Moduli of K3 surfaces

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[see §5 in Huybrechts & the Stacks project]

In the past we've studied a $\boxed{\text{single K3 surface}}$ X over a field k.

Today, we'd like to study $\boxed{\text{families of K3 surfaces}}$ over general schemes.

<u>Defn</u>: Given a scheme S, then a $\boxed{\text{K3 surface over } S}$ is a scheme X with a $\boxed{\text{proper and smooth}}$ morphism $f: X \longrightarrow S$ such that for all geometric points $\operatorname{Spec} k \longrightarrow S$, X_k is a $\boxed{\text{K3 surfaces.}}$

<u>Ex</u>: The Fermat quartic $X_0^4 + X_1^4 + X_2^4 + X_3^4$ inside $\mathbb{P}^3$ defines a K3-surfaces over $\operatorname{Spec}(\mathbb{Z}[1/2])$.

In particular, the $\boxed{\text{moduli space } \mathcal{M}_d}$ of (polarized) K3 surfaces (of degree $2d$) is the <u>universal family</u> of K3 surfaces.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d] \\ S \arrow[r] & \mathcal{M}_d \end{tikzcd}$$

Today we'll discuss $\mathcal{M}_d$ by looking at:

(I) F.O.P. perspective.

(II) Hilbert schemes.

(III) Using Hilbert schemes to approximate $\mathcal{M}_d$.

(IV) Discuss some results from the literature.

$\boxed{\mathcal{M}_{d,\mathbb{C}}}$ $\quad$ $M_{d,S}$ $\quad$ $\mathcal{M}_{d,S}$

(I) Functor of points perspective

$\boxed{\cancel{\text{Fix}} \text{ a scheme } S}$

Given an S-scheme X, then X defines a functor

$$h^X : Sch_S^{op} \longrightarrow Sets, \quad (T \to S) \longmapsto Hom_S(T, X).$$

This is functorial, ie, given $X \to Y$ on S, we obtain $h^X \to h^Y$.

In fact, $h^{(-)}$ defines an embedding of categories

$$Sch_S \overset{y}{\hookrightarrow} \underbrace{Fun(Sch_S^{op}, Sets)}; \quad X \longmapsto h^X$$

(It's the Yoneda embedding!)

Our job now is to define things here, and show they come from here. In this case, we say our functor is $\boxed{\text{representable}}$. Otherwise a functor F can be $\boxed{\text{coarsely representable}}$, meaning there is there is a scheme X and an initial map $F \to X$ which is a bijection on geometric points.

(i) $F(k) \xrightarrow{\sim} X(k) \quad h^X \quad k = \bar{k}$

(ii) $\exists\, F \to Y$, $\exists!\, X \to Y$
st $F \to X$

$\boxed{(i)\ F \simeq h^X}$

$\boxed{(i)\ F \simeq h^X}$ (ii) $\exists\ F \to Y$, $\exists!\ X \to Y$ s.t.
$$\begin{tikzcd} F \arrow[r] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

<u>Example</u>: $\boxed{\mathbb{P}^N}$ [exercise!]

<u>Example</u>: We want to study the functor $\mathcal{M}_d$,

$$\mathcal{M}_d : Sch_S^{op} \longrightarrow Sets \ ; \ (T \to S) \longmapsto \boxed{\left\{ \begin{array}{l} (\underbrace{X \xrightarrow{f} T}_{\text{K3 surface over } T}, L) \ \text{line bundle on } X \\ \text{such that } L \text{ is ample,} \\ \text{primitive on geometric fibres,} \\ \text{and } L_k^2 = 2d, \ k = \bar{k}. \end{array} \right\} \Big/ \simeq}$$

produces a small enough moduli functor.

This is the moduli functor of <u>polarised</u> K3 surfaces of <u>degree</u> 2d.

<u>Q</u>: When is it ~~representable~~? Or coarsely representable?

(II) Hilbert schemes

Let $X \to S$ be a morphism of schemes.

<u>Defn</u>: Let $\mathrm{Hilb}_{X/S}$ be the functor $Sch_S^{op} \longrightarrow Sets$ defined by

$$(T \to S) \longmapsto \boxed{\left\{ \begin{tikzcd} Z \subseteq_{\text{closed}} X_T = X \times_S T \arrow[d] \arrow[dr, bend right, "\text{flat + proper}"'] & \\ & T \end{tikzcd} \right\}}$$

<u>Theorem</u> [Grothendieck]

If $X \to S$ is $\boxed{\text{projective}}$ then $\mathrm{Hilb}_{X/S}$ is representable.

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Pf: ∘ Reduce to the case $X = \mathbb{P}^N_S \to S$ by some flat base-change arguments.

