Four lectures on stable pairs *(Hong Kong, April 2013)*

I would like to thank Professors N. C. Leung, K. Chan, and Z. Hua for the opportunity to visit the Chinese University of Hong Kong. In the coming four lectures, I would like to speak about stable pair theory on surfaces and applications. In the first lecture, I will survey part of the general theory of stable pairs developed by R. Pandharipande and R. P. Thomas. In the second lecture, we will specialize to stable pairs on surfaces and discuss perfect obstruction theories and virtual fundamental classes. In the third and fourth lecture, I will discuss several applications: (1) curve counting on surfaces and the Göttsche-Yau-Zaslow formula, (2) cases of the GW/PT correspondence, and (3) Seiberg-Witten invariants. The latter uses work of H.-l. Chang.

1 Stable pair theory *(after Pandharipande-Thomas)*

Enumerative geometry is a beautiful topic with a rich history. The area saw a lot of activity in the 19th century by work of H. Schubert and others. A typical problem is the determination of the Severi degrees. Let $S = \mathbb{P}^2$ and consider δ-nodal curves $C \subset S$. I.e. reduced planar curves with exactly δ nodes and no further singularites. In the linear system $|\mathcal{O}(d)|$ of all degree d divisors, the δ-nodal curves form a locally closed subset $V_{d,\delta}$ of codimension δ. Its closure (compactification) $\overline{V}_{d,\delta}$ is known as a Severi variety. The number of δ-nodal curves of degree d going through

$$\dim(\overline{V}_{d,\delta}) = \dim(|\mathcal{O}(d)|) - \delta = \frac{d^2+3d}{2} - \delta$$

points on $\mathbb{P}^2$ in general position is equal to the degree $N_{d,\delta}$ of $\overline{V}_{d,\delta}$. These numbers are known as Severi degrees. We come back to this in lecture 3.

The following are more modern moduli spaces of curve-like objects:

(i) Let X be a smooth projective variety, $\beta \in H_2(X)$ a curve class, and $g, m \geq 0$. Suppose we are interested in finding a moduli space of maps $f : C \to X$ with C a smooth genus g curve with m marked points and $f_*[C] = \beta$. Unless we allow for certain "degenerations" of f, we cannot expect the moduli space to be compact. It is well-known that the stack $\overline{M}_{g,m}(X, \beta)$ of stable maps $f : C \to X$, where C is a connected nodal curve of arithmetic genus g with m marked points satisfying $f_*[C] = \beta$ is a proper Deligne-Mumford stack[1]. Note that f is not just an embedding, but can have degree > 1, can contract components etc.

(ii) Instead of looking at curves as being parametrized as in (i), we can also look at them from the perspective of defining equations like in the case of Severi varieties. I.e. we can consider subschemes $C \subset X$ of reduced curves with $[C] = \beta$. Again, we need to allow "degenerations" to allow for a compact moduli space. It is well-known that the moduli space $I_\chi(X, \beta)$ of ideal sheaves $I \subset \mathcal{O}_X$ of subschemes $C \subset X$ of dimension ≤ 1 such that $[C] = \beta$ and $\chi(\mathcal{O}_C) = \chi$ is a projective coarse moduli scheme. Note that C can have components of dimension 0 and 1, and can be very complicated (it can be non-reduced, have embedded components etc.).

[1] There is also a coarse moduli scheme version of this stack, but the stack has the advantage of having a universal curve over it.

I will talk about a more recent moduli space of curve-like objects, namely stable pairs. The theory is developed in [PT1]. This first lecture is an exposition of part of their work.

1.1 The moduli space

Let X be a smooth projective variety with polarization L. A pair (F, s) on X consists of[2]

(i) F is pure dimension 1 sheaf on X. This means F is a coherent sheaf and the support of any subsheaf $0 \neq E \subset F$ is 1-dimensional. In particular, the support of F is 1-dimension.

(ii) $s \in H^0(F)$ a section.

We can only expect a moduli space of finite type if we fix some topological parameters. Therefore, we fix a curve class $\beta \in H_2(X)$, an integer $n \in \mathbb{Z}$, and require $[C_F] = \beta$ and $\chi(F) = \chi$, where C_F is the scheme-theoretic support of F. We can only expect the moduli space to be realizable as a GIT quotient if we impose a stability condition. The Hilbert polynomial of F is

$$\chi(F(t)) = t \int_\beta c_1(L) + \chi =: P_F(t), \text{ where } F(t) := F \otimes L^{\otimes t}.$$

Fix $q(t) \in \mathbb{Q}[t]$ a polynomial with positive leading coefficient. A pair (F, s) is called (semi)stable (w.r.t. L, q) if for any proper subsheaf $0 \neq G \subset F$ we have

$$\frac{P_G(t)}{r(G)} (\leq) \frac{P_F(t) + q(t)}{r(F)}, \ \forall t \gg 0$$

and if $s : \mathcal{O}_X \to F$ factors through G we require

$$\frac{P_G(t) + q(t)}{r(G)} (\leq) \frac{P_F(t) + q(t)}{r(F)}, \ \forall t \gg 0.$$

Here $r(F), r(G)$ are the linear coefficients of $P_F(t)$, $P_G(t)$, which are > 0 by purity of F.

A projective fine moduli space $P_\chi^{L,q}(X, \beta)$ of such objects has been constructed by J. Le Potier [LeP]. For q linear, this simplifies: (F, s) is semistable if and only if (F, s) is stable if and only if s has 0-dimensional cokernel [PT1, Lem. 1.3]. Note that this is independent of L. We denote the resulting moduli space by $P_\chi(X, \beta)$ and refer to its elements as *stable pairs*. From now on, we consider this moduli space only.

Given a stable pair (F, s), we can form an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow Q \longrightarrow 0.$$

We can say more about the sheaf K. We claim the scheme theoretic support of $\operatorname{im} s$ is C_F [PT1, Lem. 1.6] so $K = I_{C_F}$. The proof is as follows. Since the question is local, we can work over an open affine subset $\operatorname{Spec} R \subset X$. The scheme theoretic supports of F and $\operatorname{im} s$ are $\operatorname{Ann} F$ and $\operatorname{Ann} s(1)$. Clearly, $\operatorname{Ann} F \subset \operatorname{Ann} s(1)$. Moreover

[2] Le Potier's coherent systems are more general [LeP].

Ann $s(1).F \subset F$ must have 0-dimensional support since coker s has 0-dimensional support, so Ann $s(1).F = 0$ by purity.

Note that purity of F implies C_F is Cohen-Macaulay; i.e. it is pure dimensional with no embedded components. Also note that Q is a 0-dimensional sheaf supported on C_F. If you are optimistic, you might think Q is always the structure sheaf of a 0-dimensional subscheme $Z \hookrightarrow C_F$ and a stable pair is nothing but a pair (Z, C) with C a Cohen-Macaulay curve and $Z \hookrightarrow C$ a 0-dimensional subscheme. This is not far off and certainly gives the right intuition, but it is not true in general. Let me give a few examples of stable pairs.

(i) The simplest example comes from a closed embedding of a 1-dimensional subscheme $\iota : C \hookrightarrow X$. Then $\mathcal{O}_X \to \iota_* \mathcal{O}_C$ is surjective and $\mathcal{O}_C$ is pure if and only if C is Cohen-Macaulay. So Cohen-Macaulay curves give rise to stable pairs.

(ii) Let C be a Cohen-Macaulay curve and $D \subset C$ an effective Cartier divisor. In this case, we have an induced canonical section $s_D \in H^0(\mathcal{O}(D))$, which induces a stable pair $\mathcal{O}_X \to \iota_* \mathcal{O}_C \to \iota_* \mathcal{O}_C(D)$. This is the prototype stable pair.

(iii) Let $C = \{xy = 0\} \subset \mathbb{C}^2$ be the node and let $C_1 = \{y = 0\} \subset C$ and $C_2 = \{x = 0\} \subset C$. Denote by $\iota_i : C_i \hookrightarrow C$ the closed embedding. For points $p_i \in C_i$, the maps $\mathcal{O}_{C_i}(-p_i) \hookrightarrow \mathcal{O}_{C_i}$ induce a stable pair

$$\mathcal{O}_C \longrightarrow \iota_{1*}\mathcal{O}_{C_1}(p_1) \oplus \iota_{2*}\mathcal{O}_{C_2}(p_2).$$

At the level of modules, and after some identifications, this is

$$\mathbb{C}[x, y]/(xy) \xrightarrow{(x\ y)} \mathbb{C}[x, y]/(y) \oplus \mathbb{C}[x, y]/(x).$$

The cokernel Q is the 3-dimensional vector space spanned by, for example, $(1, 0)$, $(0, 1)$, $(0, y)$. As a $\mathbb{C}[x, y]/(xy)$-module it does not have one generator so cannot be a structure sheaf.

1.2 Deformation theory

From now on, X is a 3-fold. The original motivation of Pandharipande and Thomas was to consider $P_\chi(X, \beta)$ as a union of components of the moduli space of objects in $D^b(X)$ (the derived category of bounded complexes of coherent sheaves on X). To a stable pair (F, s) one can associate a 2-term complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$$

concentrated in degrees 0,1. This complex has determinant $\mathcal{O}_X$. For two stable pairs (F, s), (F', s') and associated complex $I^\bullet$, $I'^\bullet$, one has: (F, s), (F', s') are isomorphic if and only if $I^\bullet$, $I'^\bullet$ are quasi-isomorphic [PT1, Prop. 1.21].

The deformation and obstruction spaces are:

$$\begin{aligned} (F, s) &: \ \mathrm{Ext}^0(I^\bullet, F), \mathrm{Ext}^1(I^\bullet, F), \\ I^\bullet &: \ \mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \mathrm{Ext}^2(I^\bullet, I^\bullet)_0. \end{aligned}$$

where the latter are the deformation and obstruction space for deforming the complex $I^\bullet$ keeping the determinant $\mathcal{O}_X$ fixed. Here $(\cdot)_0$ stands for trace free part. Consider the exact triangle

$$F[-1] \longrightarrow I^\bullet \longrightarrow \mathcal{O}_X.$$

This induces maps

$$\mathrm{Ext}^i(I^\bullet, F) \longrightarrow \mathrm{Ext}^{i+1}(I^\bullet, I^\bullet)_0 \subset \mathrm{Ext}^{i+1}(I^\bullet, I^\bullet).$$

The following is a non-trivial theorem [PT1, Thm. 2.7].

