ORDINARY K3 / $\mathbb{F}_q$

§ Intro: Abelian Varieties

- There is an equivalence of categories:

$$\left\{\begin{matrix}\text{Abelian varieties} \\ \text{over } \mathbb{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-lattices } L \text{ (in } \mathbb{C}^g) \\ \text{s.t. } \mathbb{C}^g/L \text{ is} \\ \text{algebraic}\end{matrix}\right\}$$

$$A \longmapsto H_1(A, \mathbb{Z})$$

- Does not work in positive char!

because there are supersingular elliptic curves.

- Nevertheless, Deligne proved:

$$\left\{\begin{matrix}\text{ordinary AV} \\ \text{over } \mathbb{F}_q\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}(M, F) : M \text{ is} \\ \text{a } \mathbb{Z}\text{-lattice,} \\ F \in \operatorname{End}_{\mathbb{Z}}(M) \\ + \text{ axioms}\end{matrix}\right\}$$

- A <u>ordinary</u> if $A[p](\bar{\mathbb{F}}_q) \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^g$
  $g = \dim A$

- If $\dim A = 1$ (Elliptic curve) then A is ordinary iff $|A(\mathbb{F}_q)| \not\equiv 1 \mod p$.
  $p = \mathrm{char}(\mathbb{F}_q)$

$$\{ AV^{ord}/\mathbb{F}_q \} \rightsquigarrow \{ (M, F) \}$$

- The functor:

$$A/\mathbb{F}_q \underset{\substack{\text{Serre-Tate} \\ \text{theory}}}{\mapsto} A_{can}/W \rightsquigarrow$$

$W = \mathrm{Witt}(\mathbb{F}_q)$

A_{can} is characterized by $\mathrm{End}_{\mathbb{F}_q}(A) = \mathrm{End}_W(A_{can})$ ⚠

Fix $\iota : \mathrm{Frac}(W) \hookrightarrow \mathbb{C}$

$\rightsquigarrow A_{\mathbb{C}} := A_{can} \otimes_{\iota} \mathbb{C}$

Set $M := H_1(A_{\mathbb{C}}, \mathbb{Z})$ $F :=$ induced Frobenius.

§ K3

• **THM** (Jenny Taelman 2020, Niels Nygaard 1983, Jeng-Daw Yu 2012)

[N.],[Y.] There is a fully faithful functor between the groupoids of

i) ordinary K3 surf / $\mathbb{F}_q$, and

ii) triples $(M, F, \mathcal{K})$ where:

a) M is an integral lattice,

b) F is an endomorphism of M,

c) $\mathcal{K} \subset M \otimes \mathbb{R}$ convex

satisfying certain axiom (M1)-(M5)

If every K3 surf over $\mathrm{Frac}(\mathrm{Witt}(\mathbb{F}_q))$

[T.] satisfies ⊛ then the functor is also essentially surjective.

- The functor:

- $X/\mathbb{F}_q \rightsquigarrow X_{can}/W \rightsquigarrow$

ordinary

$W = \text{Witt}(\mathbb{F}_q)$

Serre-Tate theory

N. Y. uses Kuga-Satake

$\rightsquigarrow X_{\mathbb{C}} := X_{can} \otimes_{\iota} \mathbb{C}$

$\iota : \text{Frac}(W) \hookrightarrow \mathbb{C}$

- Set $M := H^2(X_{\mathbb{C}}, \mathbb{Z})$

$F :=$ " Frobenius endomorphism of M "

convex

$\mathcal{K} \subset M \otimes \mathbb{R} \leftrightsquigarrow$ ample line bundles

§ Serre-Tate theory for K3 surf

- Λ a complete noeth. local ring with perfect residue field of charact. $p > 0$.

- $\mathcal{X}$ a K3 surf over $\operatorname{Spec}(\Lambda)$

- $\mathrm{Art}_\Lambda :=$ category of Artinian local Λ-algebras w/ perfect residue field.

