction map
\[
\mathcal{C}_{i,j} : \Gamma T_{r,s}M \to \Gamma T_{r-1,s-1}M
\]
by point-wise contraction:
\[
(\mathcal{C}_{i,j}S)_m := C_{T_m M, i,j}((S_m))_m.
\]
Let \( \varphi : M \to N \) be diffeomorphism. Then it is readily seen that
\[
\varphi^* \circ \mathcal{C}_{N,i,j} = C_{M,i,j} \circ \varphi^* \text{ on } \Gamma T_{r,s}N.
\]

**Lemma 5** Let \( X \) be a smooth vector field on \( M. \) Then
\[
\mathcal{L}_X \circ \mathcal{C}_{i,j} = \mathcal{C}_{i,j} \circ \mathcal{L}_X
\]
on \( \Gamma T_{r,s}M. \)

**Proof:** Let \( S \in \Gamma T_{r,s}M \) and \( m \in M. \) Then
\[
(\mathcal{L}_X \circ \mathcal{C}_{i,j}S)_m = \frac{d}{dt} \bigg|_{t=0} (\varphi^t)_* \mathcal{C}_{i,j}S)_m
= \frac{d}{dt} \bigg|_{t=0} C_{T_m M, i,j}((\varphi^t)_* S)_m
= C_{T_m M, i,j} \frac{d}{dt} \bigg|_{t=0} ((\varphi^t)_* S)_m
= (\mathcal{C}_{i,j} \mathcal{L}_X S)_m.
\]
Here the interchange of \( d/dt \) and \( C_{T_m M, i,j} \) is allowed by linearity of the latter map. \( \square \)

**Lemma 6** The Lie derivative \( \mathcal{L}_X \) defines a derivation of order 0 of the graded algebra \( E(M) \) which commutes with the exterior differentiation \( d. \)

**Proof:** We observed already that \( \mathcal{L}_X \) maps the subspace \( E_k(M) \) to itself, for each \( k. \) Let \( \omega, \eta \in E(M). \) Then, for \( m \in M, \)
\[
\mathcal{L}_X (\omega \wedge \eta)_m = \frac{d}{dt} \bigg|_{t=0} ((\varphi^t)_* (\omega \wedge \nu)_m
= \frac{d}{dt} \bigg|_{t=0} [((\varphi^t)_* \omega)_m \wedge ((\varphi^t)_* \nu)_m].
\]
Now apply Lemma 4 to see that \( \mathcal{L}_X \) is a derivation.

Let now \( \omega \in E(M). \) Then we must show that \( \mathcal{L}_X d\omega = d\mathcal{L}_X \omega. \) We first assume that \( \omega = f \in C^\infty(M). \) Fix \( m \in M \) and \( Y_m \in T_m M \) and extend \( Y_m \) to a smooth vector field on \( M. \) Then it suffices to show that
\[
(\mathcal{L}_X df)_m Y_m = d(\mathcal{L}_X f)_m Y_m.
\]
The expression on the left-hand side equals
\[ \frac{d}{dt} \bigg|_{t=0} (\varphi^t_* df)_m Y_m = \frac{d}{dt} \bigg|_{t=0} (d\varphi^t_* f)_m Y_m \]
= \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} \varphi^t_* f (\psi^s(m)),
where \( \psi^s := \varphi^s_X \). In the last expression the derivatives with respect to \( s \) and \( t \) may be interchanged. From this we see that the expression equals
\[ \frac{d}{ds} \bigg|_{s=0} (\mathcal{L}_X f)(\psi^s(m)) = (d\mathcal{L}_X f)_m Y_m, \]
and the result for \( \omega = f \) follows.

For general \( \omega \) we may now obtain the result by applying the method of the proof of Lemma 1. □

Lemma 7  Let \( X, Y \) be smooth vector fields on \( M \). Then \( \mathcal{L}_X Y = [X,Y] \).

Proof: Let \( f \in C^\infty(M) \). Then \( Yf = df(Y) \) equals the contraction \( C_{1,1} \) of \( Y \otimes df \). It follows that
\[ XYf = \mathcal{L}_X(Yf) = \mathcal{L}_X C_{1,1}(Y \otimes df) = C_{1,1} \mathcal{L}_X(Y \otimes df) = C_{1,1}[(\mathcal{L}_X Y) \otimes df + Y \otimes d(\mathcal{L}_X f)] = (\mathcal{L}_X Y)f + Y(Xf). \]
The result follows. □

Lemma 8  (Cartan’s formula)  Let \( X \) be a smooth vector field on \( M \). Then on \( E(M) \),
\[ \mathcal{L}_X = i(X) \circ d + d \circ i(X). \]

Proof: As in Warner, it is seen that the right-hand side of the expression is a derivation of \( E(X) \) of order 0, which commutes with \( d \). The same was seen to be true for the operator on the left-hand side. It follows that the equality needs only be checked when applied to a function \( f \in C^\infty(M) \). Now \( i(X)f = 0 \) and
\[ \mathcal{L}_X f(m) = Xf(m) = df \cdot X_m = (i(X)df)_m \]
so that the result follows. □

Lemma 9  Let \( \omega \in E_k(M) \) and let \( X_0, \ldots, X_k \) be smooth vector fields on \( M \). Then
\[ X_0[\omega(X_1, \ldots, X_k)] = \mathcal{L}_{X_0} \omega(X_1, \ldots, X_k) + \sum_{j=1}^k \omega(X_1, \ldots, [X_0, X_j], \ldots X_k). \]

Proof: Viewing \( \omega \) as an alternating tensor field in \( \Gamma T_{0,k}M \), we observe that
\[ \omega(X_1, \ldots, X_k) = C_{1,1} \cdots C_{1,1} \omega \otimes X_1 \otimes \cdots \otimes X_k. \]
The result now follows by applying Lemmas 2, 5 and 7. □
Lemma 10  Let $\omega \in E_k(M)$ and let $X_0, \ldots, X_k$ be smooth vector fields on $M$. Then

$$d\omega(X_0, \ldots, X_k) = \sum_{j=0}^{k} (-1)^j X_j \omega(X_0, \ldots, \widehat{X}_j, \ldots, X_k)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k).$$

Proof: First of all, if $k = 0$, then $\omega$ is a function and the equation is obvious. Note that in this situation the second sum on the right-hand side equals zero. We now proceed by induction. Thus, let $k > 0$ and assume the result has been established for strictly smaller values of $k$. Let $\omega \in E_k(M)$ and let $X_0, \ldots, X_k$ be smooth vector fields. Then

$$d\omega(X_0, \ldots, X_k) = (i(X_0) \circ d)\omega(X_1, \ldots, X_k)$$

$$= ([L_{X_0} - d \circ i(X_0)]\omega)(X_1, \ldots, X_k)$$

$$= X_0\omega(X_1, \ldots, X_k) - \sum_{j=1}^{k} \omega(X_1, \ldots, [X_0, X_j], \ldots X_k) +$$

$$-[d(i(X_0)\omega)](X_1, \ldots, X_k).$$

Now apply the induction hypothesis to $i(X_0)\omega$ to complete the proof. $\square$