Theorem 1.1 (Pandharipande-Thomas). *Let $0 \to \mathfrak{a} \to A \to A_0 \to 0$ be a small extension of local Artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$ and let (F_0, s_0) be an unobstructed family of stable pairs over $B_0 = \mathrm{Spec}\ A_0$. Let $I_0^\bullet = \{\mathcal{O}_{X\times B_0} \stackrel{s_0}{\to} F_0\} \in D^b(X \times B_0)$. Then*

$$\mathrm{Ext}^0(I_0^\bullet, F) \otimes_{\mathbb{C}} \mathfrak{a} \longrightarrow \mathrm{Ext}^1(I_0^\bullet, I_0^\bullet)_0 \otimes_{\mathbb{C}} \mathfrak{a}$$

is an isomorphism, so $I_0^\bullet$ is unobstructed and its extensions over $B = \mathrm{Spec}\ A$ are in 1-1 correspondence with the extensions of (F_0, s_0) over B.

A few comments: Pandharipande and Thomas prove this for any square zero extension of schemes $B \supset B_0$. Note that in the case $A_0 = \mathrm{Spec}\ \mathbb{C}$ and $A = \mathrm{Spec}\ \mathbb{C}[\epsilon]/(\epsilon^2)$, there is no obstruction (e.g. trivial deformations exist). The map in this theorem is induced by

$$\mathrm{Ext}^0(I^\bullet, F) \longrightarrow \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$$

discussed above, which is therefore an isomorphism and can be seen as an isomorphism of Zariski tangent spaces. This theorem implies $P_\chi(X, \beta)$ can be seen as an open subset of the moduli space of complexes in $D^b(X)$ with determinant $\mathcal{O}_X$.

1.3 Perfect obstruction theory and stable pair invariants

The spaces $\mathrm{Ext}^0(I^\bullet, F), \mathrm{Ext}^1(I^\bullet, F)$ and $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ can be combined in families as follow. Let $\mathbb{I}^\bullet = \{\mathcal{O} \to \mathbb{F}\}$ be the universal stable pair on $P \times X$, where $P := P_\chi(X, \beta)$. Using projection $\pi : P \times X \to P$, we can form

$$R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{F}),\ \ R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0,$$

which over a closed point $I^\bullet = \{\mathcal{O}_X \to F\} \in P$ just gives $R\,\mathrm{Hom}(I^\bullet, F)$ and $R\,\mathrm{Hom}(I^\bullet, I^\bullet)_0$.

Ignoring derived categories completely, it is not unreasonable to hope that the complex $R\,\mathrm{Hom}(I^\bullet, F)$ gives rise to a perfect obstruction theory[3] on P

$$R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{F})^\vee \stackrel{?}{\longrightarrow} \mathbb{L}_P.$$

Here $\mathbb{L}_P = \tau_{\geq -1} L_P^\bullet$ is the truncated cotangent complex of P. Unfortunately, this does not seem to exist. However, we have the following theorem [PT1, Thm. 2.14].

[3]If you do not know what a perfect obstruction theory or virtual cycle is, do not worry. In the next lecture I will explain these notions in very down to earth terms.

Theorem 1.2 (Pandharipande-Thomas)**.** *There exists a perfect obstruction theory of the form $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0^\vee \to \mathbb{L}_{P_\chi(X,\beta)}$ and hence a virtual fundamental class*

$$[P_\chi(X,\beta)]^{vir} \in A_{vd}(P_\chi(X,\beta)), \text{ where } vd = \int_\beta c_1(X).$$

Note that the situation is similar in the case of Donaldson-Thomas theory. There, the moduli space $I_\chi(X,\beta)$ is a Hilbert scheme of curves on X and the deformation and obstruction space of 1-dimensional closed subschemes $C \subset X$ are

$$\mathrm{Ext}^0(I_C, \mathcal{O}_C), \ \mathrm{Ext}^1(I_C, \mathcal{O}_C).$$

However, the deformation and obstruction space of the ideal sheaf I_C are

$$\mathrm{Ext}^1(I_C, I_C), \ \mathrm{Ext}^2(I_C, I_C).$$

The latter are used to define the Donaldson-Thomas perfect obstruction theory on $I_\chi(X,\beta)$ [Tho].

In the case X is Calabi-Yau, we define the stable pair (or Pandharipande-Thomas) invariants of X to be

$$PT_{\chi,\beta}(X) := \int_{[P_\chi(X,\beta)]^{vir}} 1 \in \mathbb{Z}.$$

For any 3-fold X, one defines stable pair invariants by integrating the cup product of cohomology classes of total degree vd.

Why are they called invariants? Suppose we have a smooth projective family $\mathcal{X} \to B$ of 3-folds over a smooth base B. By working over base B instead of Spec $\mathbb{C}$, one can construct the moduli space $P_n(\mathcal{X}/B,\beta) \to B$ of stable pairs in the fibres of $\mathcal{X} \to B$. This admits a relative perfect obstruction theory over B and virtual cycle

$$[P_\chi(\mathcal{X}/B,\beta)]^{vir} \in A_{vd+\dim B}(P_\chi(\mathcal{X}/B,\beta)).$$

For any $b \in B$, let $i_b : P_\chi(\mathcal{X}_b,\beta) \hookrightarrow P_\chi(\mathcal{X}/B,\beta)$ be inclusion. One can show that $i_b^![P_\chi(\mathcal{X}/B,\beta)]^{vir} = [P_\chi(\mathcal{X}_b,\beta)]^{vir}$ for all $b \in B$ [PT1, Thm. 2.15]. This implies *deformation invariance*

$$PT_{\chi,\beta}(\mathcal{X}_b) = PT_{\chi,\beta}(\mathcal{X}_{b'}), \ \forall b, b' \in B.$$

1.4 Conjectures

What are the relations of stable pair invariants to existing curve counting theories? On any smooth projective variety X and for any curve class $\beta \in H_2(X)$, the moduli stacks $\overline{M}_{g,m}(X,\beta)$ carry the Gromov-Witten perfect obstruction theory [Beh, LT1]. In the case X is a Calabi-Yau 3-fold and $m = 0$, the virtual cycle is 0-dimensional and we can define Gromov-Witten invariants

$$GW_{g,\beta}(X) := \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}.$$

On any smooth projective 3-fold X, the moduli space $I_\chi(X,\beta)$ carries the Donaldson-Thomas perfect obstruction theory [Tho]. Again, in the case X is a Calabi-Yau 3-fold, the virtual cycle is 0-dimensional and we can define Donaldson-Thomas invariants

$$DT_{\chi,\beta}(X) := \int_{[I_\chi(X,\beta)]^{vir}} 1 \in \mathbb{Z}.$$

We have three curve counting theories

$$GW_\beta(X) := \sum_g GW_{g,\beta}(X)u^{2g-2},\ DT_\beta(X) := \sum_\chi DT_{\chi,\beta}(X)q^\chi,\ PT_\beta(X) := \sum_\chi PT_{\chi,\beta}(X)q^\chi.$$

These are all formal Laurent series. The famous MNOP conjecture (or GW/DT correspondence) [MNOP1] states:

Conjecture 1.3 (GW/DT correspondence)**.** *$DT'_\beta(X)$ is a rational function function of q invariant under $q \leftrightarrow q^{-1}$ and $DT'_\beta(X) = GW'_\beta(X)$ for $-q = e^{iu}$.*

There are two subtleties. (1) The prime in $GW'_\beta(X)$ means we should allow stable maps with disconnected domain curve but no contracted connected components. (2) $DT'_\beta(X) = DT_\beta(X)/DT_0(X)$ where $DT_0(X)$ is the point part of DT theory. For stable pairs, we have the following conjecture [PT1, Conj. 3.2, 3.3].

Conjecture 1.4 (GW/PT correspondence)**.** *$PT(X)$ is a rational function function of q invariant under $q \leftrightarrow q^{-1}$ and $PT_\beta(X) = GW'_\beta(X)$ for $-q = e^{iu}$.*

It should be noted that all these conjectures can be stated for general 3-folds with insertion classes [PP2]. Combining these conjectures predicts a DT/PT correspondence: $DT'_\beta(X) = PT_\beta(X)$. This has been proved by T. Bridgeland [Bri] (see also Y. Toda [Tod] for the Euler characteristic version). The GW/DT correspondence was proved in the toric case by [MOOP]. Recently, many new cases (including all complete intersections in products of projective spaces) of the GW/PT correspondence have been proved by R. Pandharipande and A. Pixton [PP1, PP2].

Finally, we mention the Gopakumar-Vafa invariants, which we will come back to in lecture 4. As for any collections of Laurent series with rational coefficients indexed by non-zero effective curve classes, the following system of equations

$$\Big\{GW_\beta(X) = \sum_{g\in\mathbb{Z}} \sum_{\substack{\beta = d\beta' \\ d>0}} \frac{n^{GW}_{g,\beta'}}{d}\Big(2\sin(du/2)\Big)^{2g-2}\Big\}_{\beta\neq 0 \text{ effective}}$$

has a unique solution $n^{GW}_{g,\beta'} \in \mathbb{Q}$. Note that these are just identities of formal power series with no geometric content. However, according to R. Gopakumar and C. Vafa [GV1, GV2], the numbers $n^{GW}_{g,\beta'}$ have an interpretation in M-theory and are *integer*. The latter prediction is known as the Gopakumar-Vafa integrality conjecture. Likewise, on the stable pair side, we can write

$$\Big\{PT^{conn}_\beta(X) = \sum_{g\in\mathbb{Z}} \sum_{\substack{\beta = d\beta' \\ d>0}} (-1)^{(d+1)(g-1)}\frac{n^{PT}_{g,\beta'}}{d}\Big(q^{\frac{d}{2}} + (-1)^{d+1}q^{-\frac{d}{2}}\Big)^{2g-2}\Big\}_{\beta\neq 0 \text{ effective}}.$$

Here $PT^{conn}_\beta(X)$ is the *connected* stable pair generating series which is obtained from

$$\log\Big(1 + \sum_{\beta\neq 0 \text{ effective}} PT_\beta(X)v^\beta\Big).$$

Again, the numbers $n^{PT}_{g,\beta'}$ are just defined through identities of formal power series. However, from the fact that stable pair invariants are integer, Pandharipande and Thomas prove [PT1, Thm. 3.19]:

Theorem 1.5 (Pandharipande-Thomas)**.** *The numbers $n^{PT}_{g,\beta'}$ are all integer.*

2 Stable pairs on surfaces

From now on, we study stable pair theory in the case $X = K_S$ is the total space of the canonical bundle of a smooth projective surface S. This is joint work with R. P. Thomas and some parts also with D. Panov [KT1, KT2].

