# Geometric Invariant Theory (GIT)

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Setting:

$G \curvearrowright X$

alg. gp.     scheme

<u>Def</u>. $(Sch/k)^{o} \longrightarrow (Sets)$

$$B \longmapsto \frac{Hom(B,X)}{Hom(B,G)}$$

If this functor is co-representable (?) by $Y$ then $Y$ is a <u>categorical quotient</u>.

If you have this:

get: $X \xrightarrow{\pi} Y$     (take $B = X$, $[id] \in \frac{Hom(X,X)}{Hom(X,G)}$).

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Suppose $G$ reductive gp. (e.g. $\mathbb{G}_m$, $GL(n,k)$, $SL(n,k)$, $PGL(n,k)$); suppose $X = \operatorname{Spec} R$

then: * $R^G \subseteq R$ finitely generated (Nagata)

* $X \longrightarrow Y := \operatorname{Spec} R^G$ is a categorical quot.

<u>Def.</u> Let $\pi : X \to Y$ is a categorical quotient. Then $\pi$ is a geom. quotient if:

* $\{\text{fibres of } \pi\} = \{G\text{-orbits}\}$
* $\pi$ is $G$-invariant
* ...

<u>Def.</u> Let $L$ be a $G$-equiv. line bdl. $/X$. Take $x \in X$.

Then:

1) $x$ is semistable $\iff$

$\exists\ \sigma \in \Gamma(X, L^{\otimes n})^G$ :

$\{\sigma \neq 0\}$ affine & $x \in \underline{\underline{\{\sigma \neq 0\}}}$

2) $x$ is stable $\iff$

if moreover: $G_x$ finite

$G$-orbit of $x$

on $\{\sigma \neq 0\}$ is

<u>closed</u>.

$\rightsquigarrow\ X^s(L) \underset{\text{open}}{\subseteq} X^{ss}(L) \underset{\text{open}}{\subseteq} X$.

<u>Thm.</u> $\exists$ categorical quot.

$X^{ss}(L) \xrightarrow{\pi} Y$, $Y$ is quasi-proj.

$\exists\ \underset{\text{open}}{V \subseteq Y}$ : $\pi^{-1}(V) = X^s(L)$ :

$\pi : X^s(L) \longrightarrow V$ is geom. quot.

Notation: $Y = X^{ss}(L) /\!/ G$

$$V = X^s(L)/G.$$

Facts: * $X$ is projective
$\Rightarrow Y$ is projective.

* for $x \in X$ s.t. $G_x = \{1\}$

then $\pi$ is a principal

$G$-bundle in ngh. of $\pi(x)$

(Luna)

$$\begin{tikzcd}
P \arrow[d] & P|_U \arrow[l] \arrow[d] \arrow[r, phantom, "\cong"] & U \times G \arrow[dl, bend left] \\
X & U \arrow[l, phantom, "\supseteq"] &
\end{tikzcd}$$

Bach thm: $X$ proj. $k$-scheme.
$\mathcal{O}(1)_X$ very ample

$P$ poly.

consider: $\underline{M}_P$ $\qquad X \times B$

$$\underline{M}_P(B) = \{ [F] \mid F \ B\text{-flat}, \ P_{F_b} = P \}$$

$\cup$

$$\underline{M}_P^{ss}(B) \supseteq \underline{M}_P^{s}(B)$$

Gieseker

**Def.** $E \in Coh(X)$

$E$ is semi stable (w.r.t. $\mathcal{O}_X(1)$) stable

$\Leftrightarrow$ $\forall\, 0 \neq F \subsetneq E$:

$$P_F(t) \leq P_E(t) \quad (<)$$

for $t \gg 0$.

$E$ pure &

**Def.** $E \in Coh(X)$ is pure

if $\forall\, 0 \neq F \subseteq E$:

$$\dim \operatorname{Supp} F = \dim \operatorname{Supp} E$$

(e) $\operatorname{Supp} E = X$, $E$ pure, then

$E$ is $\boxed{\text{torsion free.}}$

**Boundedness thm.:**

$\forall$ $E$ $\boxed{\text{semistable}}$ sheaf on $X$:
w/ $P_E = P$

$\exists$ $\underline{\underline{m}} > 0$:

$\ast$ $E(m)$ is globally generated $\quad \mathcal{O}_X \otimes V \twoheadrightarrow E(m)$

$\ast$ $H^{>0}(X, E(m)) = 0$.

