Numerical Bifurcation Analysis of Maps

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Aim of the course

codim 2 bifurcations are organizing centers in bifurcation diagrams

Part 1 Analysis of codim 2 bifurcations:
Normal forms, Center Manifolds, Unfoldings
→ Get a feeling of dynamical behaviour.

Part 2 Bifurcations of invariant tori:
KAM Resonance tongues, Bubble analysis, homoclinic bifurcations.
→ Get a feeling of fine details near torus bifurcations

Infinite sequences of bifurcations
Consider a map

\[ F : x \mapsto F(x, \alpha) \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m. \]

Study dynamics near a fixed point of the \( k \)-th iterate of the map. Fixed points satisfy \( F(x^0, \alpha^0)^k - x^0 = 0 \) and have multipliers

\[ \{\mu_1, \mu_2, \ldots, \mu_n\} = \sigma(A), \]

where \( A = F_x(x^0, \alpha^0) \).

\( k \) is the period of the fixed point.

W.l.o.g. \( k = 1, x_0 = 0, \alpha_0 = 0. \)
Notation

- variables $x \in \mathbb{R}$ and $z \in \mathbb{C}$
- multi-index For a multi-index $\nu$ we have $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_i \in \mathbb{Z}_{\geq 0}$
  
  $\nu! = \nu_1! \nu_2! \ldots \nu_n!$
  
  $|\nu| = \nu_1 + \nu_2 + \ldots + \nu_n$ and
  
  $\tilde{\nu} \leq \nu$ if $\tilde{\nu}_i \leq \nu_i$ for all $i = 1, \ldots, n$.

- $\langle u, v \rangle = \bar{u}^T v$ is the standard scalar product in $\mathbb{C}^n$ (or $\mathbb{R}^n$).
Decompose phase space ($W$) near steady solution:

$$W = W_u \oplus W_s \oplus W_c$$

Manifolds $W_i$ invariant under the mapping $F$. 
Center Manifold $W_c \ ( \mod (\mu) = 1)$

Bifurcations occur on $W_c$.
Normal form determines locally properties of the solutions.

Check:
1. Nondegenerate : Coefficients nonzero?
   Predict the presence of heteroclinic/homoclinic structures and invariant circles.
2. Transversal : Depends on parameters
   Transversality allows to switch to new branches.
Center Manifold: Invariance

\[ \mathbb{R}^{n_0} \ni w \xrightarrow{H} u \in \mathbb{R}^n \]

\[ \dot{w} \xrightarrow{G} \dot{u} \]

HOMOLOGICAL EQUATION:

\[ F(H(w)) = H(G(w)) \]

where \( F \) Critical Map, \( G \) Normal Form

Center Manifold \( x = H(w) \)
Center Manifold and Normal Forms

Center Manifold Reduction: Ansatz

Let

\[ F(x) = Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) \]
\[ + \frac{1}{24}D(x, x, x, x) + \frac{1}{120}E(x, x, x, x, x) + \cdots \]

and expand the functions \( G, H \) into Taylor series with unknown coefficients,

\[ G(w) = \sum_{|\nu| \geq 1} \frac{1}{\nu!} g_{\nu} w^{\nu}, \quad H(w) = \sum_{|\nu| \geq 1} \frac{1}{\nu!} h_{\nu} w^{\nu}, \]
Insert this into the homological equation and collect the coefficients of the $w^\nu$-terms in the homological equation. This gives a linear system for $h_\nu$:

$$L_\nu h_\nu = R_\nu.$$  

where $L_\nu = (A - \mu^\nu I)$ with the multipliers $\mu$. 

Singular if $\mu^\nu = 1$. Interpretation: These terms are needed in the normal form.

- Iterative solutions for higher order terms.
- Critical coefficients come from singular systems.
- If necessary singular systems are solved by bordered systems.
- Parameters can be included in this reduction process
- Method by Elphick et.al.(1987)
Center Manifold Reduction: ODE’s

Center Manifold $W_c \, (\Re(\lambda) = 0)$

Homological equation:

$$F(H(w)) = (D_w H)G(w)$$

$$L_\nu = (A - \langle \nu, \lambda \rangle I)$$
Vectorfield Approximation

Observation: composition \( A \circ F \) is close to the identity.