2.1 Perfect obstruction theories

Let M be a $\mathbb{C}$-scheme of finite type. In general, M can have many (possibly non-reduced) irreducible components of different dimension, so its fundamental class $[M] \in A_*(M)$ is not a very well-behaved object. We should expect this to be the case for most moduli spaces M by Murphy's Law of R. Vakil [Vak]. Now suppose $M = s^{-1}(0) \subset A$, where A is a smooth projective ambient variety and s is a section of a rank r vector bundle E on A.

I: Regular case. Suppose the section s is regular, i.e. $\operatorname{codim}(M) = r$. In this case, we have a nice formula for the fundamental class

$$\iota_*[M] = c_r(E),$$

where $\iota : M \hookrightarrow A$ is inclusion. Let $I \subset \mathcal{O}_A$ be the ideal sheaf of M, then $I/I^2|_M$ is locally free and

$$N_{M/A} := (I/I^2|_M)^* \cong E|_M.$$

However, a word of warning: it might still be the case that the right exact sequence

$$I/I^2|_M \xrightarrow{d} \Omega_A|_M \longrightarrow \Omega_M \longrightarrow 0$$

is *not* left exact. An example is $A = \mathbb{A}^1$ and $M = \{x^2 = 0\} \subset A$. Then M is cut out by a regular section, but the element $0 \neq x^3 \in I/I^2|_M$ maps to $3x^2dx = 0 \in \Omega_A|_M$.

II: Regular and smooth case. Suppose M is *in addition* smooth. Then the above sequence is also left exact. After dualizing, we get the familiar short exact sequence of locally free sheaves

$$0 \longrightarrow T_M \longrightarrow T_A|_M \longrightarrow N_{M/A} \longrightarrow 0.$$

In this case, one gets a formula for the topological Euler characteristic of M

$$e(M) = \int_M c_{top}(T_M) = \int_M \frac{c_\bullet(T_A)}{c_\bullet(E)}\bigg|_M = \int_A \iota_*[M]\frac{c_\bullet(T_A)}{c_\bullet(E)} = \int_A c_r(E)\frac{c_\bullet(T_A)}{c_\bullet(E)}. \qquad (1)$$

This formula will play a key role in the proof of the Göttsche conjecture in the final lecture.

Now suppose all we have is $M = s^{-1}(0) \subset A$ for a section s of a vector bundle E on a smooth projective variety A, but s need not be regular. In this case, we still have an interesting commutative diagram (D)

$$\begin{tikzcd} E^*|_M \arrow[r, "d\circ s^*"] \arrow[d, "s^*"'] & \Omega_A|_M \arrow[d, equal] \\ I/I^2|_M \arrow[r, "d"] & \Omega_A|_M, \end{tikzcd}$$

where $s : \mathcal{O}_A \to E$ induces $s^* : E^* \to I$. The bottom row is the truncated cotangent complex of M (concentrated in degrees $-1, 0$)

$$\mathbb{L}_M := \tau_{\geq -1} L_M^\bullet \cong \{I/I^2|_M \to \Omega_A|_M\}.$$

As an object of $D^b(M)$, $\mathbb{L}_M$ is independent of choice of embedding $\iota : M \to A$ into a smooth ambient variety. The top complex $E^\bullet = \{E^*|_M \to \Omega_A|_M\}$ is a 2-term complex of locally free sheaves concentrated in degrees $-1, 0$. The diagram provides a morphism $\phi : E^\bullet \to \mathbb{L}_M$ such that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism. Such data is known as a perfect obstruction theory[4] on M.

Definition 2.1 (Behrend-Fantechi)**.** Let M be a $\mathbb{C}$-scheme of finite type, $E^\bullet$ a 2-term complex of locally free sheaves on M concentrated in degrees $-1, 0$, and $\phi : E^\bullet \to \mathbb{L}_M$ a morphism in $D^b(M)$ such that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism. Then $\phi : E^\bullet \to \mathbb{L}_M$ is called a perfect obstruction theory on M. $\oslash$

This definition does not need the embedding into a smooth ambient variety, though every perfect obstruction theory is locally of the form (D).

The first great thing about a perfect obstruction theory on M is that it encodes the deformation theory of objects in M. Let $0 \to \mathfrak{a} \to A \to A_0 \to 0$ be a small extension of local Artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Set $B_0 := \mathrm{Spec}\, A_0$, $B := \mathrm{Spec}\, A$, and suppose we have a morphism $f : B_0 \to M$. The fundamental questions in deformation theory are: when does f lift to a morphism $\overline{f} : B \to M$?; if a lift exists, can one classify all such lifts? These questions can be answered using $L_M^\bullet$, but also using $E^\bullet$ [BF, Thm. 4.5].

Theorem 2.2 (Behrend-Fantechi)**.** *Suppose $\phi : E^\bullet \to \mathbb{L}_M$ is a perfect obstruction theory. Let the setup be as above, then there exists a natural obstruction class*

$$\mathfrak{o}(f) \in h^1(f^* E^{\bullet\vee}) \otimes \mathfrak{a}$$

and it vanishes if and only if a lift exists. Moreover, if a lift exists, then the collection of lifts is a torsor under $h^0(f^* E^{\bullet\vee}) \otimes \mathfrak{a}$.

Two remarks are in order. Firstly, the theorem of Behrend and Fantechi is actually more general and works for any square zero extension. Moreover, their theorem has a converse too.

The second great thing about a perfect obstruction theory is that it provides a virtual fundamental class.

Theorem 2.3 (Behrend-Fantechi)**.** *Suppose $\phi : E^\bullet \to \mathbb{L}_M$ is a perfect obstruction theory. Then there exists a natural class $[M]^{vir} \in A_{vd}(M)$, where $vd := \mathrm{rk}\, E^\bullet = \mathrm{rk}\, E^0 - \mathrm{rk}\, E^{-1}$ is the virtual dimension.*

[4]The original definition [BF] calls this a perfect obstruction theory with global resolution. Moreover, the original definition [BF] works with the full cotangent complex, but the whole construction works using the truncated cotangent complex only. Obstruction theories and virtual cycles were discovered by J. Li and G. Tian [LT1], and K. Behrend and B. Fantechi [BF]. We follow the appoach of the latter.

Let us go back to the setting $M = s^{-1}(0) \subset A$ with s not necessarily regular. In this case, there are two important formulae for the virtual fundamental class

$$\begin{aligned} [M]^{vir} &= \{c_\bullet(E|_M) s_\bullet(C_{M/A})\}_{vd}, \\ \iota_*[M]^{vir} &= c_r(E), \end{aligned}$$

where $C_{M/A}$ is the normal cone of $M \subset A$ [Ful], which equals $N_{M/A}$ in the case of a regular section. The first formula is due to B. Siebert [Sie] and V. Pidstrigach [Pid]. In the case of a regular section, it gives $[M]^{vir} = [M]$. In the case M is smooth but not necessarily of the expected dimension and the obstruction sheaf $Ob := h^1(E^{\bullet\vee})$ is locally free, it gives

$$[M]^{vir} = \left\{ \frac{c_\bullet(E|_M)}{c_\bullet(N_{M/A})} \right\}_{vd} = \left\{ c_\bullet(T_M) \frac{c_\bullet(E|_M)}{c_\bullet(T_A|_M)} \right\}_{vd} = c_{top}(Ob) \cap M,$$

where the last equality follows from the exact sequence

$$0 \longrightarrow h^0(E^{\bullet\vee}) \longrightarrow T_A|_M \longrightarrow E|_M \longrightarrow Ob \longrightarrow 0,$$

and $h^0(E^{\bullet\vee}) \cong T_M$.

2.2 Hilbert schemes of curves

Let S be a smooth projective surface and $\beta \in H_2(S)$. Denote by $H_\beta := \mathrm{Hilb}_\beta(S)$ the Hilbert scheme of effective divisors on S with class β. The Abel-Jacobi map is

$$H_\beta \longrightarrow \mathrm{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(C)],$$

where $\mathrm{Pic}^\beta(S)$ is the Picard variety of isomorphism classes of line bundle L on S with $c_1(L) = \beta$. The latter is a complex torus of dimension $q(S) = h^{0,1}(S)$ (irregularity of the surface). The fibre over $[L] \in \mathrm{Pic}^\beta(S)$ is the linear system $|L|$. Over the locus of line bundle L with $h^1(L) = h^2(L) = 0$, this is a projective bundle of dimension

$$\chi(\beta) - 1 + q(S) = \frac{\beta(\beta - k)}{2} + p_g(S),\ k := c_1(\mathcal{O}(K_S)) \in H^2(S, \mathbb{Z}),$$

where $p_g(S) = h^{0,2}(S)$ is the geometric genus of S. However, this is only the expected dimension of H_β. In general, H_β is not of expected dimension and could be singular. However, we can always embed H_β in a smooth Hilbert scheme as follows.

Fix a sufficiently ample divisor $A \subset S$ such that $h^1(L) = h^2(L) = 0$ for all $L \in \mathrm{Pic}^\gamma(S)$, where $\gamma := [A] + \beta$. Then H_γ is a projective bundle over $\mathrm{Pic}^\gamma(S)$. Hence H_γ is smooth. Moreover, we have a closed embedding

$$H_\beta \hookrightarrow H_\gamma,\ C \mapsto A + C.$$

(In what follows, I often identify H_β with its image under this map.) We claim H_β is the zero locus of a section of a tautological sheaf on H_γ. An element $D \in H_\gamma$ lies in the image of $H_\beta \hookrightarrow H_\gamma$ if and only if

$$s_D|_A = 0 \in H^0(\mathcal{O}_A(D)),$$

where $s_D : \mathcal{O} \to \mathcal{O}(D)$ is the canonical section. The family version of this is as follows. Let $\mathcal{D} \subset H_\gamma \times S$ be the universal curve and let $\pi : H_\gamma \times A \to H_\gamma$ be projection. The sheaf $F := \pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ has a tautological section s coming from $\mathcal{O} \to \mathcal{O}(\mathcal{D})$ and it is not hard to see that $H_\beta \cong s^{-1}(0)$. However, it is not a priori clear whether F is a vector bundle. Assume

$$h^2(L) = 0 \text{ for all effective } L \in \mathrm{Pic}^\beta(S), \ \text{(AS1)}.$$

For any $C \in H_\beta$, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}(A+C) \longrightarrow \mathcal{O}_A(A+C) \longrightarrow 0.$$

Ampleness of A and (AS1) imply $h^1(\mathcal{O}_A(A+C)) = 0$. Continuity and base change imply $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) = 0$ in a Zariski open neighbourhood of H_β inside H_γ. Note that we can only conclude F is a vector bundle on a Zariski open neighbourhood of H_β. However, diagram (D) still provides a perfect obstruction theory $F^{red\bullet} \to \mathbb{L}_{H_\beta}$ on H_β.

