S. Marseglia, K3 seminar UU, 03 Feb 2021

KUGA SATAKE CONSTRUCTION

Idea:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{integral polarized}\\ \text{Hodge str. of weight 1}\end{array}\right\} \arrow[d, leftrightarrow] & \left\{\begin{array}{c}\text{integral Hodge}\\ \text{structure of K3 type}\end{array}\right\} \arrow[l, "\text{KSC}"'] \\
\{\text{ab. varieties}\} & \{\text{K3 surf}\} \arrow[u] \arrow[l, dashed]
\end{tikzcd}$$

Application: Deligne's proof of the Weil conjectures for K3 surf over $\mathbb{F}_q$!

§ Clifford Algebras & Spin groups.

- Commutative ring K (2 not a zero-div.)

  (eg $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$)

- V a free K-module of finite rank

- the tensor algebra of V is

$$T(V) := \bigoplus_{i \geq 0} V^{\otimes i} \quad \text{with} \quad V^{\otimes 0} = K$$

- $$T^+(V) = \bigoplus_{i \geq 0} V^{\otimes 2i} \qquad \underline{\text{even}}$$

$$T^-(V) = \bigoplus_{i \geq 0} V^{\otimes 2i+1} \qquad \underline{\text{odd}}$$

- q a quadratic form on V
  $\rightsquigarrow$ bilinear form
  $$q(v,w) := \frac{1}{2}\left[q(v+w) - q(v) - q(w)\right]$$

- $I(q) :=$ two-sided ideal of $T(V)$ gen by $v \otimes v - q(v)$, $v \in V$
  $\uparrow$ even.

- <u>Def</u> The <u>Clifford Algebra</u>
  $$Cl(V) = Cl(V,q) := \frac{T(V)}{I(q)} \quad (K\text{-alg.})$$

$$\rightsquigarrow \quad Cl(V) = \underbrace{Cl^+(V)}_{\text{even Clifford sub-algebra}} \oplus \underbrace{Cl^-(V)}_{\text{sub-mod.}}$$

- Notation: mult. on $Cl(V)$ is denoted by $v \cdot w$ $\quad v, w \in Cl(V)$

- We can embed $V \hookrightarrow C\ell(V)$ in a natural way:

$$V \subset T(V) \twoheadrightarrow C\ell(V)$$

induces an inclusion $V \subseteq C\ell^-(V)$

- $C\ell(V)$ satisfies the following Universal Prop

given (A, j) A unital, ass., K-algebra

$j : V \to A$ K-linear

st $j(v)^2 = q(v) 1_A$ $\forall v \in V$

then

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "j"'] & C\ell(V) \arrow[d, dashed, "\exists !"] \\ & A \end{tikzcd}$$

- the construction

$$(V, q) \longmapsto C\ell(V, q)$$

is functorial

- In part if L/K ext $\to$ $C\ell(V)_L \xrightarrow{\sim} C\ell(V_L)$

Dimension of $Cl(V)$:

k a field. Choose an O.N. basis of V

$v_1, \dots, v_n$

We get an isomorphism

$$Cl(V) \xrightarrow{\sim} \Lambda^* V$$

$$v_{i_1} \cdots v_{i_k} \longmapsto v_{i_1} \wedge \dots \wedge v_{i_k}$$

$Cl(V, q \equiv 0)$

$$\rightsquigarrow \quad Cl(V) \cong \bigoplus_{a_n \in \{0,1\}} K \, v_1^{a_1} \cdots v_n^{a_n}$$

$$\rightsquigarrow \quad \dim Cl(V) = 2^n$$

Huge!

Example

$V = \mathbb{R}^3$, $q((v_1, v_2, v_3)) = -v_1^2 - v_2^2 - v_3^2$.

If e_1, e_2, e_3 is the standard basis of V, then then one computes in $C\ell(V,q)$:

$$e_i \cdot e_j = \begin{cases} -e_j \cdot e_i & i \neq j \\ -1 & i = j \end{cases}$$

That is, $C\ell(V,q)$ is the Hamilton's real quaternions.

