The Global Torelli Theorem for K3 surfaces

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The goal is to prove:

Thm (Global Torelli):

Let X, X' be two complex K3 surfaces. Then:

$$\boxed{X \underset{\text{isomorphic}}{\cong} X' \iff H^2(X,\mathbb{Z}) \underset{\text{Hodge isometry}}{\cong} H^2(X',\mathbb{Z})}$$

Deformation theory:

Df: Two compact complex manifolds X_1, X_2 are deformation equivalent if there exists a smooth proper holomorphic morphism

$$X \to B \qquad \text{s.t.}$$

(i) B is connected
(ii) $\exists t_1, t_2 \in B$ s.t. $X_1 \cong X_{t_1}$, $X_2 \cong X_{t_2}$.

Thm: Any two complex K3 surfaces are deformation equivalent.

Proof:

Step I: If $\mathrm{Pic}(X)$ is generated by a line bundle L

s.t. $L^2 = 4$, then X is a quartic surface (see Valentijn's talk).

<u>Step II</u>: Given a K3 surface X_0, consider the universal deformation $\mathcal{X} \to Def(X_0)$. We show there exists a fiber with Pic as (i).

Fix an isometry $H^2(X, \mathbb{Z}) \simeq \Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

<u>Fact</u>: Let $0 \neq \alpha \in \Lambda$. Then:

$$\bigcup_{g \in O(\Lambda)} g(\alpha)^{\perp} \cap D \subseteq D \quad \text{is dense.}$$

· Take a primitive $\alpha \in \Lambda$ s.t. $\alpha^2 = 4$. Since the period map $P: Def(X_0) \to D$ is a local isomorphism, we conclude that:

$$P(Def(X_0)) \cap \ell^{\perp} \neq \emptyset$$

for an $\ell \in \Lambda$ with $\ell^2 = 4$, ℓ primitive.

Therefore, for a general point $t \in P(Def(X_0)) \cap \ell^{\perp}$, we have that $Pic(X_t)$ is generated by ℓ.

<u>Step III</u>: Any two smooth quartics in $\mathbb{P}^3$ are deformation equivalent. In fact, they are parametrized by:

$$\{\text{smooth quartics in } \mathbb{P}^3\} \longrightarrow \underset{\substack{\text{open}\\ \text{connected}}}{U} \subseteq |\mathcal{O}_{\mathbb{P}^3}(4)|$$

$\square$

Corollary: Every complex K3 surface is simply connected.

Proof: Deformation equivalent compact manifolds are diffeomorphic, therefore we just need to prove it for a smooth quartic, which follows by a direct computation of homotopy groups. ($\pi_1(X,\mathbb{Z}) \hookrightarrow \pi_1(\mathbb{P}^3_{\mathbb{C}},\mathbb{Z})$). $\square$

Moduli space of marked K3 surfaces:

The moduli space of marked K3 surfaces is:

$$N = \{ (X, \varphi) \} / \cong$$

where X is a K3 surface, $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\simeq} \Lambda$ an isometry.

N is a 20-dim. manifold, but it is not Hausdorff.

Proposition: The period map factors:

$$\begin{tikzcd} N \arrow[r, "P"] \arrow[d, two heads, "\varepsilon"'] & D \subseteq \mathbb{P}(\Lambda_{\mathbb{C}}) \\ \overline{N} \arrow[ur] & \end{tikzcd}$$

where $\overline{N}$ is Hausdorff, ε is a locally biholomorphic map and (X,φ), (X',φ') are mapped to the same point if and only if they are inseparable. Moreover, if (X,φ), (X',φ') are inseparable, then $X \cong X'$ and

$$P(X,\varphi) = P(X',\varphi') \subseteq \alpha^{\perp} \text{ for } 0 \neq \alpha \in \Lambda. \quad \square$$

Twistor lines:

Let Λ be any lattice of signature $(3, b-3)$.