At this stage it is not clear whether this perfect obstruction theory on H_β is independent of A. Let us consider what $F^{red\bullet}$ looks like over a point $p = C \in H_\beta$. Consider the short exact sequence

$$0 \longrightarrow I_A \longrightarrow \mathcal{O}_{A+C} \longrightarrow \mathcal{O}_A \longrightarrow 0, \ I_A = \mathcal{O}_C(-A).$$

The associated long exact sequence is

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_C(C)) &\longrightarrow H^0(\mathcal{O}_{A+C}(A+C)) \longrightarrow H^0(\mathcal{O}_A(A+C)) \\ &\longrightarrow \ker\{H^1(\mathcal{O}_C(C)) \to H^1(\mathcal{O}_{A+C}(A+C))\} \longrightarrow 0. \end{aligned}$$

The short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(A+C) \longrightarrow \mathcal{O}_{A+C}(A+C) \longrightarrow 0$$

gives $H^1(\mathcal{O}_{A+C}(A+C)) \cong H^2(\mathcal{O}_S)$ and the induced map $\phi : H^1(\mathcal{O}_C(C)) \to H^2(\mathcal{O}_S)$ is exactly the semi-regularity map coming from the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0, \ \text{where } N_{C/S} \cong \mathcal{O}_C(C).$$

The above exact sequence reduces to

$$0 \longrightarrow TH_\beta|_p \longrightarrow TH_\gamma|_p \longrightarrow F|_p \longrightarrow \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\} \longrightarrow 0.$$

Hence

$$\begin{aligned} h^0(F^{red\bullet\vee}|_p) &\cong H^0(N_{C/S}), \\ h^1(F^{red\bullet\vee}|_p) &\cong \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\}. \end{aligned}$$

We indeed get the right deformation space for deforming an effective divisor $C \subset S$ keeping S fixed. However, it might come as a surprise that the obstruction class lies in $\ker\phi$. This is actually well-known (in a much more general context of lci subvarieties) by work of S. Bloch [Blo], Z. Ran [Ran], D. Iacono and M. Manetti [IM], and others. We come back to this topic in the next lecture. This perfect obstruction theory was first described by M. Dürr, A. Kabanov, and C. Okonek [DKO]. It also occurs (independently) in our work on stable pair invariants [KT1, Appendix].

Theorem 2.4 (Dürr-Kabanov-Okonek, K-Panov-Thomas)**.** *For all S, β satisfying (AS1), there exists a reduced perfect obstruction theory $F^{red\bullet} \to \mathbb{L}_{H_\beta}$ on H_β with virtual cycle $[H_\beta]^{red} \in A_{vd}(H_\beta)$ of virtual dimension $vd = \frac{\beta(\beta-k)}{2} + p_g(S)$.*

We mention another perfect obstruction theory on H_β, which can be constructed directly without use of an ambient space [DKO]. Let $\mathcal{C} \subset H_\beta \times S$ be the universal divisor and let $\pi : H_\beta \times S \to H_\beta$ be projection.

Theorem 2.5 (Dürr-Kabanov-Okonek)**.** *There exists a perfect obstruction theory $F^\bullet := (R\pi_*\mathcal{O}_\mathcal{C}(\mathcal{C}))^\vee \to \mathbb{L}_{H_\beta}$ on H_β with virtual cycle $[H_\beta]^{vir} \in A_{vd}(H_\beta)$ of virtual dimension $vd = \frac{\beta(\beta-k)}{2}$.*

This perfect obstruction theory will come back in the last lecture when we discuss relations to Seiberg-Witten invariants and [CK]. For now, it suffices to note that these two perfect obstruction theories are compatible via the semiregularity map

Proposition 2.6. *The semiregularity map induces an exact triangle*

$$\begin{tikzcd} H^2(\mathcal{O}_S)^* \otimes \mathcal{O}_{H_\beta}[1] \arrow[r] & F^\bullet \arrow[r] \arrow[dr] & F^{red\bullet} \arrow[d] \\ & & \mathbb{L}_{H_\beta} \end{tikzcd}$$

and the diagram commutes.

2.3 Stable pairs on surfaces

How are the previous virtual cycles relevant for stable pairs on surfaces? In the previous lecture, I mentioned stable pairs can intuitively be thought of as pairs (Z, C) with C a Cohen-Macaulay curve and $Z \subset C$ a 0-dimensional closed subscheme. However, we also saw that the cokernel Q of a stable pair (F, s) need not be a structure sheaf. Nevertheless, on surfaces we have the following [PT3, Prop. B.8].

Theorem 2.7 (Pandharipande-Thomas)**.** *Let S be a surface, $\beta \in H_2(S)$ a curve class, and let $2h - 2 = \beta(\beta + k)$ be the arithmetic genus of divisors with class β. Let $\mathcal{C} \to H_\beta$ be the universal curve. Then*

$$P_\chi(S, \beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta),$$

where $\chi = 1 - h + n$ and $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ is the relative Hilbert scheme parametrizing 0-dimensional subschemes of length n on the fibres of $\mathcal{C} \to H_\beta$.

Let me sketch the isomorphism. Given a stable pair (F, s), we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F|_C \longrightarrow Q|_C \longrightarrow 0$$

on C, where $C := C_F$ is the scheme theoretic support of F. (Since we are on a surface, C is not just Cohen-Macaulay but even Gorenstein.) Dualizing gives a long exact sequence

$$0 \longrightarrow (Q|_C)^* \longrightarrow (F|_C)^* \longrightarrow \mathcal{O}_C \longrightarrow \mathscr{E}xt^1(Q|_C, \mathcal{O}_C) \longrightarrow \mathscr{E}xt^1(F|_C, \mathcal{O}_C) \longrightarrow 0.$$

Here $(Q|_C)^*$ is zero since $\mathcal{O}_C$ is pure and Q is 0-dimensional. One can show that purity implies $\mathscr{E}xt^1(F|_C, \mathcal{O}_C) = 0$. Therefore $\mathscr{E}xt^1(Q|_C, \mathcal{O}_C)$ is the structure sheaf of a 0-dimensional closed subscheme Z of C. *Conclusion: on a surface, stable pairs are pairs (Z, C) with $C \subset S$ a Gorenstein curve and $Z \subset C$ a 0-dimensional closed subscheme.*

In the case $n = 0$, we discussed how to get perfect obstruction theories on $P_\chi(S, \beta) \cong \mathrm{Hilb}^0(\mathcal{C}/H_\beta) \cong H_\beta$. Now let us consider the case $n > 0$. Let $S^{[n]}$ be the Hilbert scheme of n points on S. We first observe that $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ is naturally an incidence scheme inside $S^{[n]} \times H_\beta$

$$\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta.$$

Again, we can realize $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ as the zero locus of a section of a vector bundle on $S^{[n]} \times H_\beta$. This works as follows. For any pair (Z, C), we have $Z \subset C$ if and only if

$$s_C|_Z = 0 \in H^0(\mathcal{O}_Z(C)),$$

where $s_C : \mathcal{O} \to \mathcal{O}(C)$ is the canonical section. The family version goes as follows. Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme. Consider

$$\mathcal{O}_{\mathcal{Z} \times H_\beta}(\mathrm{Hilb}^n(S) \times \mathcal{C}) \text{ on } S^{[n]} \times S \times H_\beta.$$

We denote this sheaf somewhat sloppily by $\mathcal{O}_{\mathcal{Z}}(\mathcal{C})$. Pushing forward to $S^{[n]} \times H_\beta$ gives a rank n vector bundle $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$ with fibre $H^0(\mathcal{O}_Z(C))$ over (Z, C). It is not hard to construct a tautological section cutting out $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$.

However, this does *not* automatically give a perfect obstruction theory on $S^{[n]} \times H_\beta$. The Hilbert scheme of points $S^{[n]}$ is smooth of expected dimension, but this does not need to be the case for H_β! Therefore, this construction only gives a *relative* perfect obstruction theory on $P_S := P_\chi(S, \beta) \cong S^{[n]} \times H_\beta$

$$\mathcal{E}^\bullet := \{(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C}))^*|_{P_S} \to \pi^*_{S^{[n]}}(\Omega_{S^{[n]}})|_{P_S}\} \to \mathbb{L}_{P_S/H_\beta},$$

where $\pi_{S^{[n]}} : S^{[n]} \times H_\beta \to S^{[n]}$ is projection. All these perfect obstruction theories can be put together:

Theorem 2.8 (K-Panov-Thomas). *Consider $\pi : P_S \to H_\beta$. There exist natural morphisms $\mathcal{E}^\bullet[-1] \to \pi^* F^\bullet$, $\mathcal{E}^\bullet[-1] \to \pi^* F^{red\bullet}$, such that the following diagrams commute*

$$\begin{tikzcd} \mathcal{E}^\bullet[-1] \arrow[r] \arrow[d] & \pi^* F^\bullet \arrow[d] \\ \mathbb{L}_{P_S/H_\beta}[-1] \arrow[r] & \pi^* \mathbb{L}_{H_\beta} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^\bullet[-1] \arrow[r] \arrow[d] & \pi^* F^{red\bullet} \arrow[d] \\ \mathbb{L}_{P_S/H_\beta}[-1] \arrow[r] & \pi^* \mathbb{L}_{H_\beta}. \end{tikzcd}$$

Consequently, the mapping cones $E^\bullet$, $E^{red\bullet}$ define perfect obstrucion theories on P_S

$$\begin{tikzcd} \pi^* F^\bullet \arrow[r] \arrow[d] & E^\bullet \arrow[r] \arrow[d] & \mathcal{E}^\bullet \arrow[d] \\ \pi^* \mathbb{L}_{H_\beta} \arrow[r] & \mathbb{L}_{P_S} \arrow[r] & \mathbb{L}_{P_S/H_\beta} \end{tikzcd} \qquad \begin{tikzcd} \pi^* F^{red\bullet} \arrow[r] \arrow[d] & E^{red\bullet} \arrow[r] \arrow[d] & \mathcal{E}^\bullet \arrow[d] \\ \pi^* \mathbb{L}_{H_\beta} \arrow[r] & \mathbb{L}_{P_S} \arrow[r] & \mathbb{L}_{P_S/H_\beta} \end{tikzcd}$$

of virtual dimensions $\frac{\beta(\beta-k)}{2} + n$ and $\frac{\beta(\beta-k)}{2} + p_g(S) + n$. We denote the corresponding virtual cycles by $[P_S]^{vir}$, $[P_S]^{red}$.