∘ Study $\mathrm{Hilb}_{\mathbb{P}^N_S}(S) = \{\text{subschemes } Z \subseteq \mathbb{P}^N_S\} \hookrightarrow \mathrm{Gr}\big(\mathcal{O}_S[X_0, \dots, X_N]\big)$,

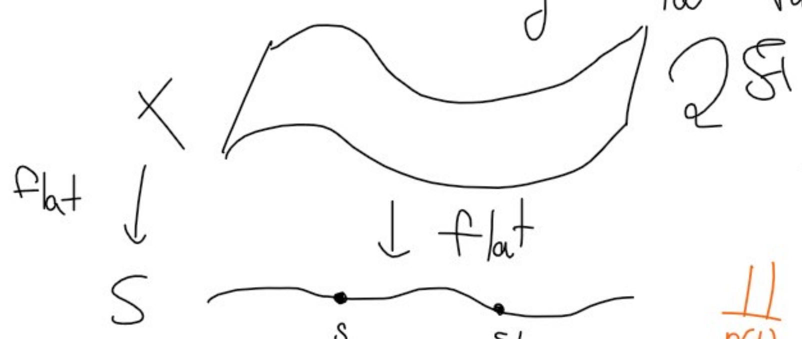
and show it's image is a closed & bounded subscheme. "□"

Def^n: Given a projective $X_{/k}$ and a coherent sheaf $\mathcal{F}$ on X, we define the $\boxed{\text{Hilbert polynomial of } \mathcal{F},}$ $P_{\mathcal{F}}(t) = \boxed{\chi(X, \mathcal{F}(t))}$.

If $\mathcal{F} = \mathcal{O}_X$, we write $P_X(t)$.

Facts: • This is a polynomial.

• It's invariant amongst flat families:



Then $P_{\mathcal{F}_s}(t) = P_{\mathcal{F}_{s'}}(t)$

$\coprod_{P(t)} \mathrm{Hilb}^{P(t)}_{X/S}$

//

- For projective $X \to S$, $\mathrm{Hilb}_{X/S}$ splits as a disjoint union of $\mathrm{Hilb}_{X/S}^{P(t)}$ of closed subschemes Z with fixed $P_Z(t) = P(t)$, (our geometric fibres.).

(- Classical result of Hartshorne says $\mathrm{Hilb}_{\mathbb{P}^N}^{P(t)}$ is connected.)

That's enough for now.

(III) Approximating $\mathcal{M}_d$ using Hilbert schemes

Given a field k and $(X, L) \in \mathcal{M}_d(k)$, then

$$P(t) = P_L(t) = \chi(X, L(t)) \overset{R.R.}{=} \frac{L(t)^2}{2} + 2 = \frac{2dt^2}{2} + 2 = \boxed{dt^2 + 2}.$$

Now Saint-Donat says $L^{\otimes 3}$ is very ample, so we have $X \hookrightarrow \mathbb{P}^N_k$ and $N = h^0(L^3) - 1$

$\mathrm{Hilb} \quad \mathrm{Hilb}^{P(3t)}$

and $N = h^0(L^3) - 1$
$= P(3) - 1$, so .... $\boxed{\mathrm{Hilb} := \mathrm{Hilb}^{P(3t)}_{\mathbb{P}^N_S/S}}$,

$\boxed{P_{L^3}(t) = P_L(3t)}$

This comes with a universal $\mathcal{Z} \hookrightarrow \mathrm{Hilb} \times \mathbb{P}^N$ (with $\mathbb{P}^N_{\mathrm{Hilb}} = \mathrm{Hilb} \times \mathbb{P}^N$) s.t. geometric fibres $\mathcal{Z}_s \subseteq \mathbb{P}^N_s$ have $P_{\mathcal{Z}_s}(t) = P(3t)$.

$$\begin{tikzcd} \mathcal{Z}_T \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Z} \arrow[d] \\ & \mathbb{P}^N_{\mathrm{Hilb}} \arrow[d] \\ T \arrow[r] & \mathrm{Hilb} \end{tikzcd}$$

<u>Prop (5.2.1)</u>: There is $H \subseteq \mathrm{Hilb}$ (open) subscheme (locally closed) s.t. $T \longrightarrow \mathrm{Hilb}$ factors through H iff:

K3 surfaces (i) $\mathcal{Z}_T \xrightarrow{f} T$, is a K3 surface over T,

nice degree 2d polarisations:

(ii) Writing $p : \mathcal{Z}_T \to \mathbb{P}^N_T$, then $p^*\mathcal{O}(1) \simeq L^3 \otimes \underline{f^* L_0}$, $L_0 \in \mathrm{Pic}(T)$, $L \in \mathrm{Pic}(\mathcal{Z}_T)$