$r = \dim H^0(X, E(m))$

Construction:

define: $V := k^{\oplus P(m)}$

$\underline{\mathcal{H}} := \mathcal{O}_X(-m) \otimes V$ $\ast$

$\forall E$ as above:

$\exists [\mathcal{H} \twoheadrightarrow E] \in \mathrm{Quot}_X(\mathcal{H}, P) =: Q$

$$Q \overset{\text{open}}{\supseteq} \underbrace{R^{ss}}_{E \text{ semistable}} \overset{\text{open}}{\supseteq} \underbrace{R^{s}}_{E \text{ stable}}$$

$G := PGL(V) \curvearrowright Q \ni [\mathcal{O}(-m)\otimes V \twoheadrightarrow E]$

$\rotatebox{90}{$\in$}$ $\quad *$

$g : V \to V$ $\qquad\qquad\qquad\qquad\qquad\qquad \| \; p$

$g \cdot p := [\mathcal{O}(-m)\otimes V \xrightarrow{g} \mathcal{O}(-m)\otimes V \twoheadrightarrow E]$

we saw the ideas:

$R^{ss}/G$, $R^{s}/G$ parametrize

$E$ (semi)stable on $X$ w/ $P_E = P$

Let $\mathcal{V} \boxtimes \mathcal{O}_{Q\times X} \twoheadrightarrow \mathbb{E}$ be

the universal quot on $Q\times X$
$$\downarrow \pi$$
$$Q$$

$\det \pi_*(\mathbb{E}(l)) = L$ is

a $G$-equiv. line bdl.

Thm. $\forall\, l \gg 0$:

GIT (semi)stability = Gieseker

cat. quot. (semi)stability

$\Rightarrow \quad R^{ss} \longrightarrow R^{ss} /\!\!/_{\!d}\, G =: M_P^{ss}$ proj.

corresponding $\underline{M}_P^{ss}$

$R^s \longrightarrow R^s/G =: \underline{M}_P^s$ quasi-proj.

geom. quot.

corresponding $\underline{M}_P^s$.

(2)

pts of $M_P^{ss}$ do <u>not</u> correspond to isom. classes of semistable sheaves

pts of $M_P^s$ do correspond to isom. classes of stable sheaves

[ $E$ stable $\Rightarrow$ $E$ simple
$End(E) = k$

Facts: • $R^s \longrightarrow R^s/G$
is principal $G$-bdl.

• $T_{[E]} M_P^s$

$$\begin{array}{c} \| \wr \\ \mathrm{Hom}(D, M_P^s) \cong \mathrm{Ext}^1(E,E) \\ pt \mapsto [E] \end{array}$$

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Exclude strictly semistables:

$$P = \sum_{i=0}^{d} \frac{\alpha_i}{i!} t^i \quad \text{w/} \quad \alpha_i \in \mathbb{Z}$$

$(d = \dim X)$

$$\rightsquigarrow \quad rk := \frac{\alpha_d}{\alpha_d(\mathcal{O}_X)}$$

$= 1$ ($X$ reduced)

$$\deg := \alpha_{d-1} - rk \cdot \alpha_{d-1}(\mathcal{O}_X)$$

fact: ($X$ reduced)

$$\gcd(\underline{rk}, \deg) = 1$$

$\implies$ stable = semistable

(for sheaves w/ Hilb. poly. $P$)

$\forall\ [E] \in M_P^s$ :

exp. dim.

$$\mathrm{ext}^1(E,E)_0 - \mathrm{ext}^2(E,E)_0 \leq \dim_{[E]} M_P^s \leq \mathrm{ext}_X^1(E,E)_0$$

$\mathrm{ext}^2(E,E) = 0 \Rightarrow M_P^s$ is smooth at $[E]$.

In general:

$$\mathrm{Ext}_X^i(E,E) \overset{\mathrm{tr}}{\twoheadrightarrow} H^i(X,\mathcal{O}_X)$$

$$\ker \mathrm{tr} := \mathrm{Ext}_X^i(E,E)_0 .$$

Suppose: $b_1(X) = 0$

$(\mathrm{Pic}^0(X) = 0)$

then: $\mathrm{Ext}^1(E,E) = \mathrm{Ext}^1(E,E)_0$

above: can replace $\mathrm{Ext}^2(E,E)$
by $\mathrm{Ext}^2(E,E)_0$.

