Moduli of sheaves (general)

§ X proj. scheme/ $k = \bar{k}$

$\mathcal{O}_X(1)$ very ample line bdl.

Def. For B base scheme;
a flat family of coherent sheaves over B is a coherent sheaf $\mathcal{F}$ on $X \times B$ s.t. $\mathcal{F}$ is <u>B-flat</u>.

Def. Let $\mathcal{E}$ be a coh. sheaf on X;

$$\sum_i (-1)^i h^i(\mathcal{E}(t))$$

$$P_{\mathcal{E}}(t) := \chi(X, \mathcal{E}(t)) = \sum_{i=0}^{d} \frac{1}{i!} \alpha_i(\mathcal{E})\, t^i$$

where $\alpha_d(\mathcal{E}) \neq 0$

$P_\mathcal{E}(t)$ is the Hilbert poly of $\mathcal{E}$.

Reduced Hilb. poly. :

$$p_\mathcal{E}(t) = \frac{P_\mathcal{E}(t)}{\alpha_d(\mathcal{E})}.$$

Facts: $\ast$ $d = \dim(\mathcal{E})$
$:= \dim \mathrm{Supp}(\mathcal{E})$

$\mathrm{Supp}(\mathcal{E}) := \{x \in X : \mathcal{E}_x \neq 0\}$

$\ast$ $\mathcal{F}$ B-flat family

then

$\{P_{\mathcal{F}_b}(t) \mid b \in B\}$ locally constant.

$\mathcal{F}_b := \mathcal{F}\big|_{X \times \{b\}}$

Def Moduli: M : (Sch/$\mathbb{C}$)^{o} → Sets

Def: Define: $\underline{M}_P$: (Sch/k)^{op} → Sets

fix poly.

$$\underline{M}_P(B) := \left\{ [\mathcal{F}] \;\middle|\; \begin{array}{l} \mathcal{F} \text{ } B\text{-flat family,} \\ P_{\mathcal{F}_b} = P \quad \forall b \in B \end{array} \right\}$$

$\mathcal{F} \sim \mathcal{F}' \iff \mathcal{F} \cong \mathcal{F}' \boxtimes L \quad \exists L \in \mathrm{Pic}(B)$

$\underline{M}_P$: moduli functor.

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§ Special case: $B = \operatorname{Spec} A$

w/ $A \in \mathrm{Artin}/k$

$\quad$ local Artinian k-alg. w/ residue field k

covariant

$$\mathrm{Artin}/k \xrightarrow{\mathcal{D}_{[E]}} \mathrm{Sets}$$

$$A \longmapsto \left\{ [F] \in \underline{M}_P(\operatorname{Spec}(A)) \;\middle|\; \right.$$

$$\mathcal{F}\Big|_{X \times \operatorname{Spec} k} \cong \mathcal{E} \Big)$$

for some $\mathcal{E} \in \underline{M}_p(\operatorname{Spec} k)$

$\mathcal{D}_{[\mathcal{E}]}$: deformation functor.

First interesting case:

$A = \frac{k[\epsilon]}{(\epsilon^2)} =: D$, then

$\mathcal{D}_{[\mathcal{E}]}(D)$: first order deformations.

$\|$ fact

$\operatorname{Ext}^1_X(\mathcal{E}, \mathcal{E})$.

"Sketch of pf":

Given $\mathcal{F}$ 1st order def.;

$$0 \to k \xrightarrow{\cdot \epsilon} D \xrightarrow{ev_0} k \to 0$$

apply $- \otimes_D F$

$0 \to C(E) \to D \to D/C(E) \to 0$

T

$$\Rightarrow \quad 0 \to \underbrace{\mathcal{F} \otimes_D k}_{\cong\, \mathcal{E}} \to \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

Given : s.e.s.

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{E} \longrightarrow 0$$

$\mathcal{F}$ ↑ $\mathcal{O}_X$-module

for $\mathcal{F}$ to be a $\mathcal{O}_X \otimes D$ module : define $\epsilon : \mathcal{F} \to \mathcal{F}$

$$\epsilon = \alpha \circ \beta \, .$$

$\square$

Higher order :

small extensions

$$0 \to \mathfrak{a} \to A' \to A \to 0 \quad \text{s.e.s.}$$

where
- $A, A' \in \mathrm{Art}/k$
- $\mathfrak{a} \cdot \mathfrak{m}' = 0$ (*) , $\mathfrak{a}$ principal

Q1: Given $\mathcal{F} \in \mathcal{D}_{[\mathcal{E}]}(A)$

$\exists\, \mathcal{F}' \in \mathcal{D}_{[\mathcal{E}]}(A')$: $\mathcal{F}' \otimes_{A'} A \cong \mathcal{F}$?