Note that the reduced theories are only defined when (AS1) is satisfied, whereas the non-reduced theory is always defined. Our next task is to describe how these two perfect obstruction theories are related to the ones of the previous lecture.

3 Universality

3.1 Reducing

In this lecture, we relate the stable pair theory on $X = K_S$ of lecture 1 to the perfect obstruction theories of lecture 2. Consider the moduli space of stable pairs $P_\chi(X, \beta)$ with $\beta \in H_2(S)$ a curve class pushed forward along the zero section $S \subset X$. The associated perfect obstruction theory for stable pairs is

$$E^\bullet_{PT,X} := R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0^\vee \longrightarrow \mathbb{L}_{P_\chi(X,\beta)}, \ vd = \int_\beta c_1(X) = 0.$$

There are two issues. Firstly, X is non-compact, so $P_\chi(X, \beta)$ need not be compact. We can deal with this by lifting the $\mathbb{C}^*$-action on the fibres of X to $P_\chi(X, \beta)$ and using the localization formula of T. Graber and R. Pandharipande [GP] to define our invariants. However, there is a more fundamental issue.

Suppose we have a smooth deformation $(\mathcal{S} \to B, 0)$ of S over a smooth simply connected base B with central fibre $\mathcal{S}_0 \cong S$. The Poincaré dual of β lies in $H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ and can be transported to any fibre $\mathcal{S}_b$ to give a class $\beta_b \in H^2(\mathcal{S}_b, \mathbb{Z})$. However, there is no reason for β_b to still be of Hodge type $(1, 1)$. The Noether-Lefschetz locus of β is defined as

$$NL_\beta = \{b \in B \mid \beta_b \text{ is of type } (1,1)\} \subset B.$$

It is a complex analytic closed subset. Whenever $NL_\beta \subset B$ is positive codimensional, we can deform S to a surface where β is not of type $(1, 1)$ anymore. Suppose this is the case and consider the induced deformation of canonical bundles $(\mathcal{X} \to B, 0)$. Then by deformation invariance all Gromov-Witten and stable pair invariants of X and S with class β are zero. In order to get interesting Gromov-Witten and stable pair invariants, one should remove part of the obstruction bundle corresponding to deformations of X, S outside the Noether-Lefschetz locus. This problem has been considered by many people. Let me only mention a few key references. For Gromov-Witten theory on Kähler surfaces (differential geometry setting): [Don, Lee]. For Gromov-Witten theory on abelian and K3 surfaces: [BL1, BL2, Li]. For stable pair theory on canonical bundles of K3 surfaces: [MPT].

I will now present a quite general approach to this problem. Let me show how to "reduce" Gromov-Witten theory on a surface S. The method also works for Gromov-Witten and stable pair theory on $X = K_S$. We make an assumption

$$H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S) \text{ is surjective, (AS2)}.$$

For any $C \subset S$, the map $\cup\beta$ factors through the semi-regularity map

$$H^1(T_S) \longrightarrow H^1(T_S|_C) \longrightarrow H^1(N_{C/S}) \longrightarrow H^2(\mathcal{O}_S),$$

so (AS2) implies (AS1). A typical case when neither (AS1) nor (AS2) is satisfies is when $\beta = k$ and $|K_S| \neq \varnothing$. In this case, the usual Gromov-Witten/stable pair theory is non-trivial. We come back to the non-reduced theories in lecture 4. Next, let $(\mathcal{S} \to B, 0)$ be a versal deformation of S over a smooth base B. (A versal deformation over a smooth base need not exist. However, all we really need is a *first order* versal deformation, which does always exist. We will not go into the details of this.) For any family $(\mathcal{S} \to B, 0)$ (not necessarily versal) and for any $\beta \in H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$ (not necessarily satisfying (AS1)) consider

$$T_0 B \xrightarrow{KS_0} H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S),$$

where KS_0 is the Kodaira-Spencer map. The kernel of this composition is $T_0 NL_\beta$. Our assumptions imply this composition is surjective so $T_0 NL_\beta \subset T_0 B$ has codimension $p_g(S)$. In fact, one can show this implies $NL_\beta \subset B$ is smooth of codimension $p_g(S)$ at 0. Therefore, we can take a small polydisc $\Delta \subset B$ of dimension $p_g(S)$ intersecting NL_β transversally at 0. Let $(\mathcal{T} \to \Delta, 0)$ be the induced family. Let $\overline{M}_{g,m}(\mathcal{T}/\Delta, \beta)$ be the (relative) moduli stack of stable maps mapping to the fibres of $\mathcal{T} \to \Delta$. One can show that the natural inclusion

$$\overline{M}_{g,m}(S,\beta) \hookrightarrow \overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)$$

is an isomorphism of stacks. At the level of $\mathbb{C}$-points, this map is clearly a bijection since the central fibre of $\mathcal{T} \to \Delta$ is (by construction) the only fibre where β is of type $(1,1)$. As for any smooth family, $\mathcal{T} \to \Delta$ has a relative Gromov-Witten theory

$$\mathcal{E}^\bullet_{GW,S} \longrightarrow \mathbb{L}_{\overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)/\Delta}.$$

There is a standard procedure to make a relative perfect obstruction theory over a smooth base absolute (cf. [KT1, Sect. 2]). Applied to this setting, this gives a perfect obstruction theory on $\overline{M}_{g,m}(S,\beta)$

$$E^{red\bullet}_{GW,S} \longrightarrow \mathbb{L}_{\overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)} \cong \mathbb{L}_{\overline{M}_{g,m}(S,\beta)},$$

which we refer to as a reduced Gromov-Witten theory.

Why is this the right theory? One can show that the corresponding invariants are only invariant under deformations of S *inside* the Noether-Lefschetz locus of β. Over an embedded curve $C \subset S$ corresponding to a point $p \in \overline{M}_{g,m}(S,\beta)$, the deformation and obstruction space are

$$h^0(E^{red\bullet\vee}_{GW,S}|_p) \cong H^0(N_{C/S}),\ h^1(E^{red\bullet\vee}_{GW,S}|_p) \cong \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\},$$

where $\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)$ is the semi-regularity map as before. These are the same deformation and obstruction space as for the perfect obstruction theory $F^{red\bullet}$ on H_β in the previous lecture. Note the difference with the usual Gromov-Witten theory

$$h^0(E^{\bullet\vee}_{GW,S}|_p) \cong H^0(N_{C/S}),\ h^1(E^{\bullet\vee}_{GW,S}|_p) \cong H^1(N_{C/S}).$$

Theorem 3.1 (K-Thomas)**.** *Let S,β satisfy (AS2) and let $X = K_S$. Then there exist natural reduced obstruction theories*

$$E^{red\bullet}_{GW,S} \longrightarrow \mathbb{L}_{\overline{M}_{g,m}(S,\beta)},$$
$$E^{red\bullet}_{GW,X} \longrightarrow \mathbb{L}_{\overline{M}_{g,m}(X,\beta)},$$
$$E^{red\bullet}_{PT,X} \longrightarrow \mathbb{L}_{P_\chi(X,\beta)},$$

and their virtual dimensions are $p_g(S)$ bigger than the virtual dimensions of the non-reduced theories.

3.2 Localizing

The moduli spaces $\overline{M}_{g,m}(X,\beta)$, $P_\chi(X,\beta)$ are in general non-compact. Therefore, we cannot define invariants directly on them. However, the natural action of $\mathbb{C}^*$ on the fibres of X lifts to $\overline{M}_{g,m}(X,\beta)$, $P_\chi(X,\beta)$ and the fixed loci are compact

$$\begin{aligned}\overline{M}_{g,m}(X,\beta)^{\mathbb{C}^*} &\cong \overline{M}_{g,m}(S,\beta),\\ P_\chi(X,\beta)^{\mathbb{C}^*} &\cong P_\chi(S,\beta) \sqcup \text{other components}.\end{aligned}$$

Here "other components" contain stable pairs thickened in the fibre direction.

Let M be a compact $\mathbb{C}$-scheme of finite type with $\mathbb{C}^*$-action. Assume M is $\mathbb{C}^*$-equivariantly embeddable in a smooth variety. Let $E^\bullet \to \mathbb{L}_M$ be a $\mathbb{C}^*$-equivariant perfect obstruction theory. Then each component $M_i^{\mathbb{C}^*} \subset M^{\mathbb{C}^*}$ of the fixed point locus has a natural induced perfect obstruction theory $(E^\bullet|_{M_i^{\mathbb{C}^*}})^f \to \mathbb{L}_{M_i^{\mathbb{C}^*}}$ by taking the fixed part [GP]. Since we work $\mathbb{C}^*$-equivariantly,

$$[M]^{vir} \in A_*^{\mathbb{C}^*}(M),$$

where $A_*^{\mathbb{C}^*}(M)$ is the $\mathbb{C}^*$-equivariant Chow group. Let $t := c_1(\mathcal{O} \otimes \mathfrak{t})$, where $\mathfrak{t}$ is a primitive character of $\mathbb{C}^*$, then

$$A_*^{\mathbb{C}^*}(M) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}] \cong A(M^{\mathbb{C}^*}) \otimes_{\mathbb{Q}} \mathbb{Q}[t,t^{-1}].$$

The virtual localization formula is [GP]

$$\int_{[M]^{vir}} \alpha = \sum_i \int_{[M_i^{\mathbb{C}^*}]^{vir}} \frac{1}{e(N_i^{vir})} \alpha|_{M_i^{\mathbb{C}^*}}.$$

Here the virtual normal bundle N_i^{vir} is defined as the moving part of $E^{\bullet\vee}|_{M_i^{\mathbb{C}^*}}$ and $e(\cdot)$ is the $\mathbb{C}^*$-equivariant total Chern class.

Back to the moduli spaces $\overline{M}_{g,m}(X,\beta)$, $P_\chi(X,\beta)$.

