K3 surfaces talk 10.

Derived categories & Fourier-Mukai transforms

References: - Huybrechts, ch. 16.
- Fourier-Mukai transforms in algebraic geometry.

We assume all schemes are smooth projective and work over a field k of arbitrary characteristic.

Plan:
- Derived categories
- Derived functors
- Equivalences and FM-transforms.
- Examples of equivalences.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^{-1} \arrow[r, "d"] & A^0 \arrow[r, "d"] & A^1 \arrow[r] & \cdots
\end{tikzcd}$$
$d^2 = 0$

Def Let $\mathcal{A}$ be an abelian category. Denote by $\mathrm{Kom}(\mathcal{A})$ the category of complexes in $\mathcal{A}$. The <u>derived category</u> $D(\mathcal{A})$ is $\mathrm{Kom}(\mathcal{A})$ where we have formally inverted all quasi-isomorphisms: maps that induce isomorphisms on cohomology.

<u>Example</u> Let $A \in \mathcal{A}$ be an object and $A \to I^0 \to I^1 \to I^2 \to \cdots$ be an injective resolution. Then

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[dd] & 0 \arrow[r] \arrow[dd] & A \arrow[r] \arrow[dd] & 0 \arrow[r] \arrow[dd] & 0 \arrow[r] \arrow[dd] & \cdots \\
 & & & A \arrow[dd, "\mathrm{id}"] & 0 \arrow[dd] & 0 \arrow[dd] & \\
 & 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\
 & & & A & 0 & 0 &
\end{tikzcd}$$

is a quasi-iso.

<u>Example</u> Suppose $\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$ is an exact complex. Then we have a quasi-iso:

$$\begin{tikzcd}
\cdots \arrow[r] & E^{-1} \arrow[r] \arrow[dd] & E^0 \arrow[r] \arrow[dd] & E^1 \arrow[r] \arrow[dd] & E^2 \arrow[r] \arrow[dd] & \cdots \\
 & 0 \arrow[dd] & 0 \arrow[dd] & 0 \arrow[dd] & 0 \arrow[dd] & \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

$E^{\circ} \cong 0$

We can think of morphisms as "roofs"

$$\begin{tikzcd} W \arrow[r, "\sim"] \arrow[dr] & X \arrow[l, bend right] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "\sim"'] \arrow[dr] & \\ X \arrow[ur, bend left] & & Y \end{tikzcd}$$

Facts:
- every morphism can be represented by a single roof.
- The derived categories we consider will be locally small.

There is a natural embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$.

$$A \longmapsto \cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots$$

There is a natural functor $\mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$

$$E^\circ \longrightarrow E^\circ$$

and there are natural cohomology functors $H^i : D(\mathcal{A}) \to \mathcal{A}$.

$$E^\bullet \longrightarrow H^i(E^\bullet) = \frac{\ker d^i}{\mathrm{im}\, d^{i-1}}$$

We will also be interested in bounded complexes. (I.e. A° with $A^i = 0$ for $i \gg 0$ and $i \ll 0$.)

Lemma A complex $A^\bullet$ is quasi-isomorphic to a bounded complex iff $H^i(A^\bullet) = 0$ for $i \gg 0$ or $i \ll 0$

Def $D^b(X)$ is the full subcategory of $D(\mathrm{Coh}(X))$ consisting of complexes quasi-iso to bounded complexes.

Example: $D^b(\mathrm{Spec}(k))$ $\qquad \cdots 0 \to V \xrightarrow{f} W \to 0 \to 0 \to \cdots$

$$\begin{array}{c} \mathrm{im}(f) \xrightarrow{\sim} \mathrm{im}(f) \\ \oplus \\ \ker(f) \xrightarrow{0} \mathrm{coker}(f) \end{array} = 0 \qquad \begin{array}{c} V \\ \parallel \\ \ker(f) \oplus \mathrm{im}(f) \to \mathrm{im}(f) \oplus \mathrm{coker}(f). \end{array}$$

$\Leftarrow$ all differentials are zero. $\cdots \to V^0 \xrightarrow{0} V^1 \xrightarrow{0} V^2 \to \cdots$

Triangulated structure.

Derived categories are additive and k-linear, but not abelian. Instead, they are triangulated.