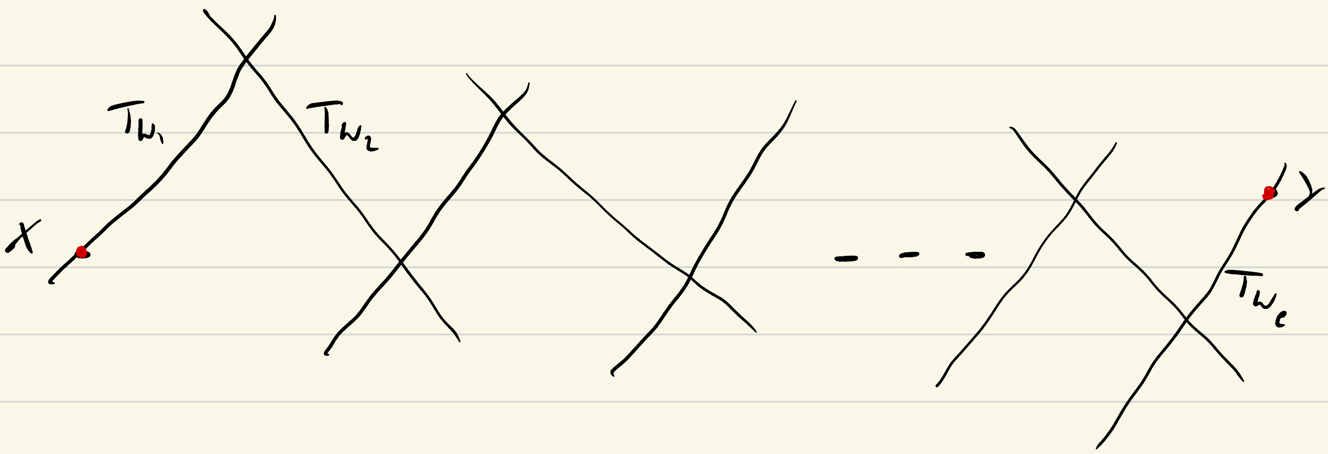
A subspace $W \subset \Lambda_{\mathbb{R}}$ is called a positive 3-space if the restriction of the pairing is positive definite. We associate the twistor line:

$$W \rightsquigarrow \underset{\mathbb{P}^1_{\mathbb{C}}}{\overset{}{\underset{\cong}{T_W}}} := D \cap \mathbb{P}(W_{\mathbb{C}}) \subseteq \underset{\mathbb{P}^2_{\mathbb{C}}}{\underset{\cong}{\mathbb{P}(W_{\mathbb{C}})}}$$

as T_W is a smooth quadric in $\mathbb{P}^2_{\mathbb{C}}$.

A twistor line T_W is called generic if $W^{\perp} \cap \Lambda = 0$ or, equivalently if $\exists\, w \in W$ with $w^{\perp} \cap \Lambda = 0$.

