Continuation of point-to-cycle connections in 3D ODEs

Yuri A. Kuznetsov

joint work with E.J. Doedel, B.W. Kooi, and G.A.K. van Voorn
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- Truncated BVP's with projection BC's
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- Examples
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Previous works


2. Truncated BVP’s with projection BC’s

- Some notations
- Isolated families of connecting orbits
- Truncated BVP
- Error estimate
Consider the (local) flow $\varphi^t$ generated by a smooth ODE

$$\frac{du}{dt} = f(u, \alpha), \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$
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Let $O^- = \xi$ be a hyperbolic equilibrium with $\dim W^-_u = n^-_u$. 

Let $O^+$ be a hyperbolic limit cycle with $\dim W^+_s = m^+_s$. 
Consider the (local) flow $\varphi^t$ generated by a smooth ODE

$$\frac{du}{dt} = f(u, \alpha), \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$ 

Let $O^- = \xi$ be a hyperbolic equilibrium with $\dim W^-_u = n^-_u$.

Let $O^+$ be a hyperbolic limit cycle with $\dim W^+_s = m^+_s$.

If $x^+(t)$ is a periodic solution (with minimal period $T^+$) corresponding to $O^+$, then $m^+_s = n^+_s + 1$, where $n^+_s$ is the number of eigenvalues $\mu^+$ of the monodromy matrix

$$M^+ = D_x \varphi^{T^+}(x) \bigg|_{x = x^+(0)},$$

satisfying $|\mu^+| < 1$. 

Isolated families of connecting orbits

- Necessary condition: \( p = n - m_s^+ - n_u^- + 2 \) (Beyn, 1994).
Isolated families of connecting orbits

- Necessary condition: \( p = n - m_s^+ - n_u^- + 2 \) (Beyn, 1994).
- Two types of point-to-cycle connections in \( \mathbb{R}^3 \):

\[
\begin{align*}
&\text{(a) } \dim W_u^- = 1 \\
&\text{(b) } \dim W_u^- = 2
\end{align*}
\]
Truncated BVP

- The connecting solution \( u(t) \) is *truncated* to an interval \([\tau_-, \tau_+]\).
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The points $u(\tau_-)$ and $u(\tau_+)$ are required to belong to the linear subspaces that are tangent to the unstable and stable invariant manifolds of $O^-$ and $O^+$, respectively:

\[
\begin{cases}
L^{-}(u(\tau_-) - \xi) = 0, \\
L^{+}(u(\tau_+) - x^+(0)) = 0.
\end{cases}
\]
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L^-(u(\tau_-) - \xi) &= 0, \\
L^+(u(\tau_+) - x^+(0)) &= 0.
\end{align*}
$$

Generically, the truncated BVP composed of the ODE, the above *projection BC’s*, and a *phase condition* on $u$, has a unique solution family $(\hat{u}, \hat{\alpha})$, provided that the ODE has a connecting solution family satisfying the pahase condition and Beyn’s equality.
Error estimate

If \( u \) is a generic connecting solution to the ODE at parameter value \( \alpha \), then the following estimate holds:

\[
\| (u|_{[\tau_- , \tau_+]}, \alpha) - (\hat{u}, \hat{\alpha}) \| \leq Ce^{-2 \min(\mu_-|\tau_-|, \mu_+|\tau_+|)} ,
\]

where

- \( \| \cdot \| \) is an appropriate norm in the space \( C^1([\tau_- , \tau_+], \mathbb{R}^n) \times \mathbb{R}^p \),
- \( u|_{[\tau_- , \tau_+]} \) is the restriction of \( u \) to the truncation interval,
- \( \mu_{\pm} \) are determined by the eigenvalues of the Jacobian matrix \( D_u f \) at \( \xi \) and the monodromy matrix \( M^+ \).

(Pampel, 2001; Dieci and Rebaza, 2004)
3. The defining BVP in 3D

It has equilibrium-, cycle-, and connection-related parts.
If $n_u^- = 1$, we use $u(\tau_-) = \xi + \varepsilon v$, where

\[
\begin{align*}
    f(\xi, \alpha) &= 0, \\
    f_\xi(\xi, \alpha)v - \lambda_u v &= 0, \\
    \langle v, v \rangle - 1 &= 0.
\end{align*}
\]
Equilibrium-related equations

- If $n_u^- = 1$, we use $u(\tau_-) = \xi + \varepsilon v$, where

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\langle v, v \rangle - 1 &= 0.
\end{aligned}
\]

- If $n_u^- = 2$, we use $\langle v, u(\tau_-) - \xi \rangle = 0$, where

\[
\begin{aligned}
f(\xi, \alpha) &= 0, \\
f_\xi^T(\xi, \alpha)v - \lambda_s v &= 0, \\
\langle v, v \rangle - 1 &= 0,
\end{aligned}
\]