Theorem 3.2 (K-Thomas). *The usual Gromov-Witten complex $E^\bullet_{GW,X}$ induces the usual Gromov-Witten theory $E^\bullet_{GW,S}$ on $\overline{M}_{g,m}(S,\beta)$. If (AS2) is satisfied, then the reduced Gromov-Witten complex $E^{red\bullet}_{GW,X}$ induces the reduced Gromov-Witten complex $E^{red\bullet}_{GW,S}$ on $\overline{M}_{g,m}(S,\beta)$. The usual stable pairs complex $E^\bullet_{PT,X}$ induces the complex $E^\bullet$ constructed via Hilbert schemes on $P_\chi(S,\beta)$. If (AS2) is satisfied, then the reduced stable pair theory $E^{red\bullet}_{PT,X}$ induces the complex $E^{red\bullet}$ constructed via Hilbert schemes on $P_\chi(S,\beta)$.*

Since the moduli spaces we consider can be non-compact, we *define* the various reduced/non-reduced Gromov-Witten and stable pair invariants via the localization formula. For any $\sigma_1,\ldots,\sigma_m \in H^*(S,\mathbb{Z})$

$$\begin{aligned}\mathcal{R}^{red}_{g,\beta}(X,\sigma_1\cdots\sigma_m) &:= \int_{[\overline{M}_{g,m}(S,\beta)]^{red}} \frac{1}{e(N^{vir})} \prod_{i=1}^m \mathrm{ev}_i^*\sigma_i,\\ \mathscr{P}^{red}_{\chi,\beta}(X,\sigma_1\cdots\sigma_m) &:= \int_{[P_\chi(X,\beta)^{\mathbb{C}^*}]^{red}} \frac{1}{e(N^{vir})} \prod_{i=1}^m \tau_0(\sigma_i),\\ \mathscr{P}^{red}_{\chi,\beta}(S,\sigma_1\cdots\sigma_m) &:= \int_{[P_\chi(S,\beta)]^{red}} \frac{1}{e(N^{vir})} \prod_{i=1}^m \tau_0(\sigma_i),\end{aligned}$$

and similarly for non-reduced invariants. Here $\mathrm{ev}_i : \overline{M}_{g,m}(X,\beta) \to X$ is evaluation on the ith marked point and $\tau_0(\cdot)$ are primary descendent insertions, which are defined as follows. On $X \times P_\chi(X,\beta)$, we have a universal stable pair $(\mathbb{F}, s)$. Denoting projection on the first and second factor by π_X, π_{P_X}, one can define

$$\tau_0(\cdot) := \pi_{P_X *}(\pi_X^*(\cdot) \cap \mathrm{ch}_2(\mathbb{F})) \in H^*(P_\chi(X,\beta), \mathbb{Z}).$$

The universal sheaf $\mathbb{F}$ has $\mathrm{ch}_2(\mathbb{F})$ as its lowest non-zero Chern character. Higher descendents are defined by using higher Chern characters of this sheaf [PP2]. Higher descendents are the stable pair analog of ψ-classes.

3.3 Universality

Whenever (AS1) is satisfied, we have invariants $\mathscr{P}^{red}_{\chi,\beta}(S, \sigma_1 \cdots \sigma_m)$. We are interested in two types of insertions. Let $\gamma_1, \ldots, \gamma_{2q(S)}$ be an integral oriented basis of $H_1(S)/\text{torsion}$ and consider the point class $[pt] \in H_0(S)$. Consider invariants

$$\begin{aligned} &\mathscr{P}^{red}_{\chi,\beta}(S, [\gamma_1] \cdots [\gamma_{2q(S)}][pt]^m), \\ &\mathscr{P}^{red}_{\chi,\beta}(S, [pt]^m), \end{aligned}$$

where $\chi = 1-h+n$ and $2h-2 = \beta(\beta+k)$ as before. Roughly speaking, the H_1-insertions ensure only stable pairs in a fixed linear subsystem $|L| \subset H_\beta$ contribute to the invariant and putting in $[pt]^m$ requires the curves to lie in a codimension m linear subsystem $\mathbb{P} \subset |L|$. These H_1-insertions also feature in the work of J. Bryan and C. Leung on reduced Gromov-Witten invariants of abelian surfaces [BL2]. We now want to exploit our hard work by computing these invariants on the ambient smooth projective variety

$$\iota : P_\chi(S,\beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta \hookrightarrow S^{[n]} \times H_\gamma,$$

There is one issue though. Recall that $H_\beta \hookrightarrow H_\gamma$ is cut out by a section of the sheaf $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$. (AS1) implies $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is a vector bundle on a Zariski open neighbourhood of H_β, but $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ can be non-zero outside this Zariski open neighbourhood! Instead of (AS1), we assume the following stronger assumption

$$h^2(L) = 0 \text{ for all } L \in \mathrm{Pic}^\beta(S), \text{ (AS3)}.$$

Then $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is a vector bundle on H_γ. Since $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ is cut out by sections of vector bundles, one can show

$$\iota_*[P_\chi(S,\beta)]^{red} = c_{top}(\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) \oplus \pi_*\mathcal{O}_{\mathcal{Z}}(\mathcal{D} - A)). \tag{2}$$

(Note: $\mathcal{D}|_{H_\beta \times S} = \mathcal{C} + A$.) This formula for the virtual cycle allows us to compute the above invariants in the following sense. Via wedging together and integrating over S, the classes $\beta, k \in H^2(S,\mathbb{Z})$, and $1 \in H^4(S,\mathbb{Z})$ give elements

$$[\beta], [k] \in \Lambda^2 H^1(S,\mathbb{Z})^*, \text{ and } [1] \in \Lambda^4 H^1(S,\mathbb{Z})^*.$$

Wedging together any combination produces an element

$$\Lambda^a[\beta] \wedge \Lambda^b[k] \wedge \Lambda^c[1] \in \Lambda^{2q(S)} H^1(S,\mathbb{Z})^* \cong \mathbb{Z}, \text{ where } a+b+2c = q(S).$$

The canonical isomorphism comes from choosing any integral basis of $H^1(S,\mathbb{Z})/\text{torsion} \subset H^1(S,\mathbb{R})$ compatible with the orientation coming from the complex structure.

Theorem 3.3 (Universality, K-Thomas). *Fixing q, p_g, m, n, there exist universal functions $F_{q,p_g,m,n}(\mathbf{x})$, $G_{q,p_g,m,n}(\mathbf{y})$ with variables*

$$\mathbf{x} := (x_1, x_2, x_3, x_4, t), \ \mathbf{y} := (y_1, y_2, y_3, y_4, \{y_{abc}\}_{a+b+2c=q}, t),$$

such that for any S with $q(S) = q$, $p_g(S) = p_g$, and $\beta \in H^2(S, \mathbb{Z})$ satisfying (AS3)

$$\mathscr{P}^{red}_{\chi,\beta}(S, [\gamma_1] \cdots [\gamma_{2q(S)}][pt]^m) = F_{q,p_g,m,n}(\beta^2, \beta.k, k^2, c_2(S), t),$$
$$\mathscr{P}^{red}_{\chi,\beta}(S, [pt]^m) = G_{q,p_g,m,n}(\beta^2, \beta.k, k^2, c_2(S), \{\Lambda^a[\beta] \wedge \Lambda^b[k] \wedge \Lambda^c[1]\}_{a+b+2c=q}, t).$$

A few remarks on this theorem:

(1) The strategy of the proof is as follows. By virtue of formula (2), one can express the invariants as integrals of Chern classes of tautological bundles on $\mathrm{Hilb}^n(S) \times H_\gamma$. Integrals of Chern classes of tautological bundles on $\mathrm{Hilb}^n(S)$ have been studied by G. Ellingsrud, L. Göttsche, and M. Lehn [EGL]. They are known to produce universal expressions in $\beta^2, \beta.k, k^2, c_2(S)$. Including H_γ (which is a projective bundle over $\mathrm{Pic}^\gamma(S) \cong \mathrm{Pic}^\beta(S)$) in their recursion leads to the theorem.

(2) Even though the invariants are also defined when (AS1) is satisfied, the above theorem is *false* in this more general case [Koo]. I will come back to this in the next lecture.

(3) Up to a factor $(-1)^{\cdots} t^{\cdots}$, with $\cdots$ universal, the functions F, G in the theorem are rational polynomials and independent of t.

(4) When β is irreducible $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$, so this theorem computes the full reduced stable pair invariants of X. By the GW/PT correspondence, we get universality results for the corresponding reduced Gromov-Witten invariants of X.

3.4 The Göttsche conjecture

We now apply the theory of stable pairs on surfaces to solve the Göttsche conjecture. Actually, one does not need any of virtual cycles for the proof, but the motivation of our strategy comes from stable pair theory on surfaces. The discussion will be independent of what I said so far. This was joint work with V. Shende and R. P. Thomas [KST], 2010. Let me start by rephrasing the problem of Severi degrees on any smooth projective surface S. Let $|L|$ be any linear system on S. If L is *sufficiently ample*, then the locally closed subset of δ-nodal curves inside $|L|$ has codimension δ. Moreover, the general δ-dimensional linear subsystem $\mathbb{P}^\delta \subset |L|$ hits this locus in finitely many δ-nodal curves with multiplicity 1. We denote this number of δ-nodal curves by $a^S_\delta(L)$. How ample L should be exactly will be discussed shortly. Based on experimental evidence of I. Vainsencher [Vai], and S. Kleiman and R. Piene [KP], Lothar Göttsche made the following remarkable conjecture [Got]:

Conjecture 3.4 (Göttsche, 1997). *For any $\delta \geq 1$, there exists a universal rational polynomial $T_\delta(x_1, x_2, x_3, x_4)$, such that for any S, L with L δ-very ample, one has*

$$a^S_\delta(L) = T_\delta(c_1(L)^2, c_1(L).c_1(S), c_1(S)^2, c_2(S)).$$

A few remarks on this conjecture:

(1) δ-very ample means that for any 0-dimensional subscheme $Z \subset S$ of length $\delta + 1$, the restriction map $H^0(L) \to H^0(L|_Z)$ is surjective [BS]. E.g. L is a δth power of a very ample line bundle. Göttsche's original conjecture required "more" ampleness.

(2) The first algebraic proof of this conjecture was given by Y.-j. Tzeng [Tze] in 2010. Her beautiful proof uses algebraic cobordism and degeneration methods. The case $S = \mathbb{P}^2$ was proved by Y. Choi using a recursion of Z. Ran [Cho] 1999. For $S = \mathbb{P}^2$, there also exists a proof using tropical geometry by S. Fomin and G. Mikhalkin [FM] 2009. The case of S a generic K3 surface with β primitive was proved by J. Bryan and C. Leung in [BL1] 1997 (see below). For a topological approach to the Göttsche conjecture (and generalizations), see the work M. È. Kazarian [Kaz].

(3) By now there exist many generalizations of this conjecture: other singularity types (J. Li and Y.-j. Tzeng, [LT2] and J. V. Rennemo [Ren]), a refined version (L. Göttsche and V. Shende [GS]), and a virtual version (Thm. 3.3).

