S. Marseglia K3 Seminar 17/02/2021.

PERIODS

Ch 6, §1,2,3

PLAN FOR TODAY:

- Period Domains
- local Period map $\mathcal{P}$
- local Torelli
- Deform. theory
- global $\mathcal{P}$
- surj of $\mathcal{P}$
- global Torelli

§1 Period domains

- Λ a lattice + symmetric bilinear form

$$(\ ,\ ): \Lambda \times \Lambda \to \mathbb{Z}$$

assumed to be non-degenerate, with signature (n_+, n_-).

- We assume that $n_+ \geq 2$

- Def the period domain associated to Λ is

$$D := \{ x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,/\, \underset{\underset{(x,x)}{\parallel}}{(x)^2} = 0 , (x, \bar{x}) > 0 \}$$

- Note: $x \in \mathbb{P}(\Lambda_{\mathbb{C}})$ $\quad (x, \bar{x}) \equiv (\lambda x, \bar{\lambda} \bar{x}) = \underset{\underset{\mathbb{R}_{>0}}{\cap}}{\lambda \bar{\lambda}} (x, \bar{x})$

- Note: defines a smooth quadric in $\mathbb{P}(\Lambda_{\mathbb{C}})$.

$D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ is open inside the quadric.

• **Prop** There exists a natural bijection:

$$D \longleftrightarrow \left\{ \begin{array}{l} \text{Hodge structures of K3 type on } \Lambda \text{ s.t.} \\ \forall\, \sigma \neq 0 \text{ in } (2,0)\text{-part we have} \\ \quad i)\ (\sigma)^2 = 0 \\ \quad ii)\ (\sigma, \bar{\sigma}) > 0 \\ \quad iii)\ \Lambda^{1,1} \perp \sigma \end{array} \right\}$$

Pf

H. str. of K3 type $\Rightarrow \dim_{\mathbb{C}} \Lambda^{2,0} = 1$.

i.e a line in $\Lambda_{\mathbb{C}}$, $\sigma \cdot \mathbb{C}$.

If i), ii) hold then $\exists\, x \in D$ st $\ell_x = \sigma \cdot \mathbb{C}$

Conversely, given $x \in D$ there $\exists$ an Hodge structure with ℓ_x as $(2,0)$-part.

Considition iii) guarantees uniqueness:

$$\Lambda^{1,1} = \left(\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma) \rangle_{\mathbb{C}} \right)^{\perp}$$

Example: X a complex K3 surf.

The natural Hodge structure satisfyes i) ii) iii).

- D comes with a natural action of $O(\Lambda)$.
  The action is properly discontinuous only if $n_+ = 2$.

  $\forall x \in D$ $\exists U_x$ s.t. $g(U_x) \cap U_x \neq \phi \Rightarrow g = e$

  $D \to O(\Lambda) \setminus D$ is a covering space projection.

- If $n_+ > 2$ then $O(\Lambda) \setminus D$ is not Hausdorff.

- We assume $n_+ = 2$.
- Let $\Gamma \subset O(\Lambda)$ an arithmetic subgroup of finite index

- Prop $\exists$ a subgroup $\Gamma' \subset \Gamma$ of finite index which is torsion-free.

- Thm (Baily-Borel)
  If $\Gamma \subset O(\Lambda)$ is torsion-free, then $\Gamma \setminus D$ is a smooth quasi proj. variety.

§2 Local Period Map

- $f : X \to S$ a smooth proper family of K3 with S a connected complex manifold. with a distinguished point $o \in S$.
- Such a family is called non-isotrivial if the fibres X_t at $t \in S$ are not all isomorphic.
- The locally constant system $R^2 f_* \mathbb{Z}$ with fibre $H^2(X_t, \mathbb{Z})$ at $t \in S$ corresponds to representations of $\pi_1(S)$ acting on $H^2(X_o, \mathbb{Z})$.
- If S is simply connected then $R^2 f_* \mathbb{Z} \simeq H^2(X_o, \mathbb{Z})$
- $R^2 f_* \mathbb{Z}$ induces a flat hol. vector bundle
$$R^2 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_S \simeq R^2 f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S$$
- At $t \in S$
$$H^{2,0}(X_t) \subset H^2(X_t, \mathbb{C})$$
- Lemma: these lines glue together
$$f_* \Omega^2_{X/S} \subset R^2 f_* \mathbb{C} \otimes \mathcal{O}_S.$$

- Assume S simply connected, with marked $0 \in S$.

- Fix a <u>marking</u> of X_0 i.e. a choice of isomorphism
$$\varphi : H^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$$
$\rightsquigarrow$ induce a marking on every fibre.