Q2: If lifts exist, can we classify them ?

Q2: (fact)

$\{$lifts$\}$ form a $\mathrm{Ext}^1_X(\mathcal{E},\mathcal{E}) \otimes_k \Omega$ torsor.

Q1: $\exists\, ob_{\mathcal{F}} \in \mathrm{Ext}^2_X(\mathcal{E},\mathcal{E}) \otimes_k \Omega$:

$ob_{\mathcal{F}} = 0 \iff$ lifts exist.

"Sketch": $\mathcal{E}$ simple ($\underline{\underline{\mathrm{End}(\mathcal{E})}} \cong k$)

Given: $F \in \mathcal{D}_{\mathcal{E}}(A)$

$$0 \to \Omega \to A' \to A \to 0$$

small extension

We have:

$$0 \to \mathfrak{m} \to A \to k \to 0 \qquad \boxed{*}$$
$$0 \to \mathfrak{m}' \to A' \to k \to 0$$
$$0 \to \mathfrak{a} \to A' \to A \to 0$$

$$\rightsquigarrow \quad 0 \to \mathfrak{a} \to \mathfrak{m}' \to \underline{\underline{\mathfrak{m}}} \to 0 \qquad (*)$$

is short exact sequence of $A \cong \cancel{A'}/\mathfrak{a}$ - modules
(b/c $\mathfrak{a} \cdot \mathfrak{m}' = 0$)

$\boxed{*}$:

$$0 \to \mathcal{F} \otimes_A \mathfrak{m} \to \mathcal{F} \to \mathcal{E} \to 0$$

$\rightsquigarrow$: apply $\mathrm{Hom}(-, \mathcal{E} \otimes_k \mathfrak{a})$

$$\cdots \to \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{E} \otimes_k \mathfrak{a}) \ni \xi$$
$$\to \mathrm{Ext}^1_X(\mathcal{F} \otimes_A \mathfrak{m}, \mathcal{E} \otimes_k \mathfrak{a})$$
$$\to \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) \otimes_k \mathfrak{a} \to \cdots \ni ob_{\mathcal{F}} \quad \underline{\underline{\text{def}}}$$

$(*)$: apply $\mathcal{F} \otimes_A -$:

$$0 \to F \otimes_A \mathfrak{m} \to F \otimes_A \mathfrak{m}' \to F \otimes_A \mathfrak{m}'' \to 0$$

$$\begin{matrix} \wr\wr \\ \xi \otimes_k \mathfrak{m} \end{matrix}$$

defines ξ

Upshot: if $d_F = 0$, then

$\exists$ $0 \to \xi \otimes_k \mathfrak{m} \to \boxed{F'} \to F \to 0$

mapping to ξ

Fact: F' can be made onto an A'-module.

(Artamkin).

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§

<u>Def.</u> Given a contravariant functor: $\underline{M} : (Sch/_k)^{o} \to Sets$

We say that $\underline{M}$ is representable

if $\exists$ ...

If $M \in Sch/k$

$\exists\ \Phi : \underline{M} \xrightarrow{\sim} Hom(-, M)$

natural transf.

If so: * M fine moduli space for $\underline{M}$

$$* \quad \begin{array}{ccc} Hom(M, M) & \xleftarrow{\sim} & \underline{M}(M) \\ \Psi & & \Psi \\ id_M & \longleftrightarrow & \mathcal{U} \end{array}$$ univ. fam.

$\forall\ F \in \underline{M}(B)\ \exists!\ f : B \to M$:

$f^*\mathcal{U} = F$

$$\begin{array}{ccc} \underline{M}(B) & \longrightarrow & Hom(B, M) \\ \uparrow\ \underline{M}(f) =: f^* & & \uparrow - \circ f \\ \underline{M}(M) & \longrightarrow & Hom(M, M) \\ \mathcal{U} & & id_M \end{array}$$

of fine moduli space

E.g. 1) $Gr(r, V)$, $\dim_k V = n$

families / B:

$$V \otimes_k \mathcal{O}_B \twoheadrightarrow \underset{\substack{\text{locally free} \\ \text{w/ fibres of} \\ \text{dim. } n-r}}{\underbrace{Q}}$$

2) $\mathcal{E} \in Coh(X)$, P some polyn.

$$Quot_X(\mathcal{E}, P) := \{[\mathcal{E} \twoheadrightarrow Q] \mid P_Q = P\} / \cong$$

families / B:
(on $X \times B$)

$$\mathcal{E} \boxtimes \mathcal{O}_B \twoheadrightarrow \underset{\substack{B\text{-flat}, \\ P_{Q_b} = P \quad \forall b \in B.}}{\underbrace{Q}}$$

Grothendieck, $Quot_X(\mathcal{E}, P)$

is a proj. scheme.