Theorem (Takens, Neimark): Suppose \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism and \( D\Phi(0) \) has all eigenvalues on the unit circle. Denote by \( S \) the semi-simple part of \( D\Phi(0) \). Then there exists a diffeomorphism \( \Psi \) and a vectorfield \( X \) such that

\[
\Psi \circ \Phi \circ \Psi^{-1} = \phi_X(t = 1) \circ S
\]

in the sense of Taylor series.


Remark:

- \( \Phi \) is the time-1 map of the flow of the vectorfield \( X \).
- Parameters can be included.
Reduced ODEs for codim 2 bifurcations

- **Cusp** \( \dot{x} = \beta_1 + \beta_2 x + x^3 \)
- **Bautin** \( \dot{x} = x(\beta_1 + \beta_2 x^2 + x^4) \)
- **Bogdanov-Takens** \( \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_2 \\ \beta_1 + \beta_2 x_1 + x_1^2 - x_1 x_2 \end{pmatrix} \)
- **Zero-Hopf** \( \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \beta_1 + x_1^2 + sx_2^2 \\ x_2(\beta_2 + \theta x_1 + x_2^2) \end{pmatrix} \)
- **Double Hopf** \( \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1(\beta_1 - x_1^2 - \theta x_2^2) \\ x_2(\beta_2 - \delta x_1^2 \pm x_2^2) \end{pmatrix} \)
Fold-Hopf: Normal form

\[
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix} = 
\begin{pmatrix}
\beta_1 + x^2 + s|z|^2 \\
(\beta_2 + i\omega)z + (\theta + i\vartheta)xz + x^2z
\end{pmatrix}
\]

Introduce cylindrical coordinates \((x, z) = (x_1, x_2 e^{i\phi})\), scalings then give amplitude system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = 
\begin{pmatrix}
\beta_1 + x_1^2 + s x_2^2 \\
x_2(\beta_2 + \theta x_1 + x_2^2)
\end{pmatrix}
\]

Bifurcation curves:

- fold \(\beta_1 = 0\)
- Hopf \(\beta_1 = -\left(\frac{\beta_2}{\theta}\right)^2\)
- Torus If \(s\theta < 0, \beta_2 = 0, \theta \beta_1 < 0\).
- Heteroclinic If \(s < 0 < \theta, \beta_2 = \frac{\theta}{3\theta - 2} \beta_1, \beta_1 < 0\).
Fold-Hopf Unfolding $s = 1, \theta > 0$
Fold-Hopf Unfolding $s = -1, \theta < 0$
Fold-Hopf Unfolding $s = -1, \theta > 0$
Fold-Hopf Unfolding $s = 1, \theta < 0$
Double Hopf: Normal form

\[
\begin{pmatrix}
\dot{w}_1 \\
\dot{w}_2
\end{pmatrix} = \left( (i\omega_1(\beta) + \beta_1)w_1 + f_{2100}w_1|w_1|^2 + f_{1011}w_1|w_2|^2 \\
(i\omega_2(\beta) + \beta_2)w_2 + g_{1110}w_2|w_1|^2 + g_{0021}w_2|w_2|^2 \right) + O(\| (w_1, w_2) \|^4),
\]

There are always two curves of Neimark-Sacker bifurcations.
Intermezzo: ODE’s
Double Hopf

Simple case: parameter diagrams
Simple case: phase portraits
Difficult case: parameter diagrams
Difficult case: phase portraits
Fold and Period-doubling

1. **Fold**: The fixed point has a simple eigenvalue $\lambda_1 = 1$ and no other eigenvalues on the unit circle, while the restriction of $F$ to a one-dimensional center manifold at the critical parameter value has the form

$$
\xi \mapsto \xi + \frac{1}{2} a \xi^2 + O(\xi^3),
$$

where $a \neq 0$. When the parameter crosses the critical value, two fixed points coalesce and disappear. If $Av = F_x v$ and $B(u, v) = F_{xx}[u, v]$ are evaluated at the critical fixed point, then

$$
a = \langle q^*, B(q, q) \rangle,
$$

where $Aq = q$, $A^T q^* = q^*$, and $\langle q^*, q \rangle = 1$.