Together with $\langle u(\tau_-) - \xi, u(\tau_-) - \xi \rangle - \varepsilon^2 = 0$. 
Cycle-related equations

- Periodic solution:

\[
\begin{align*}
\dot{x}^+ - f(x^+, \alpha) &= 0, \\
x^+(0) - x^+(T^+) &= 0.
\end{align*}
\]
Cycle-related equations

- Periodic solution:

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\begin{align*}
\dot{x}^+ - f(x^+, \alpha) &= 0, \\
 x^+(0) - x^+(T^+) &= 0.
\end{align*}
\]

- Adjoint eigenfunction: \( \mu = \frac{1}{\mu_u^+} \)

\[
\begin{align*}
\dot{w} + f_u^T(x^+, \alpha)w &= 0, \\
w(T^+) - \mu w(0) &= 0, \\
\langle w(0), w(0) \rangle - 1 &= 0.
\end{align*}
\]
Cycle-related equations

- Periodic solution:

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\dot{x}^+ - f(x^+, \alpha) &= 0, \\
x^+(0) - x^+(T^+) &= 0.
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\]

- Adjoint eigenfunction: \( \mu = \frac{1}{\mu_u^-} \)

\[
\begin{align*}
\dot{w} + f_T^-(x^+, \alpha)w &= 0, \\
w(T^-) - \mu w(0) &= 0, \\
\langle w(0), w(0) \rangle - 1 &= 0.
\end{align*}
\]

- Projection BC: \( \langle w(0), u(\tau^-) - x^+(0) \rangle = 0. \)
Connection-related equations

- We need a phase condition to select a unique periodic solution, \textit{i.e.}, to fix a \textit{base point}

\[ x_0^+ = x^+(0) \]

on the cycle \( O^+ \).
Connection-related equations

- We need a phase condition to select a unique periodic solution, i.e., to fix a base point

$$x_0^+ = x^+(0)$$

on the cycle $O^+$. 

- Usually, an integral phase condition is used.
Connection-related equations

- We need a phase condition to select a unique periodic solution, i.e., to fix a base point
  \[ x_0^+ = x^+(0) \]
on the cycle \( O^+ \).

- Usually, an integral phase condition is used.

- For the point-to-cycle connection, we require the end point of the connection to belong to a plane orthogonal to the vector
  \[ f_0^+ = f(x^+(0), \alpha) : \]

\[
\begin{align*}
\dot{u} - f(u, \alpha) &= 0, \\
\langle f(x^+(0), \alpha), u(\tau_+) - x^+(0) \rangle &= 0.
\end{align*}
\]
The defining BVP in 3D: \( \lambda = \ln |\mu|, \ s = \text{sign} \mu = \pm 1. \)
4. Finding starting solutions with homotopy

- Adjoint scaled eigenfunction.
- Homotopies to connecting orbits.

References to homotopy techniques for point-to-point connections:

For fixed $\alpha$ and any $\lambda$, $x^+(\tau) = x^{+}_{old}(\tau), w(\tau) \equiv 0$, and $h = 0$ satisfy

\[
\begin{align*}
\dot{x}^+ - f(x^+, \alpha) &= 0, \\
x^+(0) - x^+(T^+) &= 0, \\
\int_0^1 \langle \dot{x}^+_{old}(\tau), x^+(\tau) \rangle &= 0, \\
\dot{w} + T^+ f_u^T(x^+, \alpha)w + \lambda w &= 0, \\
w(1) - s w(0) &= 0, \\
\langle w(0), w(0) \rangle - h &= 0,
\end{align*}
\]
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w(1) - s w(0) &= 0, \\
\langle w(0), w(0) \rangle - h &= 0,
\end{align*}
$$

A branch point at $\lambda_1$ corresponds to the adjoint multiplier $\mu = se^{\lambda_1}$. Branch switching and continuation towards $h = 1$ gives the eigenfunction $w$. 
Continuation in $(T, h_1)$ for fixed $\alpha$ (dim $W_u = 1$)

\[
\begin{aligned}
\left\{ \begin{array}{l}
\dot{x}^+ - T^+ f(x^+, \alpha) = 0, \\
x^+(0) - x^+(1) = 0, \\
\Psi[x^+] = 0, \\
\dot{w} + T^+ f_u^T (x^+, \alpha) w + \lambda w = 0, \\
w(1) - s w(0) = 0, \\
\langle w(0), w(0) \rangle - 1 = 0, \\
\dot{u} - T f(u, \alpha) = 0, \\
\langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle - h_1 = 0.
\end{array} \right.
\end{aligned}
\]