- For $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$ define $\mathcal{U}_A$ by

$$1 \to \mathcal{U}_A \to \mathcal{O}^\times_{\mathcal{X}_A} \to \mathcal{O}^\times_{\mathcal{X}_{A/\mathfrak{m}}} \to 0$$

- The formal Brauer group of $\mathcal{X}$ is the functor

$$\widehat{\mathrm{Br}}(\mathcal{X}) : \mathrm{Art}_\Lambda \to \mathrm{Ab}$$
$$A \mapsto H^2(\mathcal{X}_{\text{ét}}, \mathcal{U}_A)$$

- [Artin-Mazur] prove that $\hat{Br}(\mathcal{X})$ is representable by a smooth one-dimensional formal group $Spf(R)$

- Spf = formal spectrum
  R = a complete k-algebra
- the group law is given $R \to R \hat{\otimes}_k R$

can be described explicitely as follows:

Fix $R \cong k[[T]]$, so

$$\begin{tikzcd} R \arrow[r] \arrow[d, phantom, "\wr"] & R \hat{\otimes} R \arrow[d, phantom, "\parallel"] \\ k[[T]] & k[[X,Y]] \end{tikzcd}$$

$$T \longmapsto F(X,Y) = X + Y + \text{h.o.t.}$$

- One can prove that multipl. by p is ⑦

$$[p](T) = \begin{cases} 0 & (h=\infty) \\ aT^{p^h} + \text{h.o.t} \end{cases}$$

Def h is the height of $\hat{Br}(X)$

Def A K3 surf. X/k is ordinary if $\hat{Br}(X)$ has height $= 1$.

Equivalent to

- Frobenius on $H^2(X, \mathcal{O}_X)$ is a bijection

If k = finite the

- $|X(k)| \not\equiv 1 \mod p$.

- $(A, m) \in \mathrm{Art}_\Lambda$, $k = \text{res of } \Lambda$ ⑧

Assume

- $\mathcal{X}_k$ is ordinary.
- The enlarged Brauer group of $\mathcal{X}$ is the functor

$$\Psi(\mathcal{X}) : \mathrm{Art}_\Lambda \to \mathrm{Ab}$$
$$A \mapsto H^2_{\underset{\text{fppf site}}{\mathrm{fl}}}(\mathcal{X}_A, \underbrace{\mu_{p^\infty}}_{= \mathrm{colim}\, \mu_{p^n}})$$

- $\Psi(\mathcal{X})$ is a p-divisible group.

- Consider the local-étale seq:

$$0 \to \Psi^0(\mathcal{X}) \to \Psi(\mathcal{X}) \to \Psi^{et}(\mathcal{X}) \to 0$$

- We $\Psi^0(\mathcal{X}) \underset{\text{can}}{\cong} \widehat{\mathrm{Br}}(\mathcal{X})$

- $0 \to \Psi^\circ(\mathfrak{X}) \to \Psi(\mathfrak{X}) \to \Psi^{et}(\mathfrak{X}) \to 0$

X/k is ord $\iff$ $\hat{Br}(X)$ has height $= 1$

$$\hat{Br}(X) = \Psi^\circ(X)$$

$\rightsquigarrow$ $\exists!$ extension of $\Psi^\circ(\mathfrak{X})$ over Λ
$\Psi^\circ(\mathfrak{X})_{can}$

$\rightsquigarrow$ $\exists!$ extension of $\Psi^{et}(\mathfrak{X})$ over Λ
$\Psi^{et}(\mathfrak{X})_{can}$

- For every lift $\mathfrak{X}/\Lambda$ of X/k we have

$$(\dagger) \qquad 0 \to \Psi^\circ(\mathfrak{X})_{can} \to \Psi(\mathfrak{X}) \to \Psi^{et}(\mathfrak{X})_{can} \to 0$$

over Λ

- Thm [N.]

$$\begin{array}{ccc} \{\text{formal lifts } \mathfrak{X}/\Lambda \text{ of } X/k\} & & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(\Psi^{et}(\mathfrak{X})_{can}, \Psi^\circ(\mathfrak{X})_{can}) & & \Psi(\mathfrak{X}) \end{array}$$

is a bijection.

10

Def the canonical lifting $\mathfrak{X}_{can}$ of X_k is the unique lift st $\mathfrak{X}_{can} \Longleftrightarrow (\dagger)$ splits.