2. **Flip**: The fixed point has a simple eigenvalue $\lambda_1 = -1$ and no other eigenvalues on the unit circle, while the restriction of $(\？？)$ to a one-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$
\xi \mapsto -\xi + \frac{1}{6} b \xi^3 + O(\xi^4),
$$

where $b \neq 0$. When the parameter crosses the critical value, a cycle of period 2 bifurcates from the fixed point. This phenomenon is often called the

...
Neimark-Sacker

The fixed point has simple critical eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$ and no other eigenvalues on the unit circle. Assume that

$$e^{iq\theta_0} - 1 \neq 0, \quad q = 1, 2, 3, 4 \quad \text{(no strong resonances)}.$$ 

Then, the restriction of \(??\) to a two-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$\eta \mapsto \eta e^{i\theta_0} \left(1 + \frac{1}{2}d|\eta|^2\right) + O(|\eta|^4),$$

where \(\eta\) is a complex variable and \(d\) is a complex number. Further assume that

$$c = \text{Re } d \neq 0.$$ 

Under the above assumptions, a unique \textit{closed invariant curve} around the fixed point appears when the parameter crosses the critical value. One has the following expression for \(d\):

$$d = \frac{1}{2} e^{-i\theta_0} \langle v^*, C(v, v, \bar{v}) + 2B(v, (I_n - A)^{-1} B(v, \bar{v})) + B(\bar{v}, (e^{2i\theta_0} I_n - A)^{-1} B(v, v)) \rangle,$$

where \(Av = e^{i\theta_0} v, \quad A^T v^* = e^{-i\theta_0} v^*, \quad \text{and} \quad \langle v^*, v \rangle = 1.\)
List of local codim 2 bifurcations for maps

(1) \( \mu_1 = 1, b = 0 \) (cusp)
(2) \( \mu_1 = -1, c = 0 \) (generalized flip)
(3) \( \mu_{1,2} = e^{\pm i\theta_0}, \Re \left[ e^{-i\theta_0} c_1 \right] = 0 \) (Chenciner bifurcation)
(4) \( \mu_1 = \mu_2 = 1 \) (1:1 resonance)
(5) \( \mu_1 = \mu_2 = -1 \) (1:2 resonance)
(6) \( \mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3} \) (1:3 resonance)
(7) \( \mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2} \) (1:4 resonance)
(8) \( \mu_1 = 1, \mu_2 = -1 \) (fold-flip)
(9) \( \mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0} \) (“fold-Hopf for maps”)
(10) \( \mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0} \) (“flip-Hopf for maps”)
(11) \( \mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2} \) (“Hopf-Hopf for maps”)

The critical normal form is

$$w \mapsto G(w) = w + \left( \frac{1}{2} bw^2 \right) + \frac{1}{6} cw^3 + \cdots$$

on the center manifold

$$H(w) = wh_1 + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \cdots$$

The first three terms of the expansion are given by

- $w : (A - I)h_1 = 0$
- $w^2 : (A - I)h_2 = bh_1 - B(h_1, h_1)$
- $w^3 : (A - I)h_3 = ch_1 - C(h_1, h_1, h_1) - 3B(h_1, h_2)$
Codim 2 bifurcations of maps

Cusp

So we first obtain the eigenvectors such that

\[ Aq = q, A^T p = p, \langle p, q \rangle = 1, \]

Then higher order terms give

\[ b = \langle p, B(q, q) \rangle = 0, \]
\[ h_2 = -(A - I_n)^{INV} B(q, q), \]

and finally the critical normal form coefficient

\[ c = \langle p, C(q, q, q) + 3 B(q, h_2) \rangle \]
Cusp: Unfolding

\[ \begin{array}{c}
\beta_2 \\
T_1 \\
0 \\
\beta_1 \\
1 \\
2 \\
T_2 \\
0 \\
\eta \\
\Gamma \\
\end{array} \]
Degenerate Period-Doubling

\[ Aq = -q, \quad A^T p = -p, \quad \langle p, q \rangle = 1, \quad c = 0 \]