Here, e.g. $\Psi[x^+] = x_j^+ (0) - a_j$ and the initial connection $u(\tau) = \xi + \varepsilon v e^{\lambda_u T \tau}$.
Continuation in \((\alpha_1, h_2)\) for fixed \(T\) (\(\dim W_u = 1\))

\[
\begin{align*}
\dot{x}^+ - T^+ f(x^+, \alpha) &= 0, \\
x^+(0) - x^+(1) &= 0, \\
\langle w(0), u(1) - x^+(0) \rangle - h_2 &= 0, \\
\dot{w} + T^+ f_u^T(x^+, \alpha) w + \lambda w &= 0, \\
w(1) - s w(0) &= 0, \\
\langle w(0), w(0) \rangle - 1 &= 0, \\
\dot{u} - T f(u, \alpha) &= 0, \\
\langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle &= 0.
\end{align*}
\]

When \(h_2 = 0\) is located, improve connection by the continuation in \((\alpha_1, T)\) and then continue in \((\alpha_1, \alpha_2)\) with fixed \(T\) (using the primary BVP).
Continuation in \((T, h_1)\) or \((c_1, c_2, h_k)\) \((\dim W_u = 2)\)

The equilibrium-related part is replaced by the explicit BC

\[
\begin{align*}
    u(0) - \xi - \varepsilon(c_1 v^{(1)} + c_2 v^{(2)}) &= 0, \\
    c_1^2 + c_2^2 - 1 &= 0, \\
    f(\xi, \alpha) &= 0, \\
    f_\xi(\xi, \alpha)v - \lambda_u v &= 0, \\
    \langle v, v \rangle - 1 &= 0,
\end{align*}
\]

where \(v^{(1)}\) and \(v^{(2)}\) are independent unit vectors tangent to \(W_u\) at \(\xi\).

The initial connection

\[
u(\tau) = \xi + \varepsilon e^{\tau T} f_u(\xi, \alpha) v^{(1)}, \quad c_1 = 1, \quad c_2 = 0.
\]
Implementation in AUTO

\[
\dot{U}(\tau) - F(U(\tau), \beta) = 0, \quad \tau \in [0, 1],
\]

\[
b(U(0), U(1), \beta) = 0,
\]

\[
\int_0^1 q(U(\tau), \beta) d\tau = 0,
\]

where

\[
U(\cdot), F(\cdot, \cdot) \in \mathbb{R}^{n_d}, \quad b(\cdot, \cdot) \in \mathbb{R}^{n_{bc}}, \quad q(\cdot, \cdot) \in \mathbb{R}^{n_{ic}}, \quad \beta \in \mathbb{R}^{n_{fp}},
\]

The number \( n_{fp} \) of free parameters \( \beta \) is

\[
n_{fp} = n_{bc} + n_{ic} - n_d + 1.
\]

In our primary BVPs: \( n_d = 9, \quad n_{ic} = 0, \) and \( n_{bc} = 19 \) or 18
Example: \( \dim W_u = 1 \)

- Lorenz system:

\[
\begin{align*}
\dot{x}_1 &= \sigma (x_2 - x_1), \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - b x_3,
\end{align*}
\]

with the standard value \( b = \frac{8}{3} \).
Example: $\dim W^u = 1$

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\end{align*}
\]

with the standard value $b = \frac{8}{3}$.

- The bifurcation curve in the $(r, \sigma)$-plane corresponding to the point-to-cycle connection is first presented by L.P. Shilnikov (1980).
At \((r, \sigma) = (21, 10)\), there is a *saddle limit cycle* with

\[ x^+(0) = (9.265335, 13.196014, 15.997250), \quad T^+ = 0.816222, \]

that has

\[ \mu_s^+ = 0.0000113431, \quad \mu_u^+ = 1.26094. \]
Homotopy to eigenfunction

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- [Continuation] in \((\lambda, h)\) of the trivial solution of the BVP for the scaled adjoint eigenfunction \(w(\tau)\) detects a branch point at
  
  \[
  \lambda = \ln(\mu^+_u) = 0.231854.
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  \]

- From it a nontrivial branch is followed until the value \(h = 1\) is reached. This gives a nontrivial eigenfunction \(w(t)\) with
  \[
  w(0) = (0.168148, 0.877764, -0.448616)^T, \quad \|w(0)\| = 1.
  \]
Continue in \((T, h_1)\) until \(h_1 = 0:\)

\[(a) \quad T = 1.43924 \quad (b) \quad T = 1.54543 \quad (c) \quad T = 2.00352\]
Homotopy to connection

- **Continue** in \((T, h_1)\) until \(h_1 = 0\):

(a) \(T = 1.43924\)  
(b) \(T = 1.54543\)  
(c) \(T = 2.00352\)

- **Continue** in \((r, h_2)\) until \(h_2 = 0\), that occurs at \(r = 24.0720\).
Continuation of the connection