(unique up to a unique isom)

[N.] proved

Prop: $Pic(\mathfrak{X}_{can}) \Longleftrightarrow Pic(X_k)$

Cor: $\mathfrak{X}_{can}$ is algebraizable and projective.

So we put: $M := H^2(\mathfrak{X}_q, \mathbb{Z})$

§ Frobenius (F)

- $\mathbb{F}_q$, $q = p^a$; $W = \mathrm{Witt}(\mathbb{F}_q)$
- Fix $\iota : \mathrm{Frac}(W) \hookrightarrow \mathbb{C}$
- $X / \mathbb{F}_q$ ord $\rightsquigarrow \mathcal{X}_{can} / W \rightsquigarrow X_{\mathbb{C}} = \mathcal{X}_{can} \otimes_\iota \mathbb{C}$

<u>Thm</u> There exists a unique endomorphism [N. Y.] F of $H^2(X_{\mathbb{C}}, \mathbb{Z}[\frac{1}{p}])$ s.t.

(i) $\forall \ell \neq p$ prime

$$H^2(X_{\mathbb{C}}, \mathbb{Z}[\tfrac{1}{p}]) \xrightarrow{\sim} H^2_{et}(X_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell)$$

$$F \mapsto \text{geom. Frob.}$$

(ii) analogous statement with crys coh.

(iii) F preserves the rational Hodge structure on $H^2(X_{\mathbb{C}}, \mathbb{Q})$

"Pf" • By Kuga-Satake there exists an abelian variety A s.t.

$$Cl^+(H^2(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))_p \cong \mathrm{End}(H^1(A(\mathbb{C}), \mathbb{Q}))$$

even Clifford algebra.

primitive part w.r.t a fixed polarization

- same for étale and cryst.
- A is ordinary, with good reduction A_k
- In part, the Frobenius of A_k lifts to an endomorphism of A.
- Actually, in the proof one only needs to invert p.

— IV —

- Properties of F

If $X/\mathbb{F}_q$ ord $\mapsto (M, F, \overset{?}{K})$

then:

(M1) the pairing $\langle -,- \rangle$ on M is unimodular, even, and of signature $(3,19)$.

(M2) $\langle Fx, Fy \rangle = q^2 \langle x,y \rangle \; \forall x,y \in M$

From Deligne's proof of the Weil conj:

(M3) F acts semisimply on $M \otimes \mathbb{C}$ and the eigenvalues have absolute value $= q$.

Thm [T.] The endomorphism F preserves the $\mathbb{Z}$-module structure of $M = H^2(X_{\mathbb{C}}, \mathbb{Z})$. And:

(M4) we have a decomposition

$$M \otimes \mathbb{Z}_p \underset{\mathbb{Z}_p[F]}{\cong} M^0 \oplus M^1 \oplus M^2$$

with (a) $F M^s = q^s M^s$, $s = 0, 1, 2$

(b) M^0, M^1, M^2 are free $\mathbb{Z}_p$-modules of ranks 1, 20, 1 respectively.

<u>Meaning</u>: after $- \otimes_{\mathbb{C}} \mathbb{C}$, (M4) we have an integral Hodge structure on M

i.e $\quad M^s \otimes_{\mathbb{C}} \mathbb{C} = H^{s, 2-s}(X_{\mathbb{C}})$ $s = 0, 1, 2$.

§ Line bundles and ample cone (κ) (15)

- $X/\mathbb{F}_q$ ordinary $\mapsto (M, F, \overset{?}{\kappa})$

- We have seen $\mathrm{Pic}\, X \simeq \mathrm{Pic}\, X_{can}$.