I will present our proof in the next lecture. For now, I want to sketch how this conjecture leads to the famous Göttsche-Yau-Zaslow formula. Let

$$T(S,L) = \sum_{\delta=0}^{\infty} T_\delta(x_1,x_2,x_3,x_4)x^\delta.$$

By applying the Göttsche conjecture to disjoint surfaces Göttsche shows [Got]:

Theorem 3.5 (Göttsche)**.** *There exists four universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[x_1,x_2,x_3,x_4][[x]]$ such that*

$$T(S,L) = A_1^{x_1}A_2^{x_2}A_3^{x_3}A_4^{x_4}.$$

Using Gromov-Witten theory and a degeneration argument, J. Bryan and N. C. Leung [BL1], computed the generating function for any (S,L) with S a generic K3 and L primitive. Combining their formula with the previous theorem yields the famous *Göttsche-Yau-Zaslow formula.* To formulate it, define the (quasi)-modular forms

$$\Delta(\tau) = q\prod_{i>0}(1-q^i)^{24}, \; G_2(\tau) = -\frac{1}{24} + \sum_{i>0}\Big(\sum_{d \mid i} d\Big)q^i,$$

where $q = e^{2\pi i\tau}$. Also, define $D := q\frac{d}{dq}$. The ring of quasi-modular forms is closed under this operation. Then Göttsche conjectured and Tzeng proved:

Theorem 3.6 (Göttsche, Bryan-Leung, Tzeng)**.** *There exist universal power series $B_1, B_2 \in \mathbb{Q}[x_1,x_2,x_3,x_4][[q]]$ such that*

$$\sum_{\delta=0}^{\infty} T_\delta(c_1(L)^2, c_1(L).c_1(S), c_1(S)^2, c_2(S))(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(L)}B_1(q)^{c_1(S)^2}B_2(q)^{c_1(L).c_1(S)}}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}.$$

Two remarks are in order:

(i) The enumeration of rational nodal curves on a $K3$ surface is given by the Yau-Zaslow formula [YZ]. The above formula generalizes this to arbitrary g, S.

(ii) The power series $B_1(q)$, $B_2(q)$ are only known recursively by using the work of L. Caporaso and J. Harris [CH]. No closed formulae are known at present.

4 Application to the Göttsche conjecture and Seiberg-Witten invariants

4.1 A proof of the Göttsche conjecture

Let S be any smooth projective surface and L a δ-very ample line bundle on S. The first step in the proof involves the BPS spectrum of a curve. Let $C \subset S$ be any reduced curve with class $\beta \in H_2(S)$. Then the arithmetic genus $h = 1 - \chi(\mathcal{O}_C)$ of C is given by

$$2h - 2 = \beta(\beta + k).$$

The geometric genus genus $\overline{h}$ of C is defined to be the arithmetic genus of the normalization of C and it satisfies $\overline{h} \leq h$. Next, consider the generating function for Euler characteristics of Hilbert schemes of points on C (up to some signs and factors)

$$(-q)^{1-h} \sum_{i=0}^{\infty} e(C^{[i]})(-q)^i.$$

As for any integral generating function $(-q)^{1-h} \sum_{i=0}^{\infty} a_i(-q)^i$, there exist unique integers $\{n_{g,C}\}_{g=-\infty}^{h}$ such that

$$(-q)^{1-h} \sum_{i=0}^{\infty} e(C^{[i]})(-q)^i = \sum_{g=-\infty}^{h} (-1)^{1-g} n_{g,C} (q^{1/2} + q^{-1/2})^{2g-2}.$$

We should stress that this is a purely formal rewriting of the generating function. Pandharipande and Thomas [PT3] prove (see also [She]):

Theorem 4.1 (Pandharipande-Thomas, Shende)**.** $n_{g,C} = 0$ *unless* $\overline{h} \leq g \leq h$.

It should be pointed out that the proof of this theorem is not terribly complicated. For irreducible C, one can use Serre duality on the fibres of the Abel-Jacobi map to show $n_{g,C} = 0$ for all $g < 0$. With more work, the lower bound $\overline{h}$ can be found. We should also mention that Shende [She] gives an intepretation of the numbers $n_{g,C}$ in terms of multiplicities of certain Severi strata in the versal deformation of C. The numbers $\{n_{g,C}\}_{g=\overline{h}}^{h}$ are called the BPS numbers of C. Example: for C δ-nodal, $\overline{h} = h - \delta$ and $n_{h-\delta,C} = 1$.

Why is this of any use to the Göttsche conjecture? Let us analyze the geometric content of a general δ-dimensional linear subsystem $\mathbb{P}^\delta \subset |L|$: $\mathbb{P}^\delta$ contains:

(1) No curves with $> \delta$ singularities, in particular no non-reduced curves.

(2) A finite number of δ-nodal curves. For each such curve C, $n_{h-\delta,C} = 1$.

(3) (More difficult.) No other curves with δ singularities.

(4) (More difficult.) Curves with $< \delta$ singularities (e.g. smooth curves) and each such curve C has geometric genus $> h - \delta$. (E.g. for a smooth curve C the geometric genus is h so $n_{h-\delta,C} = 0$.)

Therefore, viewing $n_{h-\delta,C}$ as a constructible function on $\mathbb{P}^\delta$, we have

$$n_{h-\delta} := \int_{\mathbb{P}^\delta} n_{h-\delta,C} de := \sum_{i\in\mathbb{Z}} i.e(\{C \in \mathbb{P}^\delta \mid n_{h-\delta,C} = i\}) = a_\delta^S(L),$$

where e denotes topological Euler characteristic. Let $\mathcal{C} \to \mathbb{P}^\delta$ be the universal curve, then "integrating" over the linear system gives

$$(-q)^{1-h} \sum_{i=0}^{\infty} e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))(-q)^i = \sum_{g=h-\delta}^{h} (-1)^{1-g} n_g (q^{1/2} + q^{-1/2})^{2g-2}.$$

Inverting this formula expresses $a_\delta^S(L)$ as a $\mathbb{Q}[h]$ linear combination of $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$ for $i = 0, 1, \ldots, \delta$. So to prove the Göttsche conjecture, it suffices to show $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$, $i = 0, 1, \ldots, \delta$ are given by universal polynomials.

For computation, we again use that

$$\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta) \subset S^{[i]} \times \mathbb{P}^\delta$$

is cut out by a tautological section of $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$ (previous lecture). However, this time we do not even need virtual cycles! Temporarily considering the whole linear system and the map

$$\mathrm{Hilb}^i(\mathcal{C}/|L|) \to S^{[i]}.$$

The fibre over $Z \in S^{[i]}$ is

$$\mathbb{P}(\ker\{H^0(L) \to H^0(L|_Z)\}).$$

Since L is δ-very ample and $i \leq \delta$, the restriction map is surjective and the fibres are all isomorphic to $\mathbb{P}^{h^0(L)-1-i}$. One can deduce that $\mathrm{Hilb}^i(\mathcal{C}/|L|)$ is a projective bundle hence smooth. By Bertini's theorem, cutting down by a general $\mathbb{P}^\delta \subset |L|$, we conclude that $\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)$ is also smooth. Moreover, it is cut out by a regular section of $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$, so by formula (1)

$$e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)) = \int_{\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)} c_{top}(T_{\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)}) = \int_{S^{[i]} \times \mathbb{P}^\delta} c_i(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})) \frac{c_\bullet(T_{S^{[i]} \times \mathbb{P}^\delta})}{c_\bullet(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C}))}.$$

Applying [EGL] (as in the case of Universality Theorem 3.3) shows this expression is given by a universal polynomial in $c_1(L)^2$, $c_1(L).c_1(S)$, $c_1(S)^2$, $c_2(S)$. This finishes the proof.

4.2 Application to GW/PT correspondence (Göttsche setting)

The previous proof is in principle independent of any stable pair theory. However, the Euler characteristics $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta))$, $i = 0, 1, \ldots, \delta$ are very closely related to the reduced stable pair invariants

$$\mathscr{P}^{red}_{1-h+i}(X, [\gamma_1] \cdots [\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta}) = \mathscr{P}^{red}_{1-h+i}(S, [\gamma_1] \cdots [\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta}).$$

The insertions ensure only stable pairs mapping down to a general $\mathbb{P}^i \subset \mathbb{P}^\delta \subset |L| \subset H_\beta$ can contribute. Since $\mathbb{P}^\delta$ has no non-reduced curves, this means no thickenings can contribute. (Note: we require a slightly stronger ampleness for this part: L is $(2\delta+1)$-very ample and $h^1(L)=0$.) Writing the generating function in GV form (no stable pairs thickened in the fibre direction contribute)

$$PT_\beta^{red}(X,[\gamma_1]\cdots[\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta}) = \sum_{g=h-\delta}^{\infty} n_{g,\beta}^{PT}(q^{1/2}+q^{-1/2})^{2g-2},$$

one can show $n_{g,\beta}^{PT} = t^{\frac{\beta(\beta-k)}{2}-\delta}a_\delta^S(L)$. Moreover, writing the corresponding generating function for reduced Gromov-Witten invariants in GV form gives (no multiple covers contribute)

$$GW_\beta^{red}(X,[\gamma_1]\cdots[\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta}) = \sum_{g=h-\delta}^{\infty} n_{g,\beta}^{GW}(2\sin(u/2))^{2g-2}.$$

The leading term is

$$n_{h-\delta,\beta}^{GW} = t^{\frac{\beta(\beta-k)}{2}-\delta}\int_{[\overline{M}_{h-\delta,m}(S,\beta)]^{red}} \prod_{i=1}^{2q(S)} \mathrm{ev}_i^*([\gamma_i]) \prod_{j=1}^{\chi(L)-1-\delta} \mathrm{ev}_j^*([pt]),$$

where $m=2q(S)+\chi(L)-1-\delta$. This reduced Gromov-Witten invariant is the one which counts stable maps mapping to a linear subsystem $\mathbb{P}^\delta \subset |L| \subset H_\beta$. Note that we really need the *reduced* cycle in this setting. I.e. the degree of the insertions matches the virtual dimension of the reduced cycle but does *not* match the virtual dimension of the non-reduced cycle (unless $p_g(S)=0$). Using the facts about the content of the general linear subsystem $\mathbb{P}^\delta \subset |L|$ discussed in the proof of the Göttsche conjecture, one can show that the only stable maps of arithmetic genus $h-\delta$ mapping to $\mathbb{P}^\delta$ are normalizations of δ-nodal curves in $\mathbb{P}^\delta$. (E.g. such maps cannot be multiple covers since $\mathbb{P}^\delta$ contains no non-reduced curves.) Hence $n_{g,\beta}^{GW} = t^{\frac{\beta(\beta-k)}{2}-\delta}a_\delta^S(L)$ as well. This proves an instance of the GW/PT correspondence.