We are interested in $\underline{M}_P$.

It is not representable!

E.g.

$X = \operatorname{Spec} k$, $P = r > 1$

$$\underline{M}_P(B) = \{[\mathcal{F}] \mid \mathcal{F} \text{ locally free over } B, \text{ of rk } r\}$$

(Take $\mathcal{F}$ indecomposable loc. free sheaf of rk r / B.)

Suppose $\underline{M}_P$ is representable:

$$\underline{M}_P \xrightarrow[\sim]{\Phi} \operatorname{Hom}(-, M)$$ for some M.

$\underline{M}_p(\mathrm{Spec}(k)) = \{[\mathcal{O}]\} - \{*\}$

Note: on B:

$$\underline{M}_B(B) \xrightarrow[\Phi_B]{\sim} \mathrm{Hom}(B, M)$$

$$\Phi_B([\mathcal{O}_B^n]) : \begin{tikzcd} B \arrow[rr] \arrow[dr] & & M \\ & \{*\} \arrow[ur] & \end{tikzcd}$$

$\Phi_B([\mathcal{F}])$:

Let $U_\alpha \subseteq B$ open cover:

$\mathcal{F}|_{U_\alpha} \cong \mathcal{O}^n_{U_\alpha}$, then

$$\Phi_B([\mathcal{F}])|_{U_\alpha} : \begin{tikzcd} U_\alpha \arrow[rr] \arrow[dr] & & M \\ & \{*\} \arrow[ur] & \end{tikzcd} \quad \forall \alpha$$

$$\Rightarrow \Phi_B([\mathcal{F}]) = \Phi_B([\mathcal{O}^n])$$

$$\Rightarrow \quad \mathcal{F} \cong \mathcal{O}_B^n$$

$\underline{\text{Def}}$. Given: $\underline{M} : (Sch/k)^{\circ} \to Sets$.

$\underline{M}$ is co-representable if

$\exists$ $\Phi : \underline{M} \Rightarrow Hom(-, M)$

s.t. for any $\Phi' : \underline{M} \Rightarrow Hom(-, M')$

$\exists!$ $M \xrightarrow{f} M'$:

$$\begin{tikzcd} \underline{M} \arrow[r, Rightarrow, "\Phi"] \arrow[dr, Rightarrow, "\Phi'"'] & Hom(-, M) \arrow[d, Rightarrow, "f\circ-"] \\ & Hom(M') \end{tikzcd}$$

If so: we call M a

coarse moduli space.

M is unique up to unique iso.

Suppose: $\Sigma', \Sigma'' / X$

$\boxed{\xi} \in \mathrm{Ext}^1_X(\mathcal{E}', \mathcal{E}'') \setminus \underline{0}$ coh.

$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$

$\boxed{0}$: $\quad 0 \to \mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}'' \to \mathcal{E}'' \to 0$

take line thru $0, \xi$: $\mathbb{A}^1$

$\rightsquigarrow$ flat family $\mathcal{F}$ over $\mathbb{A}^1$:

$\mathcal{F}_0 \cong \mathcal{E}' \oplus \mathcal{E}''$

$\mathcal{F}_b \cong \mathcal{E} \quad \forall b \in \mathbb{A}^1 \setminus 0$

If $\underline{M}_p$ is corepr.

i.e. $\Phi : \underline{M}_p \xrightarrow{\sim} \mathrm{Hom}(-, M)$

suppose $\Phi_{\mathrm{Spec}\, k}$ bij.

$\Phi_{\mathbb{A}^1} : \underline{M}_p(\mathbb{A}^1) \to \mathrm{Hom}(\mathbb{A}^1, M)$

$\mathcal{F} \rightsquigarrow \Phi_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \xrightarrow[f]{} M$ mph.

$$f(t) = [\Sigma] \quad \forall t \neq 0$$

$$f(0) = [\Sigma' \oplus \Sigma'']$$

continuity $\Rightarrow$ $\Sigma \cong \Sigma' \oplus \Sigma''$.