- We have <u>antiautomorphisms</u>:

on $T(V)$

$$v_1 \otimes \dots \otimes v_k \longmapsto v_k \otimes \dots \otimes v_1$$

hence on $Cl(V)$

$$v = v_1 \cdot \dots \cdot v_k \mapsto v_k \cdots v_1 =: v^t$$

transpose

(= v^* in the book)

- $Cl(V)^* := \{ \text{units in } Cl(V) \}$

<u>Def</u> The even <u>Clifford group</u> is

$$CSpin^+(V) := \{ v \in Cl^+(V)^* : vVv^{-1} \subseteq V \}$$

Here vVv^{-1} is the image of

$$V \mapsto V$$
$$w \mapsto v \cdot w \cdot v^{-1}$$

- Study $CSpin(V)$ by representation

$$t : CSpin(V) \to O(V)$$

- $Spin(V) = \{ v \in CSpin^{+}(V) \ / \ v \cdot v^{t} = 1 \}$

- (k field) we have s.e.s.

$$0 \to k^{*} \to CSpin^{+}(V) \to SO(V) \to 0$$

$$0 \to Spin(V) \to CSpin^{+}(V) \to k^{*} \to 0$$

§ Hodge-structures and reps (revisited)

Van Geemen

Prop V a $\mathbb{Q}$-vector space. There is a bijection

$$\left\{ \begin{array}{l} \text{Hodge structures} \\ \text{of weight } k \text{ on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{algebraic representations} \\ \rho : \mathbb{C}^* \to GL(V_{\mathbb{R}}) \\ \text{s.t. } \rho(t) = t^k, \ \forall t \in \mathbb{R} \end{array} \right\}$$

Pf Start with $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$.

Any element of $V_{\mathbb{C}}$ can be written as

$v \otimes 1 + w \otimes i$ for $v, w \in V_{\mathbb{R}}$.

Fix $p \geq q$, let $\{v_r + i w_r\}_r$ be a basis of $V^{p,q}$

Put

$$V_p := \langle v_r, w_r \rangle \subseteq V_{\mathbb{R}}$$

Then:

$$V_{\mathbb{R}} = \bigoplus_{p \geq q} V_p \quad , \quad V_p \otimes_{\mathbb{R}} \mathbb{C} = V^{p,q} \oplus V^{q,p}$$

Construct ρ on each V_p:

- if $p = q$ put $\rho(a+ib)v := (a^2+b^2)^p \cdot v \quad \forall v \in V_p$
- if $p = q + \ell$, $\ell > 0$, put $\rho(a+ib) = (a^2+b^2)^q \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{\ell}$

The eigenspaces of ρ are $V^{p,q}$.

Conversely: compose $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}}) \subset GL(V_{\mathbb{C}})$

There is a basis $\{v_i\}_i$ of $V_{\mathbb{C}}$ of simultaneous eigenvectors

$$\rho(z)v_i = \lambda_i(z)v_i$$

for some hom. $\lambda_i : \mathbb{C}^* \to \mathbb{C}^*$

Note that $\lambda_i(z)$ is a polynomial $z, \bar{z}, (z\bar{z})^{-1}$

i.e. $\lambda_i(z) = z^p \bar{z}^q$ for some $p, q \in \mathbb{Z}$

Get a Hodge structure

$$V^{p,q} = \{ v \in V_{\mathbb{C}} \mid \rho(z)v = z^p \bar{z}^q v \}.$$

Note $\rho(t) = t^k$ for $t \in \mathbb{R} \Rightarrow p+q = k$.

Recall: A polarization on V (of weight k) is a morphism of structures

$$\psi : V \otimes V \to \mathbb{Q}(-k)$$

s.t. its $\mathbb{R}$-linear extension yields a pos. def. symm. form $(v,w) \mapsto \psi(v, Cw)$ on $(V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}$, where $Cw \mapsto i^{p-q} w$ is the Weil operator.

If $V \rightsquigarrow \rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$

Then $Cw = \rho(i)w$

§ From weight 2 to weight 1.

- Let V be a Hodge structure of K3 type,

i.e: $V_{\mathbb{C}} \simeq \underset{\dim=1}{V^{2,0}} \oplus V^{1,1} \oplus \underset{\dim=1}{V^{0,2}}$

endowed with a polarization $-q$

- Pick a generator $\sigma = e_1 + i e_2$ of $V^{2,0}$ with $e_1, e_2 \in V_{\mathbb{R}}$, $q(e_1) = 1$

- $q(\sigma) = 0 \Rightarrow q(e_1, e_2) = 0$, and $q(e_2) = 1$

i.e e_1, e_2 is an O.N. basis of the real part: $(V^{2,0} + V^{0,2}) \cap V_{\mathbb{R}}$.