- There is a shift functor $D(\mathcal{A}) \to D(\mathcal{A})$ $\quad A^{-1} \to A^0 \to A^1 \to \dots$

  denoted by $A^\bullet \mapsto A^\bullet[1]$ $\quad \dots \to A^0 \to A^1 \to A^2 \to \dots$

  and $A^\bullet[n]$

- There is a class of "distinguished triangles", these are special diagrams of the form

$$A \to B \to C \to A[1]$$

satisfying some axioms.

For $D(\mathcal{A})$ these are as follows: given $f: A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$,

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
A^{-1} \arrow[r, "f^{-1}"] \arrow[d] & B^{-1} \arrow[r] \arrow[d] & B^{-1} \oplus A^0 \arrow[r, two heads] \arrow[d, "\begin{pmatrix} d_B & f_0 \\ 0 & d_A \end{pmatrix}"] & A^0 \arrow[d] \\
A^0 \arrow[r, "f^0"] \arrow[d] & B^0 \arrow[r] \arrow[d] & B^0 \oplus A^1 \arrow[r, two heads] \arrow[d, "\begin{pmatrix} d_B & -f_1 \\ 0 & d_A \end{pmatrix}"] & A^1 \arrow[d] \\
A^1 \arrow[r, "f^1"] \arrow[d] & B^1 \arrow[r] \arrow[d] & B^1 \oplus A^2 \arrow[r, two heads] \arrow[d] & A^2 \arrow[d] \\
\vdots & \vdots & \vdots & \vdots \\
 & & C(f) &
\end{tikzcd}$$

Prop if $A \to B \to C \to A[1]$ is a distinguished triangle, there is a l.e.s.

$$\dots \to H^{-1}(B) \to H^{-1}(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \dots$$

Prop If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a s.e.s in $\mathrm{Kom}(\mathcal{A})$, then there is a triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$.

So: s.e.s. of complexes ⇝ L.e.s. in cohomology.

Construction of $C \to A[1]$ if $A, B, C \in \mathcal{A}$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0 & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[u, "\mathrm{id.}"'] \arrow[d] & B \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] & 0 \arrow[r, phantom, "\leftarrow \text{ quasi-iso.}"] & {} \\
& 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
A[1] \\
C \arrow[u] \arrow[u, bend left]
\end{tikzcd}$$

$\mathrm{Ext}^1_{\mathcal{A}}(C, A) \to \mathrm{Hom}_{D(\mathcal{A})}(C, A[1])$.

Fact: $\mathrm{Hom}_{D(\mathcal{A})}(C, A[1]) \cong \mathrm{Ext}^1_{\mathcal{A}}(C, A)$ for $A, C \in \mathcal{A}$.

In particular, there are compositions $\mathrm{Ext}^i(C,A) \times \mathrm{Ext}^j(A,B) \to \mathrm{Ext}^{i+j}(C,B)$.

This leads to the theory of Yoneda extensions.

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[rd, dashed, "\mathrm{id.}"'] & D(\mathcal{A}) \arrow[d, "H^0"] \\
& \mathcal{A}
\end{tikzcd}$$

Derived functors.

If $F: \mathcal{A} \to B$ is a left-exact functor, there is a derived functor

$$D^b(\mathcal{A}) \longrightarrow D(B)$$

provided $\mathcal{A}$ has enough injectives. It's also called the total derived functor.

Construction: Take $A \in D^b(\mathcal{A})$.

Find a complex $I^\bullet$ of injectives with $A \simeq I^\bullet$

Apply F pointwise to $I^\bullet$. ⇝ $RF(A)$

Take $R^iF(A) = H^i(RF(A))$

Remark: if F is exact then $RF(A^\bullet) = F(A^\bullet)$.

Prop RF is exact, i.e. it preserves the shift and exact triangles.

Remarks - Derived functors do not compose in general (i.e. $RF \circ RG = R(F \circ G)$)

- Similar for left derived. Injectives/projectives not always needed. (For example: free resolutions)

Relation with the "usual" derived functors R^iF. $R^iFA = H^i(RF(A))$

Now: if $0 \to A \to B \to C \to 0$ s.e.s. in $\mathcal{A}$

⇝ triangle $A \to B \to C \to A[1]$

⇝ triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$.