- Prop [T.] We have a natural isomorphism:
  - $\mathrm{Pic}\, X_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \mathrm{Pic}\, X_{can, \bar{K}}$, $K = \mathrm{Frac}(W)$
    which respects ampleness
  - $\forall d \geq 1$:
    $$\mathrm{Pic}\, X_{\mathbb{F}_{q^d}} \xrightarrow{\sim} \{\lambda \in \overset{H^2(X_{\mathbb{C}}, \mathbb{Z})}{\overset{\shortparallel}{M}} \;/\; F^d \lambda = q^d \lambda\}$$

Put:

$$NS(M, F) = \{x \in M \,/\, F^d x = q^d x \text{ some } d\}$$

- the real cone $\mathcal{K} \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the classes of ample line bundles on $\operatorname{Pic} X_{\overline{\mathbb{F}_q}}$ satisfies:

(M5) $\mathcal{K}$ is a connected component of

$$\left\{ \begin{array}{l} x \in NS(M,F) \otimes \mathbb{R} \ / \ \langle x, x \rangle > 0 \\ \langle x, \delta \rangle \neq 0 \quad \forall \delta \in NS(M,F) \text{ with } \delta^2 = -2 \end{array} \right\}$$

and $F\mathcal{K} = \mathcal{K}$.

"Pf" Fix H an ample line bundle.

By the structure theorem of ample line bundles: [Huybrechts Ch 8 Cor 1.6]

a line bundle L on $X_{\overline{\mathbb{F}_q}}$ is ample iff:

i) $L^2 > 0$
ii) $\forall D \in \operatorname{Pic} X_{\overline{\mathbb{F}_q}}$ with $D^2 = -2$

we have $L \cdot D \neq 0$ and $L \cdot D$ has the same sign of $H \cdot D$. ☒

§ Fully faithfulness

- Now we have a functor between groupoids (i.e. only invertible morph):

$$X/\mathbb{F}_q \text{ ord } \mapsto (M, F, \kappa) \text{ st (M1)–(M5)}$$

- [N. Y.] proved fully faith.

(Faithfulness):

Let $f, g : X_1 \to X_2$ morph. of ordinary $K3/\mathbb{F}_q$ inducing the same map on the lattices

$$H^2(X_{1,\mathbb{C}}, \mathbb{Z}) \to H^2(X_{2,\mathbb{C}}, \mathbb{Z})$$

Then $f_{\mathbb{C}} = g_{\mathbb{C}}$ by Torelli.

Hence $f = g$.

(Fullness): X_1, X_2 ord ova $\mathbb{F}_q$.

- Let $\varphi : H^2(X_{1,\mathbb{C}}, \mathbb{Z}) \to H^2(X_{2,\mathbb{C}}, \mathbb{Z})$ an isometry, commuting with F_1 and F_2, and respecting κ_1 and κ_2
- Then φ respects the Hodge structures
- By Torelly $\exists! \, f : X_{1,\mathbb{C}} \xrightarrow{\sim} X_{2,\mathbb{C}}$ inducing φ.
- One can show that $\exists! \, f : X_1 \to X_2$

§ Essential surjectivity

Def Let $\mathcal{O}_K$ be a complete DVR with frac. field K.
We say that a K3 over K sat. $\circledast$ if $\exists$ $K \subset L$ finite and an alg. space $\mathcal{X}/\mathcal{O}_L$ st:

(i) $\mathcal{X}_L \cong X_L$
(ii) $\mathcal{X}$ is regular
(iii) the special fiber is a normal crossing divisor in $\mathcal{X}$
(iv) the rel. dual. sheaf of $\mathcal{X}/\mathcal{O}_L$ is trivial.

$\circledast$ is a strong form of "potential semi-stable reduction."

$\circledast$ is expected to hold in general,

but for now it is only known under extra assumption.

(eg if the residue char of $\mathcal{O}_K$ is 0)

<u>Thm</u> [T.] If $(*)$ hold for K3's over

$$K = \operatorname{Frac} W$$

then the functor

$$X/\mathbb{F}_q \text{ ord} \longmapsto (M, F, \mathcal{K}) \text{ s.t. } (M1)\text{-}(M5)$$

is essentially surj.

Pf Pick $(M, F, \mathcal{K})$ st $(M1)$–$(M5)$

1) Construction of a complex K3 X

2) X has complex mult

3) Descend $K \underset{\iota}{\subseteq} \mathbb{C}$ $\quad \iota : K \hookrightarrow \mathbb{C}$

4) Extension to $\mathcal{O}_K$

5) Reduction to $\mathbb{F}_q$ = res field of $\mathcal{O}_K$

—————— $\boxed{\text{IMJ}}$ ——————