- Improve connection by the **continuation** in \((r, T)\):

\[
\begin{align*}
(a) \quad & (r, T) = (21.0, 2.00352); \\
(b) \quad & (r, T) = (24.0579, 3.0)
\end{align*}
\]
Continue the point-to-cycle bifurcation curve in \((r, \sigma)\):
Example: $\dim W^u = 2$

The standard tri-trophic food chain model:

\[
\begin{align*}
\dot{x}_1 &= x_1(1 - x_1) - \frac{a_1 x_1 x_2}{1 + b_1 x_1}, \\
\dot{x}_2 &= \frac{a_1 x_1 x_2}{1 + b_1 x_1} - \frac{a_2 x_2 x_3}{1 + b_1 x_2} - d_1 x_2, \\
\dot{x}_3 &= \frac{a_2 x_2 x_3}{1 + b_1 x_2} - d_2 x_3,
\end{align*}
\]

with $a_1 = 5$, $a_2 = 0.1$, $b_1 = 3$, and $b_2 = 2$. 

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\dot{x}_2 &= \frac{a_1 x_1 x_2}{1 + b_1 x_1} - \frac{a_2 x_2 x_3}{1 + b_1 x_2} - d_1 x_2, \\
\dot{x}_3 &= \frac{a_2 x_2 x_3}{1 + b_1 x_2} - d_2 x_3,
\end{align*}
\]

with \( a_1 = 5, \ a_2 = 0.1, \ b_1 = 3, \) and \( b_2 = 2. \)

At $d_1 = 0.25, d_2 = 0.0125$, we have an \textit{equilibrium}

$$\xi = (0.74158162, 0.16666666, 11.997732)$$

and a \textit{saddle limit cycle} with the period $T^+ = 24.282248$ and

$$x^+(0) = (0.839705, 0.125349, 10.55289)$$

Its nontrivial multipliers are $\mu_s^+ = 0.6440615, \mu_u^+ = 6.107464 \cdot 10^2$. 
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Continuation in $(\lambda, h)$ of the secondary branch from the branch point

$$\lambda = \ln(\mu_s^+) = -0.439961.$$ 

gives at $h = 1$ a nontrivial eigenfunction $w(t)$ with $\|w(0)\| = 1$:

$$w(0) = (0.09306, -0.87791, -4.69689)^T.$$
The initial solution \( u(\tau) \) is found by integration in CONTENT from a point in the plane tangent to \( W_u \) at distance \( \varepsilon = 0.001 \) to \( \xi \):

\[
u(0) = (0.742445, 0.166163, 11.997732).
\]

Integration interval \( T = 155.905 \).
The initial solution $u(\tau)$ is found by integration in CONTENT from a point in the plane tangent to $W_u$ at distance $\varepsilon = 0.001$ to $\xi$:

$$u(0) = (0.742445, 0.166163, 11.997732).$$

Integration interval $T = 155.905$.

Continue in $(T, h_1)$ towards a minimum of $h_1$. 
Homotopy to connection

- The initial solution $u(\tau)$ is found by integration in CONTENT from a point in the plane tangent to $W_u$ at distance $\varepsilon = 0.001$ to $\xi$:

$$u(0) = (0.742445, 0.166163, 11.997732).$$

Integration interval $T = 155.905$.

- Continue in $(T, h_1)$ towards a minimum of $h_1$.

- Continue in $(c_1, c_2, h_1)$ to get $h_1 = 0$;
The initial solution $u(\tau)$ is found by integration in CONTENT from a point in the plane tangent to $W^u$ at distance $\varepsilon = 0.001$ to $\xi$:

$$u(0) = (0.742445, 0.166163, 11.997732).$$

Integration interval $T = 155.905$.

- **Continue** in $(T, h_1)$ towards a minimum of $h_1$.
- **Continue** in $(c_1, c_2, h_1)$ to get $h_1 = 0$;
- **Continue** in $(c_1, c_2, h_2)$ to get $h_2 = 0$. 
Continuation of the connection

- Improve connection by the continuations in $T$ (and then in $\varepsilon$):

The connection with $T = 180.0$, $\varepsilon^2 = 10^{-5}$. 
Continuation in $\alpha_1 = d_1$:

\[\begin{array}{c|c|c|c|c|c|c|c|c}
\hline
x_2 & 0.05 & 0.10 & 0.15 & 0.20 & 0.25 & 0.30 & 0.35 & 0.40 & 0.45 \\
\hline
dx_3 & 9.5 & 10.0 & 10.5 & 11.0 & 11.5 & 12.0 & 12.5 & \\
\hline
\end{array}\]

LP: $d_1 = 0.280913$ and $d_1 = 0.208045$ (LPC).
Continue the point-to-cycle LP-bifurcation curve $T_{het}$ in $(d_1, d_2)$:
Open questions

- Cycle-to-cycle connections?
Open questions

- **Cycle-to-cycle connections?**

- Should all this be integrated in AUTO?
To be continued