⇝ l.es.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \dots$$

Examples for $D^b(X)$.

- Pushforward $f_* : Coh(X) \to Coh(Y)$. Problems: - Coh(X) does not have enough inj's.
  - The result may be unbounded.

$$\begin{tikzcd}
D^b(Qcoh(X)) \arrow[r, "Rf_*"] & D(Qcoh(Y)) \\
D^b(Coh(X)) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[rd, "Rf_*"'] & D(Coh(Y)) \arrow[u, phantom, "\cup"] \\
 & D^b(Coh(Y)) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

(Also get $R\Gamma$)

- Cohomology of a coherent sheaf on a projective scheme is (1) finite-dimensional
  (2) there are finitely many cohomology groups.

- Pull back $f^* : Coh(Y) \to Coh(X)$. Problems: Coh(Y) does not have enough projectives.

  $\rightsquigarrow Lf^*$

- Tensor product. $\otimes : Coh(X) \times Coh(X) \to Coh(X)$

  $\rightsquigarrow \otimes^L$

- Homs $\mathcal{H}om : Coh(X)^{op} \times Coh(X) \to Coh(X)$.

  $R\mathcal{H}om$

$R Hom = R\Gamma \circ R\mathcal{H}om$

Properties:
- compositions work as expected. Pushforwards and pullbacks compose
- "Everything is locally free": $R\mathcal{H}om(F, E) \cong F^\vee \otimes^L E$
- Projection formula: $Rf_* F \otimes^L E = Rf_*(F \otimes^L Lf^* E)$ ↳ $R\mathcal{H}om(F, \mathcal{O})$.
- Adjunction: $Lf^* \dashv Rf_*$
- Base change if

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[rd, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
Y' \arrow[r, "f"] \arrow[r, phantom, "\text{flat}"', shift right=3] & Y
\end{tikzcd}$$

then $f^* Rg_* = Rg'_* f'^*$

Serre duality

Def The Serre functor S_X is defined as: $A \mapsto A \otimes \omega_X[\dim X]$.

Thm For all complexes E, F, there is a natural isomorphism

$$\mathrm{Hom}_{D^b(X)}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(F, S_X E)^\vee$$

From this we can prove ordinary Serre duality $(n = \dim X)$

$$\begin{array}{ccc} \mathrm{Ext}^i(E,F) & & \mathrm{Ext}^{n-i}(F, E \otimes \omega_X) \\ \| & & \| \\ \mathrm{Hom}_{D(X)}(E, F[i]) & & \\ = \mathrm{Hom}(F[i], E \otimes \omega_X[n])^\vee & = & \mathrm{Hom}(F, E \otimes \omega_X[n-i]) \end{array}$$

Thm Grothendieck-Verdier duality. Suppose $f: X \to Y$ is a morphism. Define $\dim(f) = \dim(X) - \dim(Y)$ and $\omega_f = \omega_X \otimes f^* \omega_Y^\vee$. Then

$$Rf_* R\mathcal{H}om(F^\bullet, Lf^* E^\bullet \otimes \omega_f[\dim f]) \cong R\mathcal{H}om(Rf_* F, E)$$

Fourier - Mukai transforms.

Def Suppose $E^\bullet \in D^b(X\times Y)$. The Fourier-Mukai transform Φ_E is defined by:

$$A^\bullet \longmapsto R\pi_{X*}(\pi_Y^* A \otimes^L E^\bullet)$$

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$E^\bullet$ is called the kernel of the FM- transform.

Examples: Rf_*, Lf^*, the shift and $-\otimes L$ are all FM-transforms.

$$\begin{tikzcd} Rf_* & Lf^* & \text{shift} & -\otimes L \\ \Gamma_f \arrow[u] \arrow[ur, bend right] & & \mathcal{O}_\Delta[1] \arrow[u] & \Delta_* L \arrow[u] \end{tikzcd}$$

Fact: FM transforms are closed under composition. Also, they admit left and right adjoints.