Def: $x, y \in D$ are equivalent if $\exists$ chain of generic twistor lines

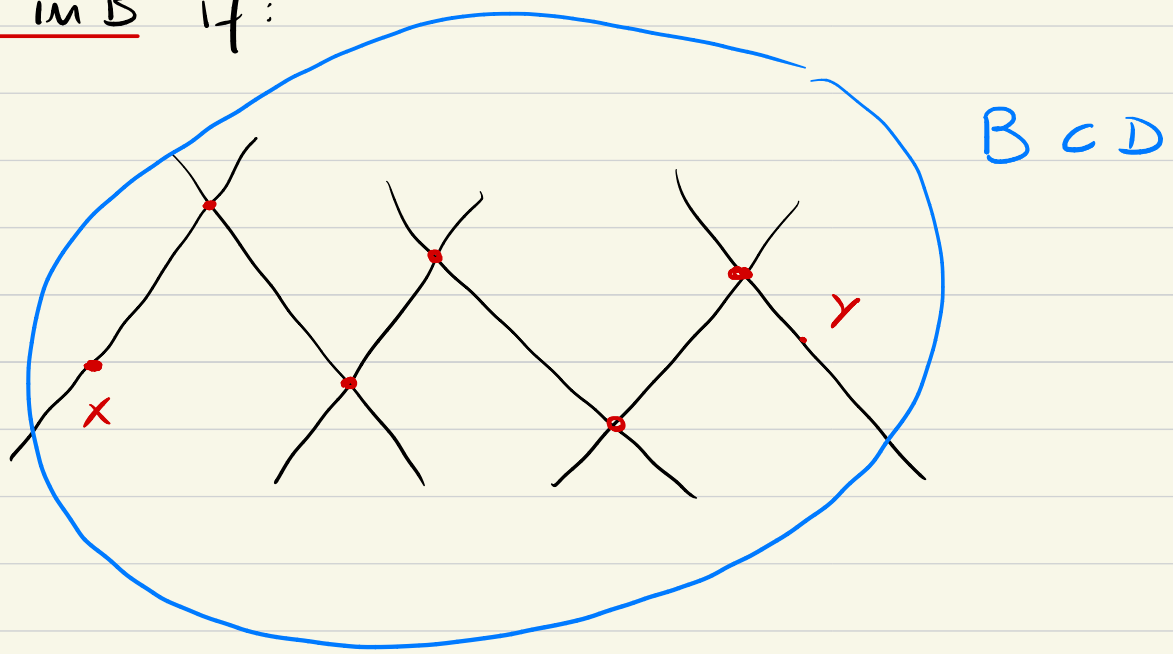


Proposition: Any two points $x, y \in D$ are equivalent. $\square$

[We just need 3 lines!]

We have a local version of the previous result.

Def: Let $B \subset D$ be a ball. $x, y \in B$ are equivalent as points in B if:



Proposition: Given $B \subset D$, all points $x, y \in B$ are equivalent as points in B.

Kähler Geometry:

Fact: Every K3 surface is a Kähler manifold (i.e. it has compatible complex, Riemannian and symplectic structures).
I g ω

[See X as (X, I)]

Thm For any Kähler class $\alpha \in H^2(X, \mathbb{R})$, $\exists$ Kähler metric g and complex structures J, K s.t.:

- g is Kähler with respect to I, J, K.
- The Kähler form $\omega_I := g(I\cdot, \cdot)$ represents α.
- We have $K = I \circ J = -J \circ I$

$\square$

In other words, every K3 surface is an hyperKähler manifold.

For each $(a,b,c) \in S^2$, $\lambda = aI + bJ + cK$ is a complex structure on X, with Kähler class $\alpha \in H^2(X,\mathbb{R})$.

Therefore we have a family of K3 surfaces (X,λ):

$$\begin{array}{ccccc} (X,\lambda) & \subseteq & \mathcal{X}(\alpha) & \simeq & X \times \mathbb{P}^1_{\mathbb{C}} \\ \downarrow & & \downarrow & \text{as manifolds} & \downarrow \\ \lambda & \in & T(\alpha) & \simeq & \mathbb{P}^1_{\mathbb{C}} \simeq S^2 \end{array}$$

Called the <u>twistor space</u>. We have the period map:

$$P : T(\alpha) \xrightarrow{\sim} T_{W_\alpha} \subseteq D \qquad \left[\sigma_I = \omega_J + i\omega_K \right]$$
$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{P}^1_{\mathbb{C}} & & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

where
$$W_\alpha := \varphi\langle [\omega_I], [\mathrm{Re}(\sigma_I)], [\mathrm{Im}(\sigma_I)] \rangle$$
$$= \varphi\Big(\underset{\substack{\cap \\ H^{1,1}(X,\mathbb{Z})_{\mathbb{R}}}}{\mathbb{R}\cdot\alpha} \oplus H^{2,0}(X)_{\mathbb{R}} \oplus H^{0,2}(X)_{\mathbb{R}} \rangle \Big)$$

We want to know when the converse is true, i.e. which twistor lines can be described with Kähler classes. In particular, we want to know which classes in $H^2(X,\mathbb{R})$ are Kähler.

Let X be a K3 surface.

$$\underbrace{K_X} \subseteq \underbrace{C_X} \subseteq H^{1,1}(X,\mathbb{R})$$

Kähler cone $\subseteq$ positive cone

$K_X :=$ component of $\{\alpha \in H^{1,1}(X,\mathbb{R});\ \alpha^2 > 0\}$ that contains one (and all) Kähler class.

<u>Proposition</u>: If $\mathrm{Pic}(X) = 0$, then $K_X = C_X$.