(iii) The L in (ii) is primitive on geometric fibres, and

(iv) For all fibres $\mathcal{Z}_s$ of f, restriction yields an isomorphism
$$H^0(\mathbb{P}^N_s, \mathcal{O}(1)) \xrightarrow{\simeq} H^0(\mathcal{Z}_s, L^3_s).$$

(1) $\widetilde{H} \subseteq \mathrm{Hilb}$ sat. (i)

<u>Pf</u>: The condition that on fibres $\mathcal{Z}_{H'_s}$ is a complete nonsingular 2-dim. variety is open, so $\exists\, H' \underset{\text{open}}{\subseteq} \mathrm{Hilb}$ w/ this property. Same for $H^1(\mathcal{Z}_{H''_s}, \mathcal{O}) = 0$, $\rightsquigarrow \exists\, \boxed{H'' \underset{\text{open}}{\subseteq} \cancel{\mathrm{Hilb}}\ H'}$. $\boxed{(\Omega^2_X \simeq \mathcal{O}_X)}$

Let us introduce $\mathrm{Pic}_{X/S} : \mathrm{Sch}_S^{op} \longrightarrow \mathrm{Sets}$

$(T \to S) \longmapsto \left(\boxed{\mathrm{Pic}(X_T)/\mathrm{Pic}(T)}\right)^{\sim}$ ← étale sheafification

$\left(\mathrm{Pic}(X')/\mathrm{Pic}(T')\right)$

Fact: If $X \to S$ is projective + flat + integral geo. fibre, then $\mathrm{Pic}_{X/S}$ is representable, (and separated!)

Then define $\boxed{H_2 \underset{\text{dsd}}{\subseteq} H''}$ via the pullback

s.t. $\boxed{\Omega^2_{Z_{H_2}} \simeq \mathcal{O}_{Z_{H_2}}}$

$$\begin{tikzcd}
H_2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & H'' \arrow[d, "(\mathcal{O}_{Z_{H''}},\, \Omega^2_{Z_{H''}})"] \\
\mathrm{Pic}_{Z_{H''}/H''} \arrow[r, hook, "\Delta"] & \left(\mathrm{Pic}_{Z_{H''}/H''}\right)^{\times^2_{H''}}
\end{tikzcd}$$

Hence $H_2 \subseteq \mathrm{Hilb}$ gets us (i). Similar tricks get (ii)–(iv).

ie, more games with Picard schemes! "□"

$H \subseteq \mathrm{Hilb}$, $\quad Z_H \to H \times \mathbb{P}^N \to H$

How close are we to M_d? Hmmm, quite?

But our $Z \to H \subseteq \mathrm{Hilb}$ has a fixed embedding $\subseteq \mathbb{P}^N$, which is too much information.

K3 + (ii) – (iv)

Now, $\boxed{PGL = PGL^{N+1}}$ acts on $\boxed{\mathbb{P}^N}$, and also on

$$PGL \times \mathrm{Hilb}^{P(3t)}_{\mathbb{P}^N} \longrightarrow \mathrm{Hilb}^{P(3t)}_{\mathbb{P}^N}$$, so on $T \to S$

this is : $(\varphi, Z \subseteq T \times \mathbb{P}^N) \longmapsto \varphi(Z) \subseteq T \times \mathbb{P}^N$, where $PGL(T) = \mathrm{Aut}_S(T \times \mathbb{P}^N)$.

Now, the data (i) - (iv) about defining $H \subseteq Hilb$ is PGL - invariant. Hence, we get $PGL \circlearrowright H$!

There is a morphism of functors $\boxed{\Theta : H \longrightarrow \mathcal{M}_d}$

$$
\begin{array}{ccc}
H(T) & \xrightarrow{\quad \Theta \quad} & \mathcal{M}_d(T) \\
\parallel & & \parallel \\
\left\{ \begin{array}{l} Z \subseteq \mathbb{P}^N_T \\ \text{s.t. its a polarized} \\ \text{K3 surface} + \cdots \end{array} \right\} & & \left\{ \begin{array}{c} (X, L) \text{ polarized} \\ \text{K3 surfaces} \end{array} \right\} \\
Z & \longmapsto & (Z, L)
\end{array}
$$

Moreover, notice Θ is PGL-equivariant, so we get

$$\boxed{\overline{\Theta} : H/PGL \longrightarrow \mathcal{M}_d} \quad \text{in } \mathrm{Fun}(\mathrm{Sch}_S^{op}, \mathrm{Sets}).$$

Prop: [2.2] Θ is injective and étale locally surjective.

[This means for each $(f : X \to T, L) \in \mathcal{M}(T)$, $\exists$ an étale cover $T = \bigcup_i T_i$ such that $(f_i : X_i \to T_i, L_i) \in \mathcal{M}(T_i)$ is hit by $\Theta(T_i)$.]