Thm (Orlov) Suppose $F: D^b(X) \to D^b(Y)$ is a fully faithful, exact and with left and right adjoints and k-linear.
Then F is the FM- transform of a unique $E \in D^b(X\times Y)$

A set of objects $\Omega \subseteq D^b(X)$ is called a spanning class if $\mathrm{Ext}^i(E,F)=0$ for all $E\in\Omega$ and all i implies $F=0$

Using Serre duality, the hypothesis is equivalent to: $\mathrm{Ext}^i(F,E)=0 \Rightarrow F=0$.

Examples
- $\{k(x) \mid x\in X\}$
- $\{L^i \mid i\in\mathbb{Z}\}$ for L ample.
- $\{E\}\cup\{F \mid \mathrm{Hom}(E,F[i])=0 \text{ for all } i\}$

Lemma A FM-transform $F: D^b(X) \to D^b(Y)$ is fully faithful iff

$$\mathrm{Hom}(E, G[i]) \longrightarrow \mathrm{Hom}(F(E), F(G)[i]) \text{ is bijective.}$$

forall $E, G \in \Omega$ and $i \in \mathbb{Z}$.

Lemma A FM-transform is an equivalence iff it is fully faithful and it commutes with the Serre functor.

Prop Suppose X is K3 and $D^b(X) \cong D^b(Y)$. Then Y is K3.

Proof sketch The equivalence between them $\nearrow$ call it F is a FM-transform. So: commutes with Serre functors. $S_X = [2]$.

$$[2] \circ F = F \circ [2] = S_Y \circ F \quad \rightsquigarrow S_Y = [2].$$

$$\to \dim Y = 2 \text{ and } \omega_Y \cong \mathcal{O}_Y.$$

$$\begin{array}{ccc}
\displaystyle\bigoplus_{q=p-1} H^p(X, \Omega^q) & = & \displaystyle\bigoplus_{q=p-1} H^p(Y, \Omega^q) \\
\shortparallel & & \rotatebox{90}{$\subseteq$} \\
H^1(X, \Omega^0) \oplus H^2(X, \Omega^1) & & H^1(Y, \Omega^0) = 0. \\
\shortparallel & & \\
0 & &
\end{array}$$

See 5.39 & 5.40 in FM transforms in AG.

Prop Let M be the moduli spaces of stable sheaves with fixed Chern character on a K3 surface X. Suppose M is fine, two-dimensional and projective. Then M is derived equivalent to X.

($\rightsquigarrow$ M is K3 surf.)

Proof

$$\begin{tikzcd} X & X \times M \arrow[l, "\pi_X"'] \arrow[d, "\pi_M"'] \arrow[d, phantom, "\text{flat}", shift left=4] & : \mathcal{E} \\ & M & \end{tikzcd}$$

Define $F : \mathcal{D}^b(M) \to \mathcal{D}(X)$ as $\Phi_{\mathcal{E}}$.

Use spanning class $\{ k([E]) \mid [E] \in M \}$.

$$\mathrm{Ext}^1(k(E_1), k(E_2)) \longrightarrow \mathrm{Ext}^1(E_1, E_2)$$

$k([E_1]) \longmapsto \pi_M^* k([E_1]) = \mathcal{O}_{X \times \{E\}}$

$\longmapsto \mathcal{E} \otimes \mathcal{O}_{X \times \{E\}} = \mathcal{E}|_{X \times \{E\}} = i_* E$ where $i : X \times \{E\} \hookrightarrow X \times M$.

$\longmapsto R\pi_{X*} i_* E = id_{X*} E = E$

Suppose $E_1 = E_2 = E$ Then
$i=0$ $\quad \mathrm{Hom}(k(E), k(E)) = k \xrightarrow{\sim} \mathrm{Hom}(E, E) = k$.
$i=1$ $\quad \mathrm{Ext}^1(k(E), k(E)) = T_{[E]} \xrightarrow{\sim} \mathrm{Ext}^1(E, E)$
$i=2$ $\quad \mathrm{Ext}^2(k(E), k(E)) \xrightarrow{\sim} \mathrm{Ext}^2(E, E)$

$$\overset{\cong}{\mathrm{Ext}^0(k(E), k(E))^\vee} \quad \cong \quad \overset{\cong}{\mathrm{Ext}^0(E, E)^\vee}$$