Hence $e_1 \cdot e_2 = - e_2 \cdot e_1$ in $C\ell(V_{\mathbb{R}})$

If we put $J := e_1 \cdot e_2$ then we see that $J \in CSpin^+(V_{\mathbb{R}})$,

and $J^2 = -id$ on $C\ell(V_{\mathbb{R}})$.

In other words we have a complex structure on $C\ell(V_{\mathbb{R}})$.

Note: the complex structure respects the decomposition $C\ell(V_{\mathbb{R}}) = C\ell^+(V_{\mathbb{R}}) \oplus C\ell^-(V_{\mathbb{R}})$.
A direct calculation shows that it is independent of the choice of e_1, e_2.

<u>Def</u> The Kuga-Satake Hodge structure is the H. str. of weight one on $C\ell^+(V)$ given by

$$\rho : \mathbb{C}^* \to GL(C\ell^+(V_{\mathbb{R}}))$$
$$x + iy \mapsto x + yJ$$

Def The Kuga-Satake variety associated an integral H. str. V of K3 type is the complex torus

$$KS(V) = \frac{Ce^+(V_{\mathbb{R}})}{Ce^+(V)}$$

Rmk: If V is instead rational then KS(V) is defined only up to isogeny.

Huge

Rmk: If $\dim_{\mathbb{Q}} V_{\mathbb{Q}} = m$ then $\dim KS(V) = 2^{m-2}$.

Want a polarization on KS(V)

Choose f_1, f_2 orthogonal vectors with $q(f_i) > 0$.

Def:

$$Q : Ce^+(V) \times Ce^+(V) \longrightarrow \mathbb{Q}(-1)$$
$$(v, w) \longmapsto \pm \, tr(f_1 \cdot f_2 \cdot v^t \cdot w)$$

(tr: is the trace by left mult)

One has: $tr(v \cdot w) = tr(w \cdot v)$

$$tr(v^t) = tr(v)$$

Prop

$$Q(v,w) = \pm\, tr(f_1 \cdot f_2 \cdot v^t \cdot w)$$

defines a polarization for $Cl^+(V)$

Pf

WTS:

- Q is a morph. of H. str.

For $z = x+iy \in \mathbb{C}^*$ we have

$$Q(\rho(z)v, \rho(z)w) = tr(f_1 \cdot f_2 \cdot (\rho(z)\cdot v)^t \cdot \rho(z) \cdot w) =$$

$$= tr(f_1 \cdot f_2 \cdot v^t \cdot (\rho(z)^t \rho(z)) \cdot w) =$$

$$= z\bar{z}\, Q(v,w)$$

$J^t = -J$

$\rho(z)^t \cdot \rho(z) = x^2+y^2$.

- Q is non-deg. (from def).

- Q is symmetric:

$$Q(v, \rho(i) w) = Q(w), \rho(i) v).$$

- <u>sign</u>: the space of Hodge structures on V has two connected components.
  the property of being pos-def is constant on each of the two components.
  If one needs to change component one might need to change the sign.

Def Let X a K3-surf or a 2-dim. complex torus. The associated Kuga-Satake-variety is the K-S-variety associated to

$H^2(X,\mathbb{Z})$ with $q=$ intersection pairing (up to a sign)

$$KS(X) := KS\left(H^2(X,\mathbb{Z})\right).$$

Facts: there are variations:

- $KS(T(X))$ has $\dim = 2^{rk(T(X))-2}$

- X is proj with $\ell \in H^2(X,\mathbb{Z})$ ample class

$$KS(X,\ell) := KS\left(H^2(X,\mathbb{Z})_p\right)$$

- We have isogenies

$$KS(X) \sim KS(X,\ell)^2 \sim KS(T(X))^{P(X)}$$

$P(X) =$ Picard number of X.

§ K-S - in some special cases

<u>Prop</u>: Let A be a complex torus of dimension 2

Then $$KS(A) \sim (A \times \hat{A})^4$$

In part, if A is an abelian surf then

$$KS(A) \sim A^8$$

if moreover (A, ℓ) is a pol. ab. surf then

$$KS(A, \ell) \sim A^4$$

<u>Prop</u>: Let X be a Kummer sf associated to the complex torus A. Then:

$$KS(X) \sim (A \times \hat{A})^{2^{18}}$$

§ Weil Conjectures

- X a smooth proj variety / $\mathbb{F}_q$, $q = p^n$.