<u>Proof</u>: Easy consequence of a deep theorem. $\square$

<u>Proposition</u>: Let (X,φ) be a marked K3 surface s.t. $P((X,\varphi)) \subseteq Tw \subset D$ is contained in a generic twistor line. Then we can lift Tw to a curve in $\bar{N}$, i.e.:

$$\begin{tikzcd}
\bar{N} \arrow[r, "P"] & D \\
& Tw \cong \mathbb{P}^1 \arrow[u, hook, "i"'] \arrow[ul, "\tilde{i}"]
\end{tikzcd}$$

<u>Proof</u>:

<u>Step I</u>: $P: \bar{N} \to D$ is locally biholomorphic (local Torelli) therefore we can lift a small disk

$$\begin{tikzcd}
\bar{N} \arrow[r] & D \\
& \Delta \subseteq Tw \arrow[u, hook, "i"'] \arrow[ul, "\tilde{i}"]
\end{tikzcd}$$

$P(X,\varphi) \in \Delta$

The lift is unique as $\overline{N}$ is Hausdorff.

Step II: As T_W is generic, for a general $t \in \Delta$, $\mathrm{Pic}(X_t) = 0$. fix such $t \in \Delta$ and denote by:

$$H^{2,0}(X, \mathbb{Z}) = \mathbb{Z} \cdot [\sigma_t]$$

a generator. Denote the marking by $\varphi_t : H^2(X_t, \mathbb{Z}) \xrightarrow{\sim} \Lambda$, then by construction,

$$\varphi_t(\sigma_t) \in W_{\mathbb{C}} \subseteq \Lambda_{\mathbb{C}}$$

Take a class $\alpha_t \in H^2(X_t, \mathbb{Z})$ s.t. $\varphi(\alpha_t) \in W$ is orthogonal to

$$\varphi(\langle \underset{H^{2,0}(X_t)}{\underset{\cap}{\mathrm{Re}(\sigma_t)}}, \underset{H^{0,2}(X_t)}{\underset{\cap}{\mathrm{Im}(\sigma_t)}} \rangle \subset W.$$

which implies that $\alpha_t \in H^{1,1}(X_t, \mathbb{Z})$ and $\alpha_t^2 > 0$, as $\alpha \in W$.

$$\Rightarrow \pm \alpha_t \in \mathcal{C}_X = \mathcal{K}_X \Rightarrow \pm \alpha_t \text{ is a Kähler class.}$$

Consider the twistor space $\mathcal{X}(\alpha_t) \longrightarrow T(\alpha_t)$, which identifies

$$T(\alpha_t) \xrightarrow{\sim} T_W,$$

Since $\varphi_t(\alpha_t, \mathrm{Re}(\sigma_t), \mathrm{Im}(\sigma_t)) = W$.

Step III: Both $T(\alpha_t)$ and $\tilde{i}(\Delta)$ contain t and map locally isomorphically to T_W, by which we conclude, again as $\overline{N}$ is Hausdorff. $\square$

Surjectivity of the Period Map:

Thm: Let $N^0 \subseteq N$ be a connected component. Then:

$$P : N_0 \twoheadrightarrow D \subseteq \mathbb{P}(\Lambda_{\mathbb{C}})$$

is surjective.

Proof: Let $x \in P(N_0)$ be in the image. Then any other point $y \in D$ is equivalent to x, i.e. "connected" by generic twistor lines, but every twistor line can be lifted to N_0, therefore $y \in P(N_0)$. $\square$

We have a "local and stronger" result:

Proposition: The period map induces a covering space

$$P : \overline{N} \twoheadrightarrow D$$

Proof: Local version of equivalence of x, y in D. $\square$

Corollary: The period map $P : N \to D$ is generically injective (i.e. on the complement of a countable union of proper analytically closed subsets) on each connected component $N^0 \subseteq N$.

Proof: If N^0 is connected, then $\overline{N^0}$ is connected and

$\overline{N^0} \twoheadrightarrow D$ is a covering of a simply conn. space by a connected space, which implies being a homeomorphism; therefore

$$N^\circ \longrightarrow \overline{N^\circ} \xrightarrow{\simeq} D$$

[locally biholomorphism]

is generically injective $\square$

<u>Rmk</u>: If $(X,\varphi),(X',\varphi') \in N^\circ \subseteq N$ with $P(X,\varphi)=P(X',\varphi')$, then $X \cong X'$ (but they could be inseparable!).

Global Torelli Theorem:

<u>Thm</u> (Global Torelli):

Let X, X' be two complex K3 surfaces. Then:

(1) $X \cong X' \iff \exists$ Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$.