- <u>Prop</u> The period map
$$\mathcal{P} : S \to \mathbb{P}(\Lambda_{\mathbb{C}})$$
$$t \mapsto [\varphi(H^{2,0}(X_t))]$$

⚠ $\mathcal{P}$ depends on X_0 & φ

is a holomorphic map with values in D

- <u>Prop</u> (Griffiths transversality)
$$d\mathcal{P}_0 : T_0 S \to T_{\mathcal{P}(0)} D \simeq \mathrm{Hom}\left(H^{2,0}(X_0), \, H^{2,0}(X_0)^{\perp} / H^{2,0}(X_0)\right)$$

can be described as the composition of the Kodaira-Spencer map $T_0 S \to H^1(X_0, \mathcal{T}_{X_0})$ and the natural map $H^1(X_0, \mathcal{T}_{X_0}) \xrightarrow{\simeq} H^1(X_0, \Omega_{X_0})$ given by contraction with a chosen $0 \neq \sigma \in H^{2,0}(X_0)$.

Review of Deformation th.

- $X \to S$ smooth + proper, $0 \in S$ distinguished $\rightsquigarrow X_0$
- If $S' \to S$, st $0' \mapsto 0$ $\rightsquigarrow$ pull-back family as the fibre-product

$$\begin{array}{ccc} X' := X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

- $X \to S$ is complete if for every other family $X' \to S'$ with $X'_0 \cong X_0$ is isomorphic to the pullback under $S' \to S$.
- If moreover $S' \to S$ is unique the $X \to S$ is called the universal deformation
- The goal of def. th. is to produce universal def's with special fibre X_0.
- If $X \to S$ is complete but only the tangent map to $S' \to S$ is unique, then $X \to S$ is called versal def.
- If the (uni)versal def exists we denote it $X \to \mathrm{Def}(X_0)$

- **Thm** – Every compact complex mf X_0 has a versal def.
  - There exist an isom. $T_0 \mathrm{Def}(X_0) \xrightarrow{\sim} H^1(X_0, T_{X_0})$
  - If $H^2(X_0, T_{X_0}) = 0$ then a smooth versal def exists.
  - If $H^0(X_0, T_{X_0}) = 0$ the a universal def exists.
  - The versal def $X \to S$ of X_0 is versal and complete for any X_{t_0} if $h^1(X_t, T_{X_t}) \equiv \text{const}$.

- **Cor** X_0 is a complex K3. Then X_0 admits a smooth univ. def $X \to \mathrm{Def}(X_0)$ with $\mathrm{Def}(X_0)$ smooth of dim 20.

Pf: $H^0 = H^2 = 0$ & $h^1 \equiv 20$. – ⊠ –

Prop (local Torelli) Let $X \to S := \mathrm{Def}(X_0)$ be the univ. def of a complex K3 X_0.

Then
$$\mathcal{P} : S \to D \subset \mathbb{P}(H^2(X_0, \mathbb{C}))$$
is a local isomorphism.

Pf Can assume S is an open disk in $\mathbb{C}^{20}$.

$h^1(X_0, T_{X_0}) \equiv 20$, so the univ. def is a univ. def for every X_t. – ⊠ –

To conclude: by Griffiths transv. $\Rightarrow$ $d\mathcal{P}_0$ is bijective.

§3 Global Period Map

We want to allow non-simply connected bases S.

- $f : X \to S$ smooth & proper, S arbitrary.
- $R^2 f_* \mathbb{Z}$ on S has fibres non-canonically isom. to
$$\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$$
- Consider the covering
$$\tilde{S} := \mathrm{Isom}(R^2 f_* \mathbb{Z}, \Lambda) \to S$$
w/ fibre $\hat{=}$ the set of Isometries $H^2(X_t, \mathbb{Z}) \simeq \Lambda$.

- The pullback of $f : X \to S$ under $\tilde{S} \to S$ yields a smooth proper family
$$\tilde{f} : \tilde{X} \to \tilde{S}$$
for which $R^2 \tilde{f}_* \mathbb{Z}$ is a constant local system
$$\rightsquigarrow R^2 \tilde{f}_* \mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}$$
- Hence the period map for $\tilde{f} : \tilde{X} \to \tilde{S}$ is well def
$$\mathcal{P} : \tilde{S} \to D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$$

- Observe that $\tilde{S} \to S$ is the natural $O(\Lambda)$-principal bundle associated w/ $R^2 \tilde{f}_* \mathbb{Z}$,
- $\tilde{S}$ has a natural $O(\Lambda)$-action w/ quotient S.
- Also: $\mathcal{P}$ is equiv. w.r.t. the natural action of $O(\Lambda)$.

$$\rightsquigarrow \quad \begin{array}{ccc} \tilde{S} & \xrightarrow{\mathcal{P}} & D \\ \downarrow & & \downarrow \\ S & \xrightarrow{\overline{\mathcal{P}}} & O(\Lambda)\backslash D \end{array}$$

- For $n_+ > 2$ the action is badly behaved and do not want to work with $\overline{\mathcal{P}}$.

