# Tentamen Voorstellingen van eindige Groepen (Exam Representations of groups)

## 6 juli 2007, 9.00-12.00 uur

• Write your name on every exam sheet you hand in. • Write on the first page also your studentnumber and e-mailaddress (for informing you about the result of this exam). • During this exam you may consult the book "Representations and characters of groups" by James and Liebeck. • Do not only give answers to the exam problems, but also show clearly by which arguments you arrive at these answers. • In case you can not answer some part of a problem, you may continue using the results formulated in this part of the problem in the *subsequent parts of the same problem.* GOOD LUCK!

### Problem 1

In this problem the group $G$ is given by generators and relations. The generators are $a$ and $b$, subject to the relations $a^7 = 1,\ b^6 = 1$ (the unit element of the group), $b^{-1}ab = a^3$. The subgroup generated by $a$ is called $H$.

a. Show that $H$ is a normal subgroup of $G$ and that ${}^{G}/_{H}$ is an abelian group.

b. List all conjugacy classes of $G$ by giving one element in each conjugacy class.

c. Determine the degrees (=dimensions) of the irreducible characters of $G$.

d. Give the complete character table of $G$.

e. Let $\psi$ be a non-trivial character of the subgroup $H$. Compute the induced character $\psi{\uparrow}_H^G$ and show that this is an irreducible character of $G$.

### Problem 2

As usual $S_4$ is the group of permutations of the set $\{1, 2, 3, 4\}$. In this problem we investigate the following two representations of $S_4$.

For the first representation we take the 4-dimensional vector space $U$ with basis $\mathsf{e}_1$, $\mathsf{e}_2$, $\mathsf{e}_3$, $\mathsf{e}_4$ and define the representation $\pi : S_4 \to \mathrm{GL}(U)$ by $\pi(s)(\mathsf{e}_i) = \mathsf{e}_{s(i)}$ for $s \in S_4$.

For the second representation we take the 6-dimensional complex vector space $V$ with basis consisting of vectors $E_{\{i,j\}}$ labeled with the 2-element subsets $\{i, j\}$ of $\{1, 2, 3, 4\}$ (note that the notation means that $\{i, j\}$ and $\{j, i\}$ are the same sets and also that $i \neq j$). We define the representation $\rho :\ S_4 \to \mathrm{GL}(V)$ by $\rho(s)(E_{\{i,j\}}) = E_{\{s(i),s(j)\}}$ for $s \in S_4$.

a. List all conjugacy classes of $S_4$ by giving one element in each conjugacy class.

b. Compute the character of the representation $\pi$; call this character $\psi$.

c. Compute the character of the representation $\rho$; call this character $\chi$.

d. How many irreducible characters are there in the decomposition of $\psi$ into irreducibles?

e. Compute $\langle \chi, \psi \rangle$.

f. Give an argument, *which does* NOT *use the character table of* $S_4$, to show that $\chi - \psi$ is a character of $S_4$.

**Problem 3**

In this problem $G$ is a finite group and $|G|$ denotes the order of $G$.
We fix an irreducible character $\chi$ of $G$ and consider the element $\mathsf{X} = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$ in the group algebra $\mathbb{C}G$.
We let $U$ be a (left) $\mathbb{C}G$-module and denote its character by $\psi$. Moreover we define the $\mathbb{C}$-linear map $\xi : U \to U$ by $\xi(v) = \mathsf{X}v$ for all $v \in U$.

a. Compute the trace (=spoor) of the $\mathbb{C}$-linear map $\xi$ in terms of the characters $\chi$ and $\psi$.

b. Prove that $h^{-1}\mathsf{X}h = \mathsf{X}$ holds for every $h \in G$.

c. Prove that $\xi$ is a $\mathbb{C}G$-homomorphism.

d. Now assume that $U$ is an irreducible (left) $\mathbb{C}G$-module.

   (a) Prove that there is a $\lambda \in \mathbb{C}$ such that $\xi(v) = \lambda v$ for all $v \in U$.

   (b) Prove $\lambda = 0$ if $\psi \neq \chi$.

   (c) Compute $\lambda$ if $\psi = \chi$.

e. Prove that $\xi(\xi(v)) = \frac{1}{\chi(1)} \xi(v)$ for every $v \in U$.

f. Prove that the relation $\mathsf{X}^2 = \frac{1}{\chi(1)} \mathsf{X}$ holds in the group algebra $\mathbb{C}G$.
*Hint:* Look at the decomposition of $\mathbb{C}G$ into irreducible $\mathbb{C}G$-modules.

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