Tentamen Voorstellingen van eindige Groepen (Exam Representations of groups)

16 June 2009, 9.00-12.00 uur

• Write your name on every exam sheet you hand in. • Write on the first page also your studentnumber and e-mailaddress. • During this exam you may consult the book "Representations and characters of groups" by James and Liebeck. • Do not only give answers to the exam problems, but also show clearly how you arrive at these answers. • You may give your answers in English or Dutch. • In case you can not answer some part of a problem, you may continue using the results formulated in this part of the problem in the *subsequent parts of the same problem.*

GOOD LUCK!

Problem 1

Let G be a finite group. Let V be a $\mathbb{C}G$-module and let χ be its character. Let U be an *irreducible* $\mathbb{C}G$-module and let ψ be its character. Let z denote the following element of the group algebra $\mathbb{C}G$:

$$z = \sum_{g \in G} \chi(g) g \,.$$

i. Show that for every $h \in G$ we have: $hzh^{-1} = z$.

ii. Define the map $\zeta : U \to U$ by $\zeta(\mathsf{u}) = z\mathsf{u}$ for every $\mathsf{u} \in U$.
Show that ζ is a $\mathbb{C}G$-homomorphism.

iii. Show that there is a number $\lambda \in \mathbb{C}$ such that $\zeta(\mathsf{u}) = \lambda \mathsf{u}$ for every $\mathsf{u} \in U$.

iv. Compute the number $\dfrac{1}{\lambda}\langle \overline{\chi}, \psi \rangle$.
Note: $\langle \overline{\chi}, \psi \rangle$ is the inner product of the characters $\overline{\chi}$ and ψ.
Hint: compute the trace of the linear map ζ in two ways.

P.T.O./ZOZ

Problem 2

In this problem the group G is given by generators a, b, c and defining relations $a^3 = 1, \quad b^3 = 1, \quad c^2 = 1, \quad ab = ba, \quad ca = a^2c, \quad cb = b^2c$; here 1 denotes the identity element of G.

It can be shown (but you do not have to do that here) that all elements of G can be written uniquely in the form $a^ib^jc^k$ with $i, j \in \{0, 1, 2\}$, $k \in \{0, 1\}$ and that the order of G is 18.

i. Show that the group G has 6 conjugacy classes $C_1, \ldots, C_6$ and give for each C_j all elements in that conjugacy class.
Remark: you should find $1 \in C_1$, $a \in C_2$, $b \in C_3$, $ab \in C_4$, $a^2b \in C_5$, $c \in C_6$.

ii. Show that there is a 1-dimensional representation of G with character χ satisfying $\chi(C_j) = 1$ for $j = 1, 2, 3, 4, 5$ and $\chi(C_6) = -1$.

iii. Show that G has precisely four irreducible characters of degree 2.

iv. Consider the elements $\alpha = (123)$, $\beta = (456)$, $\gamma = (12)(45)$ in the permutation group S_6.
Show that there is a homomorphism of groups $\varphi : G \to S_6$ such that $\varphi(a) = \alpha, \quad \varphi(b) = \beta, \quad \varphi(c) = \gamma$.

v. Let V be a 6-dimensional $\mathbb{C}$-vector space with basis $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4, \mathsf{e}_5, \mathsf{e}_6$. Show that there is a representation ρ of G such that for every $g \in G$

$$\rho(g)\mathsf{e}_j = \mathsf{e}_{\varphi(g)(j)} \quad \text{for } j = 1, \ldots, 6.$$

Note: $\varphi(g)(j)$ is the image of j under the permutation $\varphi(g)$ of $\{1, \ldots, 6\}$. Thus ρ is the restriction to G of the standard permutation representation of S_6.

vi. Compute the character χ_ρ of the representation ρ.

vii. Show that the 1-dimensional spaces $\mathbb{C}(\mathsf{e}_1 + \mathsf{e}_2 + \mathsf{e}_3)$ and $\mathbb{C}(\mathsf{e}_4 + \mathsf{e}_5 + \mathsf{e}_6)$ are $\mathbb{C}G$-submodules of V and compute their characters.

viii. Let $W \subset V$ be the linear subspace spanned by the vectors $\mathsf{v}_1 = \mathsf{e}_1 + \omega\mathsf{e}_2 + \omega^2\mathsf{e}_3$ and $\mathsf{v}_2 = \mathsf{e}_1 + \omega^2\mathsf{e}_2 + \omega\mathsf{e}_3$ where $\omega = e^{2\pi i/3} \in \mathbb{C}$.
Show that W is a $\mathbb{C}G$-submodule of V.

ix. Show that the $\mathbb{C}G$-module W is irreducible.

x. Let χ_3 denote the character of the $\mathbb{C}G$-module W.
Compute $\chi_3(C_j)$ for $j = 1, \ldots, 6$.

xi. Show that $\chi_\rho = 2\chi_1 + \chi_3 + \chi_4$ where χ_1 is the trivial character, χ_3 is the character of the $\mathbb{C}G$-module W and χ_4 is another irreducible character.

xii. Give the character table of the group G.
Hint: From the above you already know four rows of the character table. Use the orthogonality relations to find the remaining irreducible characters

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