(2) Moreover, for any Hodge isometry $\psi: H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ with $\psi(K_X) \cap K_{X'} \neq \emptyset$, $\exists!\, f: X' \xrightarrow{\simeq} X$ s.t. $f^* = \psi$.

<u>Proof</u> (1):

"$\Rightarrow$" trivial; any biholomorphic map induces a Hodge isometry.

"$\Leftarrow$" Let $\psi: H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(X',\mathbb{Z})$ be a Hodge isometry. Pick any marking on X $\varphi: H^2(X,\mathbb{Z}) \xrightarrow{\simeq} \Lambda$, let $\varphi' = \varphi \circ \psi^{-1}$ be the induced marking on X'.

Then: $\quad P(X,\varphi) = P(X',\varphi')$

Notice that $-id \in O(\Lambda)$ acts trivially on the period domain D, therefore

$$P(X', \varphi') = P(X', -\varphi')$$

<u>Claim</u>: $(X, \varphi), (X', \pm \varphi') \in N^\circ \subseteq N$

are in the same connected component $N^\circ \subseteq N$.

If the Claim holds, we conclude by the Remark above. $\square$

<u>Spoiler</u>: N has two connected components, that are interchanged by $-id \in O(\Lambda)$!

<u>Remark</u>: Since any two X, X' complex K3 are deformation equivalent, $\exists$ markings φ, φ' s.t.:

$$(X, \varphi), (X', \varphi') \in N^\circ \subseteq N$$

are in the same connected component. Therefore one should just prove for <u>one</u> K3 surface X and any two markings φ_1, φ_2 that:

$$(X, \varphi_1), (X, \pm\varphi_2) \in N^\circ \subseteq N.$$

<u>Monodromy Representation</u>:

Let X be a K3 surface, $\mathcal{X} \to S$ a proper smooth family, S connected, $\mathcal{X}_t \cong X$, $t \in S$!

The monodromy representation:

$$\pi_1(S, t) \longrightarrow O(H^2(X, \mathbb{Z}))$$
$$\overset{\Lambda}{\shortparallel}$$

Let $Mon(X) \leq O(H^2(X, \mathbb{Z}))$ be the subgroup generated by all monodromies.

<u>Proposition</u>: If

$$O(H^2(X, \mathbb{Z})) / Mon(X) = \langle -id \rangle$$

then $\mathcal{N}$ has at most two components.

<u>Sketch</u>: Any two markings of X differ by an orthogonal transformation, hence by hypothesis

$$\exists\, \psi = \pm\, \varphi_1^{-1} \circ \varphi_2 \in Mon(X)$$

Need to show: (X, φ), $(X, \varphi_1 \circ \psi)$ in the same component.

Write $\quad \psi = \psi_1 \cdot \dots \cdot \psi_n, \quad \psi_i \in Im(\pi_1(S_i, t_i) \to O(H^2(X, \mathbb{Z})))$.

For every S_i, the local system can be locally trivialized:

$$R^2 \pi_* \mathbb{Z} \cong \underline{H}^2(X, \mathbb{Z}) \qquad (\text{locally})$$

The monodromy ψ_i is obtained by following the trivializations along a closed path in S beginning in $t \in S$.

"The classifying map to $\mathcal{N}$ necessarily stays in the same connected component."

$\square$

Def: Let $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$, $s_\delta \in O(\Lambda_{\mathbb{R}})$ a reflection:

$$s_\delta : v \longmapsto v - 2\frac{(v.\delta)}{(\delta)^2}\delta \quad , \delta^2 \neq 0, \ \delta \in \Lambda_{\mathbb{R}}$$

The spinor norm is:

$$\begin{cases} +1 & \text{if } \delta^2 < 0 \\ -1 & \text{if } \delta^2 > 0 \end{cases}$$

and is extended by linearity to all $g \in O(\Lambda_{\mathbb{R}})$. Let

$$O^+(\Lambda) \subset O(\Lambda)$$

be the index 2 subgroup with spinor norm $+1$.

Proposition: $\mathrm{Mon}(X) = O^+(H^2(X, \mathbb{Z}))$

In particular, $O(H^2(X,\mathbb{Z}))/_{\mathrm{Mon}(X)} = \{\pm 1\}$

$\square$