Suppose $E_1 \neq E_2$.
$i=0$ $\quad \mathrm{Hom}(k(E_1), k(E_2)) = 0 \to \mathrm{Hom}(E_1, E_2) = 0$
$i=2$ $\quad$ Use Serre duality
$i=1$ $\quad \mathrm{Ext}^1(k(E_1)\ k(E_2)) = 0$ $\leftarrow$ for dimension reasons.

$$\underset{0}{\underset{\shortparallel}{\mathrm{ext}^0(E_1 E_2)}} - \underset{0}{\underset{\shortparallel}{\mathrm{ext}^1(E_1 E_2)}} + \underset{0}{\underset{\shortparallel}{\mathrm{ext}^2(E_1, E_2)}} = 2 - \dim M = 0$$

$\rightsquigarrow$ F is fully faithful.

$S_X = [2]$ $\qquad S_M = \otimes \omega_M [2] = [2]$.

M has a symplectic structure. $\rightsquigarrow \omega_M \cong \mathcal{O}_M$

$\hookrightarrow \omega \quad \Omega^1_M \cong T_M$

So F commutes with Serre functors.

$\to \omega_M \cong \omega_M^*$

But ω_M has a section.

Let X be $K3$.

Def $E \in D^b(X)$ is spherical if:

$$\mathrm{Ext}^i(E,E) = \begin{cases} k & \text{if } i=0,2 \\ 0 & \text{otherwise.} \end{cases}$$

Examples:
- Line bundles L.
- $\mathcal{O}_C$ if $\mathbb{P}^1 \cong C \subseteq X$.

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr] & & X \arrow[dl] \\ & \{x\} & \end{tikzcd}$$

Def $\mathcal{P}_E$ is the mapping cone of:

$$\pi_1^* E^\vee \otimes \pi_2^* E \to (\pi_1^* E^\vee \otimes \pi_2^* E)|_\Delta = R\mathcal{H}om(E^\vee, E)|_\Delta \xrightarrow{tr.} \mathcal{O}_\Delta.$$

so there is an exact triangle:

$$\pi_1^* E^\vee \otimes \pi_2^* E \longrightarrow \mathcal{O}_\Delta \longrightarrow \mathcal{P}_E \longrightarrow \text{.......} [1].$$

Prop $T_E = \Phi_{\mathcal{P}_E}$ is an equivalence.

X is K3 so $S_X = [2]$ so T_E commutes with it.

Use spanning class $\{E\} \cup \{F \mid \mathrm{Ext}^i(E,F) = 0 \ \forall i\} = E^\perp$

What is $T_E(E)$?

$$\pi_1^* E \otimes^L \pi_1^* E^\vee \otimes \pi_2^* E \to \pi_1^* E \otimes \mathcal{O}_\Delta \to \pi_1^* E \otimes \mathcal{P}_E \to \ldots$$

$$\pi_1^* E \otimes^L \pi_1^* E^\vee \otimes \pi_2^* E = \pi_1^* R\mathcal{H}om(E,E) \otimes \pi_2^* E \qquad \pi_1^* E \otimes \mathcal{O}_\Delta = \Delta_* E$$

$$R\pi_{2*} \pi_1^* R\mathcal{H}om(E,E) \otimes E \to E \to T_E(E) \to \ldots$$

$\in D^b(\operatorname{Spec} k)$

$$\begin{tikzcd}
\Gamma \mathcal{H}om(E,E) \otimes E \arrow[r] \arrow[d, phantom, "\shortparallel"] & E \arrow[r] & T_E(E) \arrow[r] & \cdots \\
(k[0] \oplus k[-2]) \otimes E \arrow[d, phantom, "\shortparallel"] & & & \\
E \oplus E[-2]. \arrow[d, phantom, "\wr"] & & & \\
E[-2] \oplus E \arrow[r] \arrow[r, bend left, dashed, "\mathrm{id}."] & E \arrow[r] & T_E(E) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
H^i(E[-2]) \oplus H^i(E) \arrow[r, two heads] & H^i(E) \arrow[r, "0"] & H^i(T_E(E)) \\
H'(T_E(E)) \arrow[u, hook] & &
\end{tikzcd}$$

$$H^{i-1}(T_E(E)) = H^i(E[-2]) = H^{i-2}(E)$$

$$\Rightarrow T_E(E) = E[-1].$$

Also $T_E(F) = F$ for $F \in E^{\perp}$