Theorem 4.2 (K-Thomas). *Let S,L be chosen such that L is $(2\delta+1)$-very ample, $h^1(L)=0$, and $\beta=c_1(L)$ satisfies (AS2). Write the generating functions*

$$GW_\beta^{red}(X,[\gamma_1]\cdots[\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta}),\ PT_\beta^{red}(X,[\gamma_1]\cdots[\gamma_{2q(S)}][pt]^{\chi(L)-1-\delta})$$

in BPS form. After the change of coordinates $-q=e^{2\pi iu}$, the leading coefficients $n_{h-\delta,\beta}^{GW}$, $n_{h-\delta,\beta}^{PT}$ coincide and are both equal to $t^{\frac{\beta(\beta-k)}{2}-\delta}a_\delta^S(L)$.

4.3 Poincaré/Seiberg-Witten invariants

I would like to end these lectures by describing a few recent results related to Seiberg-Witten invariants [CK], [Koo]. For this, I would like to come back to the setting of *any* S,β. Recall that we have discussed a *non-reduced* stable pair theory on $P_\chi(S,\beta) = \mathrm{Hilb}^n(\mathcal{C}/H_\beta)$, where $\chi = 1-h+n$, $2h-2=\beta(\beta+k)$ as before. If (AS1) is satisfied, then we also have a *reduced* stable pair theory on it. If moreover (AS3) is satisfied, the invariants are given by universal expressions. It turns out that (AS3) is really necessary:

Theorem 4.3 (K)**.** *The Universality Theorem 3.3 is not true when we replace (AS3) by (AS1). Next, fix S, β such that $p_g(S) = 0$ and neither β nor $k - \beta$ satisfies (AS3)*[5]*. If $\beta(\beta - k) < 0$, then*

$$\mathscr{P}_{\chi,\beta}(S, [pt]^m) = \mathscr{P}_{\chi,k-\beta}(S, [pt]^m) = 0.$$

If $\beta(\beta - k) \geq 0$, then $\beta(\beta - k) = 0$, $q(S) = 1$, and

$$\begin{aligned} &\mathscr{P}_{\chi,\beta}(S, [pt]^m) = \mathscr{P}_{\chi,k-\beta}(S, [pt]^m) = 0, \text{ for } m > 0 \\ &PT_\beta(S, [pt]^0) = PT_{k-\beta}(S, [pt]^0)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2k(2\beta-k)} + \frac{1}{2}[2\beta - k](q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2\beta^2}. \end{aligned}$$

In this theorem, we dropped the subscript *red* since reduced and non-reduced invariants coincide when $p_g(S) = 0$. Note that when $q(S) = 1$, $[2\beta - k] \in \Lambda^2 H^1(S, \mathbb{Z})^* \cong \mathbb{Z}$. Also note that the formula in the theorem is invariant under changing $q \leftrightarrow q^{-1}$. The failure of universality already occurs for stable pair invariants without points ($n = 0$) on certain surfaces S with $p_g(S) = 0$.

As discussed in lecture 2, in [DKO] the authors define the virtual cycle $[H_\beta]^{vir}$ for any S, β. They use $[H_\beta]^{vir}$ to define so-called Poincaré invariants as follows. Let $\mathcal{C} \to H_\beta$ be the universal curve again. Consider the two Abel-Jacobi maps

$$\begin{aligned} &AJ^+ : H_\beta \to \mathrm{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(C)] \\ &AJ^- : H_{k-\beta} \to \mathrm{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(K_S - C)]. \end{aligned}$$

Define invariants

$$\begin{aligned} &P_S^+(\beta) := AJ_*^+\Big(\sum_i c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{pt\}})^i \cap [H_\beta]^{vir}\Big), \\ &P_S^-(\beta) := (-1)^{\chi(\mathcal{O}_S) + \frac{\beta(\beta-k)}{2}} AJ_*^-\Big(\sum_i (-1)^i c_1(\mathcal{O}(\mathcal{C})|_{H_{k-\beta} \times \{pt\}})^i \cap [H_{k-\beta}]^{vir}\Big). \end{aligned}$$

In the first line, $\mathcal{C}$ denotes the universal divisor over H_β and in the second line, the universal divisor over $H_{k-\beta}$. Note that $P_S^\pm(\beta) \in H_*(\mathrm{Pic}^\beta(S), \mathbb{Z}) \cong \Lambda^* H^1(S, \mathbb{Z})$. The Poincaré invariants satisfy:

Theorem 4.4 (Blow-up formula, Dürr-Kabanov-Okonek)**.** *Let $\pi : \tilde{S} \to S$ be the blow-up of S in one point and let e denote the exceptional divisor. Using the identification $\pi^* : H^1(S, \mathbb{Z}) \xrightarrow{\cong} H^1(\tilde{S}, \mathbb{Z})$, we have*

$$P_{\tilde{S}}^\pm(\pi^*\beta + le) = \tau_{\leq \beta(\beta-k) - l(l-1)} P_S^\pm(\beta),$$

for any $\beta \in H_2(S)$ and $l \in \mathbb{Z}$. Here $\tau_{\leq \beta(\beta-k)-l(l-1)}$ denotes truncation keeping only cycles of real dimension $\leq \beta(\beta - k) - l(l - 1)$.

Theorem 4.5 (Wall-crossing formula, Dürr-Kabanov-Okonek)**.** *Fix any S, β with $p_g(S) = 0$. Then*

$$P_S^+(\beta) - P_S^-(\beta) = \sum_{i=0}^{\min\{q(S), \frac{\beta(\beta-k)}{2}\}} \frac{\left(\frac{[2\beta-k]}{2}\right)^{q(S)-i}}{(q(S) - i)!} \cap [\mathrm{Pic}^\beta(S)].$$

[5]If β or $k - \beta$ satisfies (AS3), then one of $\mathscr{P}_{\chi,\beta}(S, [pt]^m)$, $\mathscr{P}_{\chi,k-\beta}(S, [pt]^m)$ is given by the universal formula of the Universality Theorem 3.3 and the other is zero. Therefore, the interesting case is when neither β nor $k - \beta$ satisfies (AS3) (i.e. both H_β, $H_{k-\beta}$ are non-empty). Also note that (AS1) is automatic when $p_g(S) = 0$.

Compare the RHS to Poincaré's formula for the cycle class of the Brill-Noether locus in the Jacobian of a smooth projective curve. This is the origin of the name Poincaré invariants [DKO]. The blow-up formula follows from a rather straight-forward comparison of virtual cycles. The wall-crossing formula is much harder. One of the key ingredients of the proof is the surprising fact that without (AS3), the sheaf $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is still a vector bundle on the whole of H_γ even though $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero. (For the notation, see lecture 2. Surprisingly, the same is true for any S, β with β satisfying (AS1).)

Dürr, Kabanov, and Okonek designed these invariants to agree with the Seiberg-Witten invariants of S, β coming from symplectic geometry. I will not say much about Seiberg-Witten invariants, since I know little about them and there are too many experts in the audience. For compact symplectic 4-manifolds with $b_+ > 0$, one can define Seiberg-Witten invariants (originating in the work of N. Seiberg and E. Witten [Wit])

$$SW_S : H^2(S, \beta) \longrightarrow \Lambda^* H^1(S, \mathbb{Z}).$$

In the case $b_+ = 0$, the invariants depend on a chamber structure and are maps

$$SW_S^{\pm} : H^2(S, \beta) \longrightarrow \Lambda^* H^1(S, \mathbb{Z}).$$

Conjecture 4.6 (Dürr-Kabanov-Okonek)**.** *For any smooth projective surface S and $\beta \in H_2(S)$, we have*

$$P_S^+(\beta) = P_S^-(\beta) = SW_S(\beta), \text{ if } p_g(S) > 0 \qquad P_S^{\pm}(\beta) = SW_S^{\pm}(\beta), \text{ if } p_g(S) = 0.$$

Using the blow-up and wall-crossing formula, the authors gather much evidence for this conjecture [DKO]. The full conjecture was recently established by H.-l. Chang and Y.-H. Kiem [CK] using a beautiful application of cosection localization.

Theorem 4.7 (Chang-Kiem)**.** *The Poincaré invariants are the Seiberg-Witten invariants for all smooth projective surfaces.*

Let us go back to Theorem 4.3. By explicit computations on various elliptic fibrations S with $p_g(S) = 0$, it is not hard to find many examples of Seiberg-Witten invariants which are *not* given by universal functions. This establishes failure of universality. The duality formula follows from combining the wall-crossing formula with the following proposition:

Proposition 4.8 (K)**.** *For any S, β, and $m = \frac{\beta(\beta - k)}{2}$*

$$PT_\beta(S, [pt]^m) = t^m SW_S^+(\beta)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2h-2}.$$

In particular, the BPS spectrum of S, β is

$$n_{h,\beta} = t^m SW_S^+(\beta), \ n_{g,\beta} = 0 \text{ for } g \neq h.$$

This proposition is for *non-reduced* stable pair invariants. Here $\frac{\beta(\beta-k)}{2}$ is the degree of the non-reduced cycle $[H_\beta]^{vir}$. One of the essential ingredient in the proof of this proposition is the following formula for the non-reduced virtual cycle $[P_\chi(S, \beta)]^{vir} = [\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{vir}$

$$\iota_*[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{vir} = (S^{[n]} \times [H_\beta]^{vir}) \cap c_n(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})),$$

where $\iota : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta$ is the embedding [Koo].

4.4 Application to GW/PT correspondence (Taubes setting)

The previous proposition immediately implies:

Proposition 4.9 (K)**.** *Fix any* S, β *with* β *irreducible and* $m = \frac{\beta(\beta-k)}{2}$. *The GW/PT correspondence for* $GW_\beta(X,[pt]^m)$, $PT_\beta(X,[pt]^m)$ *is equivalent to the following equality*

$$GW_\beta(X,[pt]^m) = t^m SW_S^+(\beta)(2\sin(u/2))^{2h-2}.$$

In particular, setting $-q = e^{iu}$, *the leading coefficients of* $GW_\beta(X,[pt]^m)$, $PT_\beta(X,[pt]^m)$ *coincide if and only if*

$$SW_S^+(\beta) = \int_{[\overline{M}'_{h,m}(S,\beta)]^{vir}} \prod_{i=1}^m \mathrm{ev}_i^*[pt].$$

Note that this statement uses *disconnected* Gromov-Witten invariants (see lecture 1). The last equation is a simple version of the GW/SW correspondence by C. Taubes [Tau1, Tau2]. We have a similar result for any S, β with $-K_S$ nef and β sufficiently ample [Koo]. It would be nice to extent the proposition to *any* algebraic S, β without further conditions. This requires dealing with stable pairs which are not scheme theoretically supported on the zero section $S \subset X = K_S$. Such stable pairs are avoided in the applications I considered.

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