A NOTE ON EXCISION FOR $K_2$
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Summary. We consider Ruth Charney's excision theorem ([C]) for the special case of $K_2$. We describe a different proof for this special case. It follows from this proof that if one requires excision only on the $K_2$ level, one may weaken the condition somewhat. (Recall from [S] that the same holds if one requires excision only on the $K_1$ level). We also give a counterexample to a stronger statement: We give an example of an ideal $J$ with $J = J^2$ for which excision fails.

1.1. Let $J$ be an associative ring without unit and let $m$ be a positive integer. Following Charney we say that $J$ is an $m$-excision ideal for $K_2$ if the following holds: For any associative ring with unit $A$ that contains $J$ as a (2-sided) ideal, the map

$$K_2(\mathbb{Z} \otimes J, J) \otimes \mathbb{Z} \left[ \frac{1}{m} \right] \rightarrow K_2(A, J) \otimes \mathbb{Z} \left[ \frac{1}{m} \right]$$

is an isomorphism. (1)

1.2. Recall that $J \otimes J$ is the quotient of $J \otimes J$ by the subgroup generated by the elements $xy \otimes z - x \otimes yz$ with $x, y, z \in J$. Multiplication in $J$ defines a map

$$\mu : J \otimes J \otimes \mathbb{Z} \left[ \frac{1}{m} \right] \rightarrow J \otimes \mathbb{Z} \left[ \frac{1}{m} \right]$$

Theorem. (cf. [C]). If $\mu$ is a bijection then $J$ is an $m$-excision ideal for $K_2$.

1.3. Proof. Assume the hypothesis of the theorem. Let $J$ be an ideal
in $A$. We view $K_2(A,J)$ as the kernel of $\text{St}(A,J) \to \text{GL}(J)$ where $\text{St}(A,J)$ is the group $\text{St}(A^{(\infty)}, J, A^{(\infty)})$ of the appendix to [V]. It follows from [G-W] that

$$K_2(\mathbb{Z} \oplus J, J) \otimes \mathbb{Z} \left[\frac{1}{m}\right] \to K_2(A,J) \otimes \mathbb{Z} \left[\frac{1}{m}\right]$$

is surjective. Remains to show that the kernel of $K_2(\mathbb{Z} \oplus J, J) + K_2(A,J)$ is $m$-torsion. Fix an element $\alpha$ of this kernel. Put

$$J_s = \{ j \in J \mid m^s j \in J^2 \}$$

so that $J$ is the union of the increasing sequence of ideals $J_s$.

Choose $s$ so that $\alpha$ comes from $K_2(\mathbb{Z} \oplus J_s, J_s)$ and has trivial image in $\text{St}(A,J_s)$. Define $\overline{\text{St}}(\mathbb{Z} \oplus J)$ to be the quotient of $\text{St}(\mathbb{Z} \oplus J)$ by the $m$-torsion subgroup of $K_2(\mathbb{Z} \oplus J)$. The theorem follows from:

1.4. Lemma. There is a set theoretical map $\iota: \text{St}(A,J_s) \to \overline{\text{St}}(\mathbb{Z} \oplus J)$ such that the composite with $K_2(\mathbb{Z} \oplus J_s, J_s) \to \text{St}(A,J_s)$ equals $m^s$ times the natural map $K_2(\mathbb{Z} \oplus J_s, J_s) \to \overline{\text{St}}(\mathbb{Z} \oplus J)$.

**Proof of Lemma.** Let $F$ be the free group on $G(F)$, where $G(F)$ is the generating set used in the definition of $\text{St}(A^{(\infty)}, J_s, A^{(\infty)})$, and let $R = \ker(F + \text{St}(A,J_s))$. For each $x \in G(F)$, the $m^s$-th power of its matrix image lies in the image $E(\mathbb{Z} \oplus J)$ of $\text{St}(\mathbb{Z} \oplus J)$. Replacing $m$ by $m^s$ we may and shall further assume $s = 1$. Given $n$ distinct elements $x_1, \ldots, x_n$ in $G(F)$, choose an integer $N$ such that the matrix images $\text{mat}(x_i)$ all lie in $\text{GL}_N(J) \subseteq \text{GL}(J)$. Choose $y_i \in \overline{\text{St}}(\mathbb{Z} \oplus J)$ with matrix image

$$\begin{pmatrix} \text{mat}(x_1) & 0 & 0 \\ 0 & 1_{IN} & 0 \\ 0 & 0 & \text{mat}(x_1)^{-1} \end{pmatrix}$$
and define a homomorphism \( <x_1, \ldots, x_n> \rightarrow \overline{\text{St}}(\mathbb{Z} \oplus J) \) sending \( x_i \) to \( y_i \). Restrict this homomorphism to the commutator subgroup of \( <x_1, \ldots, x_n> \). This restriction \( \phi \) is characterised by the property:

Let \( x, x' \in <x_1, \ldots, x_n> \), \( y, y' \in \overline{\text{St}}(\mathbb{Z} \oplus J), M \in \mathbb{N} \), such that the matrix image of \( y \) is

\[
\begin{pmatrix}
\text{mat}(x) & 0 \\
0 & P \\
\end{pmatrix}
\]

For some \( P \in \text{GL}_M(J) \) and the matrix image of \( y' \) is

\[
\begin{pmatrix}
\text{mat}(x') & 0 & 0 \\
0 & 1_M & 0 \\
0 & 0 & Q \\
\end{pmatrix}
\]

for some \( Q \in \text{GL}(J) \). Then \( \phi([x, x']) = [y, y'] \).

(Compare the construction of Milnor's pairing in [M] §8 and use that we have factored out m-torsion in \( K_2(\mathbb{Z} \oplus J) \), including the Steinberg symbols \( \{\text{mat}(x_i), \text{mat}(x_i)\} \).

Using this characterisation we extend \( \phi \) to all of \( [F, F] \) by varying \( \{x_1, \ldots, x_n\} \). Let \( H \) be the free subgroup of \( F \) generated by \( m \)-th powers of elements of \( G(F) \). For each \( X(v, j, w) \) in \( G(F) \), choose \( p_i, q_i \in J \) so that \( \sum_i p_i q_i = mj \) (recall \( s = 1 \)) and put

\[
\psi(X(v, j, w)^m) = \prod_{i} (vp_i, 1, q_i w) \in \overline{\text{St}}(\mathbb{Z} \oplus J).
\]

This defines a homomorphism \( \psi : H \rightarrow \overline{\text{St}}(\mathbb{Z} \oplus J) \). It agrees with \( \phi \) on \( H \cap [F, F] = [H, H] \). We extend \( \phi \) to \( H[F, F] \) by putting \( \phi(xy) = \psi(x)\phi(y) \) for \( x \in H, y \in [F, F] \). Define \( \tau : F \rightarrow \overline{\text{St}}(\mathbb{Z} \oplus J) \) by \( \tau(x) = \phi(x^m) \).

One shows that \( \tau(x \circ y) = \tau(xy) \tau(r) \) for \( x, y \in F, r \in R \).

Thus if \( \tau \) annihilates \( R \), \( \tau \) factors through \( \text{St}(A, J_s) \) and the lemma easily follows. To show that \( \tau \) annihilates \( R \) indeed, one treats each of the defining relations listed in the appendix to [V]. To
deal with the third, for instance, recall the hypothesis of the theorem and use that for \( X(v,j,w) \in G(F) \) there is a homomorphism

\[
J \otimes J \to \text{St}(\mathbb{Z} \otimes J)
\]

sending \( p \otimes q \) to \( X(vp,1,qw) \).

2. The counterexample. It is commutative. Put \( R_r = \mathbb{Z}[T_r, \xi]/(T_r^2, \xi^2) \). Embed \( R_r \) into \( R_{r+1} \) by sending \( T_r \) to \( T_{r+1}^2 \), \( \xi \) to \( \xi \). Let \( R = \lim_{\to r} R_r \), \( J = \lim_{\to r} J_r \), with \( J_r = T_r R_r \). Clearly \( J = J^2 \) so that \( J \otimes J \to J \) is surjective. Nevertheless \( K_2(\mathbb{Z} \otimes J, J) \to K_2(R, J) \) is not injective: Consider \( a = \langle T_1, \xi T_1 \rangle \langle \xi T_1, -T_1 \rangle \in K_2(\mathbb{Z} \otimes J, J) \). Its image in \( K_2(R, J) \) vanishes, by an easy computation. But suppose \( a \) vanishes. Then \( \langle T_1, T_1 \xi \rangle \langle \xi T_1, -T_1 \rangle \) must vanish in \( K_2(\mathbb{Z} \otimes J_r) \) for some \( r \). However, recall that we have a Chern class \( K_2(\mathbb{Z} \otimes J_r) \to \Omega^2 \mathbb{Z} \otimes J_r \) sending \( \langle a, b \rangle \) to \( \pm(1+ab)^{-1}da \wedge db \). (The reader may choose conventions and then determine the correct signs). Straightforward computation shows that the image in \( \Omega^2 \mathbb{Z} \otimes J_r \) of our element is non-zero (This image is not even torsion).

(1) Charney has now replaced \( \mathbb{Z}[\frac{1}{m}] \) by an arbitrary subring of \( \mathbb{Q} \). Our theorem generalizes similarly.
References