- $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}$

- $F : X \to X$ absolute Frobenius $\begin{cases} \text{id on pts} \\ x \mapsto x^p \text{ on } \mathcal{O}_X \end{cases}$

- $F^n : X \to X$ is a $\mathbb{F}_q$-morph.

- $f := F^n \times \mathrm{id} : \overline{X} \to \overline{X}$ is an $\overline{\mathbb{F}_p}$-morph

- $N_r = \# X(\mathbb{F}_{q^r}) = \{ \text{pts of } \overline{X} \text{ fixed by } f^r \}$

- $Z(X, t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r} \right)$

- For $\ell \neq p$ consider

$$f^{n*} : H^*_{et}(\bar{X}, \mathbb{Q}_\ell) \to H^*_{et}(\bar{X}, \mathbb{Q}_\ell)$$

- Lefschetz fixed pt formula:

$$N_n = \sum_i (-1)^i \operatorname{tr}\left(f^{*n} \mid H^i_{et}(\bar{X}, \mathbb{Q}_\ell)\right)$$

- $$Z(X,t) = \prod_i \exp\left(\sum_n \operatorname{tr}\left(f^{n*} \mid H^i_{et}(\bar{X}, \mathbb{Q}_\ell)\right)\right)^{(-1)^i}$$

- For Y_0 a K3 surf $\mathbb{F}_q$
  - $H^i_{et} = 0$ for $i = 1, 3$
  - $H^i_{et} \cong \mathbb{Q}_\ell$ for $i = 0, 4$
  - $f^{n*} = id$ on H^0_{et}
  - $f^{n*} = q^{2n} \cdot id$ on H^4_{et}

Using
- $\exp\left(\sum \frac{t^n}{n}\right) = \frac{1}{1-t}$ we get:

$$Z(Y_0,t)^{-1} = (1-t)\cdot \overbrace{\det\left(1 - f^* t \mid H^2_{et}(\bar{Y}_0, \mathbb{Q}_\ell)\right)}^{=: P_2(t)} \cdot (1-q^2 t)$$

Thm (Weil conj for K3)

$$P_2(t) = \prod_{i=1}^{22} (1-\alpha_i t)$$

has integer coeff, indep of the choice of $\ell \neq p$,

its zeros $\alpha_i \in \overline{\mathbb{Q}}$, $|\alpha_i| = q$.

Moreover, wlog,

$$\alpha_i = \pm q \quad \text{for } i=1,\dots,2k \quad \text{even}$$

$$\alpha_i \neq \pm q, \quad i > 2k,$$

and $\alpha_{2j-1} \cdot \alpha_{2j} = q^2 \quad j > k$

- $Z(Y_0, t)$ is rational

- functional equation $Z(Y_0, \frac{1}{q^2 t}) = (qt)^{24} Z(Y_0, t)$

- Riemann hyp: zeros of $P_2(q^{-s})$ satisfy $Re(s) = 1$

Pf (sketch) $\quad Y_0$ a K3 surf $/\mathbb{F}_q$

- We can lift:

  $\exists$ a complete DVR R with $Q(R) = K$ of char $= 0$ and residue field $k/\mathbb{F}_q$ a finite ext.

- $\exists$ a polarized family $Y \to \operatorname{Spec}(R)$ with closed fiber $Y_0 \times k$

- Let X be the generic fiber $/K$

- Pick $K \hookrightarrow \mathbb{C}$ $\rightsquigarrow$ $X_{\mathbb{C}} := X \times \mathbb{C}$.

- Fact: $KS(X_{\mathbb{C}})$ descends to an abelian variety A defined over L/K finite ext. s.t. there $\exists$ a $\operatorname{Gal}(\bar{L}/L)$-inv. isom:

$$\circledast \quad Cl^+(H^2_{et}(X_{\bar{L}}, \mathbb{Z}_\ell(1))_p) \xrightarrow{\sim} \operatorname{End}_C(H^1_{et}(A_{\bar{L}}, \mathbb{Z}_\ell))$$

where $C = Cl^+(H^2(X_{\mathbb{C}}, \mathbb{Z})_p)$

- Fact: A reduces to an abelian var A_0/k and

$$Cl^+(H^2_{et}(Y_0 \times \bar{k}, \mathbb{Z}_\ell(1))_p) \simeq \operatorname{End}_C(H^1_{et}(A_0, \mathbb{Z}_\ell))$$

also $\operatorname{Gal}(\bar{k}/k)$-inv.

- Weil conj. for AV's were proved by Weil.