- $f : X \to S$ as before.
- Assume: L is ample line bundle, fiberwise primitive.

  L induce $e \in \Gamma(S, R^2 f_* \mathbb{Z})$

  and we will consider $e^{\perp}$.
- Inside $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$

  consider $\Lambda_d := (e_1 + d f_1)^{\perp}$

  where e_1, f_1 is the standard basis of the first copy of U.
- Consider the étale cover $\tilde{S}' \to S$ parametrizing isometries $e_t^{\perp} \simeq \Lambda_d$
- $\tilde{S}' \to S$ is a principal $\tilde{O}(\Lambda_d)$-bundle

  where $\tilde{O}(\Lambda_d) := \left\{ g|_{\Lambda_d} \;/\; g \in O(\Lambda) \text{ s.t. } g(e_1 + d f_1) = e_1 + d f_1 \right\}$

- Extending $e_t^{\perp} \simeq \Lambda_d$ to $H^2(X_t, \mathbb{Z}) \simeq \Lambda$

  by sending L to $e_1 + d f_1$

$\rightsquigarrow$ defines an embedding $\tilde{S}' \hookrightarrow \tilde{S}$

  Put $\mathcal{P}_d := \tilde{S}' \hookrightarrow \tilde{S} \xrightarrow{\mathcal{P}} D$

  takes values in $D_d := D \cap \mathbb{P}(\Lambda_{d\mathbb{C}})$

- $\mathcal{P}_d$ is equiv. by the action of $\tilde{O}(\Lambda_d)$

  so

$$\begin{tikzcd}
\tilde{S}' \arrow[r, "\mathcal{P}_d"] \arrow[d] & D_d \arrow[r, hook] \arrow[d] & D \arrow[d] \\
S \arrow[r, "\overline{\mathcal{P}}_d"] & \tilde{O}(\Lambda_d) \backslash D_d \arrow[r] & O(\Lambda) \backslash D
\end{tikzcd}$$

- $\tilde{O}(\Lambda_d)$ is an arithmetic subgroup of $O(\Lambda_d)$

  By Baily - Borel

  $\tilde{O}(\Lambda_d) \backslash D_d$ is a normal quasi-proj var.

- N = moduli space of marked K3s.

$$= \left\{ \begin{array}{l} \text{isom. class of pairs } (X, \varphi) \\ X \text{ a K3}, \ \varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda \end{array} \right\}$$

- Pick a K3 X_0, w/ univ. def $\mathcal{X}_0 \to \mathrm{Def}(X_0)$.

- A marking $\varphi : H^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ induces a marking on all fibres.

- By local Torelli: $\mathrm{Def}(X_0) \hookrightarrow D$ injective.

- By universality we can glue the pairs $(\mathrm{Def}(X_0), \varphi)$ along the intersections

$$\mathrm{Def}(X_0) \cap \mathrm{Def}(Y_0) \subset D.$$

$\rightsquigarrow$ global complex structure on N.

- <u>Fact</u> $\mathrm{Aut}(X) \hookrightarrow O(H^2(X, \mathbb{Z}))$ $\forall$ K3 X.

$\Rightarrow$ glue $\mathcal{X} \to \mathrm{Def}(X_0)$ $\rightsquigarrow$ global univ. family $f : \mathcal{X} \to N$ with marking $R^2 f_* \mathbb{Z} \cong \underline{\Lambda}$

$\rightsquigarrow$ global period $\mathcal{P}: N \to D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$
which is a local isom. (local Torelli).

Thm $\mathcal{P}: N \twoheadrightarrow D$ is surjective | $\overset{!}{\nabla}$ not inj.

- Similarly

N_d moduli space of (X, L, ϕ) — X: K3, ϕ: marking, L: ample line bundle of degree $2d$.

- $\tilde{O}(\Lambda_d)$ acts on N_d &

the quotient $\tilde{O}(\Lambda_d) \backslash N_d$ parametrizes all primitively pol K3s (X, L) of degree $2d$.

- Thm (global Torelli) | $\overset{!}{\nabla}$ not surj.

Get $\mathcal{P}_d : N_d \hookrightarrow D_d$ injective

$\bar{\mathcal{P}}_d : \tilde{O}(\Lambda_d) \backslash N_d \to \tilde{O}(\Lambda_d) \backslash D$ injective.

- Cor $(X, L) \simeq (X', L') \iff H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$
$e \mapsto e'$.