[ i.e. $\mathrm{ext}^2(E,E)_0 = 0$
$\implies M_P^s$ smooth at $[E]$ ]

If $X$ is K3 ($b_1(X)=0$)
($\omega_X = \mathcal{O}_X$)
and $[E] \in M_P^s$:

$$\mathrm{Ext}^2_X(E,E) \underset{SD}{\cong} \mathrm{Hom}_X(E, E\otimes\omega_X)$$

$$\underset{K3}{\cong} \mathrm{Hom}_X(E,E)$$

$$\underset{E \text{ simple}}{\cong} k$$

$= 0$ for $X$ del Pezzo

$$\mathrm{Ext}^2_X(E,E) \xrightarrow{tr} H^2(\mathcal{O}_X) \cong k$$

so $\mathrm{Ext}^2(E,E)_0 = 0$

Hence: $\boxed{M_P^s \text{ is smooth}}$

Let $X$ smooth proj. sf $/k$
$\mathcal{O}_X(1)$

instead of fixing $\underline{P}$

fix $ch = (r, c_1, \frac{1}{2}c_1^2 - c_2)$
$\in H^*(X, \mathbb{Q})$

"$ch$ determines $\underline{P}$"

$$P(E,t) = \chi(X, E\boxed{(t)}) = \int_X ch(E(t)) \cdot td_X$$

$$\underline{\underline{M^s_{ch}}} \subseteq M^s_{\underline{P}}$$

open closed

expected dim $M^s_{ch}$ :

(assuming $b_1(X)=0$)

$$\mathrm{ext}^1(E,E)_0 - \mathrm{ext}^2(E,E)_0 =$$

$$\chi(X, \mathcal{O}_X) - \chi(E,E) =$$

$$\chi(\mathcal{O}_X) - \int_X \underbrace{ch(E)^\vee ch(E)}_{ch_0 - ch_1 + ch_2} \underline{\mathrm{td}(X)}$$

$$= \boxed{2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_X)}$$

is actual dim. of $M_{ch}^s$ when $\underline{\mathrm{ext}^2(E,E)_0 \quad \forall [E] \in M_p^s}$

K3 :

$$ch = (r, c_1, \tfrac{1}{2}c_1^2 - c_2) \quad \Leftrightarrow$$

$$v = ch \cdot \sqrt{\mathrm{td}(X)} = ch + (0,0,r)$$

$(r, c_1, \tfrac{1}{2}c_1^2 - c_2)$

Mukai vector $\qquad (r, c_1, \tfrac{1}{2}c_1^2 - c_2)$

$$\mathrm{td}_X = 1 + \frac{1}{2} c_1(X) + \frac{1}{12}\left(c_1(X)^2 + c_2(X)\right)$$

$$\underset{K3}{=} 1 + \frac{1}{12} c_2(X) = 1 + 2 \cdot \mathrm{pt}$$

$$\sqrt{\mathrm{td}_X} = 1 + \mathrm{pt}$$

Mukai pairing:
on $H^{\mathrm{even}}(X, \mathbb{Q})$

$$\langle v, w \rangle := -\int_X v^\vee \cdot w$$

$$(v_0, -v_1, v_2)$$

then:

for $E, F \in \mathrm{Coh}(X)$:

$$\langle v(E), v(F) \rangle \underset{\mathrm{HRR}}{=} -\chi(E, F)$$

$$\dim M_v^s = 2r c_2 - (r-1) c_1^2 - (r^2 - 1) \cdot 2$$

$$= \langle v, v \rangle + 2$$

(ii) $\langle v, v \rangle = 0$

$\gcd(r, H \cdot c_1) = 1$

$\mathcal{O}_X(1) = \mathcal{O}_X(H)$

then $M_v^s$ is empty or smooth proj. of dim 2.

Q: * $M_v^s$ irred.?

* $\mathrm{Pic}(M_v^s)$?

In general:

($X$ K3, $M_v^s$ for some $v$):

$$\mathrm{Ext}^1_X(E,E) \cong \mathrm{Ext}^1_X(E, E \otimes \omega_X)^* \underset{K3}{\cong} \mathrm{Ext}^1_X(E,E)^*$$

$$\rightsquigarrow \mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) \xrightarrow{SD} \mathbb{C}$$

$\mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) = T_{[E]} M_v^s \times T_{[E]} M_v^s$

non-deg. bil. form.

and $\omega \in H^0(M_v^s, \Lambda^2 \Omega_{M_v^s})$

nowhere vanishing