Pf: ~~Injectivity~~ Surj.: Note that for
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & \mathbb{P}^N_T \arrow[d, "q"] \\ & T \end{tikzcd}$$
in $\mathcal{M}_d(T)$,

$$\boxed{f_*(L^3) \underset{(ii)}{\simeq} f_*\left(f^* L_0^\vee \otimes p^* \mathcal{O}(1)\right) \underset{(\text{proj.})}{\simeq} L_0^\vee \otimes \underbrace{f_* p^* \mathcal{O}(1)}_{= q_* p_*} \overset{???}{\underset{(iv)}{\simeq}} L_0^\vee \otimes \mathcal{O}_T^{N+1}}$$

(flat basechange?)

$f \to \overset{\vee}{T}$

So, $f_*(L^3)$ is locally (zariski) free of rank $N+1$, and so locally, so locally we have $\boxed{f_* L^3 \simeq \mathcal{O}_T^{N+1}}$. Also, L^3 is fibrewise very ample, so the adjoint $\mathcal{O}_X^{N+1} \twoheadrightarrow L^3$ yields $\in \mathbb{P}^N_T(X)$ $X \hookrightarrow \mathbb{P}^N_T$, our point in $H(T)$! Surjectivity ✓

Injectivity: Given $Z, Z' \subseteq \mathbb{P}^N_T$ in $H(T)$ with $\underbrace{(Z, L) \simeq (Z', L')}$ in $\mathcal{M}_d(T)$

then we want $\varphi \in PGL(T)$ s.t. $\varphi(Z) = Z'$.

$(f : Z \to T)$

[$Z \simeq Z'$ of K3 surfaces / $\boxed{\tau^* L' \simeq L \otimes f^* L_0}$

$\left[Z \xrightarrow[\varphi]{\simeq} Z' \right.$ of K3 surfaces w/ $\boxed{\varphi^* L' \simeq L \otimes f^* L_0}$ $L_0 \in \mathrm{Pic}(T)$

Given $Z, Z' \subseteq \mathbb{P}^N_T$, $\rightsquigarrow$ $f_* L^3$ and $f'_* L'^3$ are trivial, $\simeq \mathcal{O}_T^{N+1}$

$\rightsquigarrow \mathcal{O}_T^{N+1} \simeq f'_* L'^3 \simeq \underbrace{f_* L^3} \otimes L_0^3 \simeq \mathcal{O}_T^{N+1} \otimes L_0^3 \quad \in PGL(T)$

rep. of $\mathbb{P}^N$ $\rightsquigarrow$ $\Psi : \mathbb{P}^N_T \xrightarrow{\sim} \mathbb{P}^N_T$, $\quad \Psi(Z) = Z'$.

$\rightsquigarrow \quad Z = Z'$ inside $H(T)/PGL(T)$. $\quad \square$

In particular, for a field $k = \bar{k}$,

$$Q = H(k)/PGL(k) \xrightarrow{\simeq} \mathcal{M}_d(k) \quad \text{is a bijection.}$$

Moreover, if Q is representable by a scheme, then

Q is the coarse moduli space of $\mathcal{M}_d$.

[2.3]

(IV) Further results:

Theorem: Over $\mathbb{C}$, Q is represented by a quasi-projective scheme.
[Viehweg, '95]

Theorem: Over k w/ $\mathrm{char}(k) \geq 3$, ______ " ______ .
[Madapusi Pera, '74]

Péra, '14]

Prop: [5.1] $\dim(H) = 79 + N^2 + 2N = 78 + (9d+2)^2$.

Prop: [3.3] $\mathrm{Iso}((X,L))$ is finite.

(★) **Bonus**:

Theorem [7.2]: Over any (Noetherian) scheme S, Q is represented by an algebraic space, ie, a locally ringed space which is étale locally an affine scheme.

If we consider $\widetilde{\mathcal{M}}_d : \mathrm{Sch}_S^{op} \longrightarrow \mathrm{Groupoids}$

$$(T \longrightarrow S) \mapsto (\mathrm{pol}\,K3_T^d)^{\simeq}$$

2-category
$$\left\{ \begin{array}{l} \underline{\text{ob}}: \text{cat. with only } \cong\text{'s} \\ \underline{\text{1-mor}}: \text{Functors} \\ \underline{\text{2-mor}}: \text{Nat. transformations} \end{array} \right\}$$

Theorem [7.4]: $\widetilde{\mathcal{M}}_d$ is represented by a Deligne – Mumford stack, ie, a locally [illegible] of a Deligne-Mumford stack,

[7.4] ie, a locally ringed topos which is étale locally an affine scheme.

Most structured representability theorem.