The critical normal form

\[ w \mapsto G(w) = -w + \left( \frac{1}{6} cw^3 \right) + \frac{1}{120} gw^5 + \cdots \]

\[ H(w) = wq + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \frac{w^4}{24} h_4 + \frac{w^5}{120} h_5 + \cdots \]

where

\[ h_2 = -(A - I_n)^{-1} \quad B(q, q) \]
\[ h_3 = -(A + I_n)^{\text{INV}} \quad [C(q, q, q) + 3B(q, h_2)] \]
\[ h_4 = -(A - I_n)^{-1} \quad [4B(q, h_3) + 3B(h_2, h_2) + \\ 6C(q, q, h_2) + D(q, q, q, q)] \]

\[ g = \langle p, 5B(q, h_4) + 10B(h_2, h_3) + \\ 10C(q, q, h_3) + 15C(q, h_2, h_2) + \\ 10D(q, q, q, h_2) + E(q, q, q, q) \rangle \]
Degenerate Period-Doubling: Unfolding

\[ PD^1 \]
\[ LP^2 \]
The normal form $G$ (including parameters) is:

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix} \mapsto \begin{pmatrix}
  -x + y \\
  \beta_1 + (-1 + \beta_2)y + c_1 x^3 + d_1 x^2 y
\end{pmatrix} + \cdots
$$

If $c_1 < 0$ a codim 1 branch of Neimark-Sacker bifurcation of double period emanates.

Asymptotic expression of the new branch

$$
H^2 : (x^2, y, \beta_1, \beta_2) = \left(-\frac{1}{c_1}, 0, 1, \left(2 + \frac{d_1}{c_2}\right)\right) \varepsilon
$$
Unfolding $c_1 > 0$

No new local branches
Unfolding $c_1 < 0$:

New codim 1 branch $H^2$ (local bifurcation)
1:2 Resonance: normalization

Introduce (generalized) eigenvectors:

\[ Aq_0 = -q_0, \quad Aq_1 = -q_1 + q_0, \]
\[ A^T p_0 = -p_0, \quad A^T p_1 = -p_1 + p_0, \]
\[ \langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 1, \quad \langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 0. \]

Collecting the quadratic terms we get

\[
(A - I_n) h_{20} = -B(q_0, q_0) \\
(A - I_n) h_{11} = -B(q_0, q_1) - h_{20} \\
(A - I_n) h_{02} = -B(q_1, q_1) - 2h_{11} + h_{20}
\]

These are all solvable, since \( \lambda = 1 \) is not an eigenvalue of \( A \).
1:2 Resonance: Cubic normalization

We only need cubic terms to find the coefficients.

\[
c_1 = \langle p_0, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle,
\]
\[
d_1 = \langle p_0, C'(q_0, q_0, q_1) + B(q_1, h_{20}) + 2B(q_0, h_{11}) \rangle
+ \langle p_1, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle
\]

Non-degenerate if \( c_1 \neq 0 \) and \( d_1 + c_1 \neq 0 \).
Example I: GHM

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  y \\
  a - b \times x - y \times y + r \times x \times y
\end{pmatrix}
\]
Example II: Adaptive control

Golden & Ydstie (1988):

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{y}{bx+k+yz} \\ z - \frac{ky}{c+y^2}(bx+k+yz-1) \end{pmatrix}
\]

Unique fixed point

\[x = y = 1, \quad z = 1 - b - k.\]

Loses stability by Period-Doubling or Neimark-Sacker bifurcation.
Example II: Bifurcation Diagram

\[ c = 0.5 \]
Fold-flip

The hypernormal form is:

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2} a_0 x^2 + \frac{1}{2} b_0 y^2 + \frac{1}{6} c_0 x^3 + \frac{1}{2} d_0 xy^2 \\ -y + xy \end{pmatrix} + \ldots
\]

Nondegeneracy conditions:

\[a_0 \neq 0, b_0 \neq 0\]

and

\[b_0 c_0 - a_0^2 b_0 - 3a_0 b_0 - a_0 d_0 \neq 0.\]

Approximating vectorfield

\[
X(x, \mu) = \begin{pmatrix} \mu_1 + \left( -\frac{1}{2} a_0 \mu_1 + \mu_2 \right) x_1 + \frac{1}{2} a_0 x_1^2 + \frac{1}{2} b_0 x_2^2 + d_1 x_1^3 + d_2 x_1 x_2^2 \\ \frac{1}{2} \mu_1 x_2 - x_1 x_2 + d_3 x_1 x_2^2 + d_4 x_2^3 \end{pmatrix}
\]

with

\[d_1 = \frac{1}{6} \left( c_0 - \frac{3}{2} a_0^2 \right), \quad d_2 = \frac{1}{2} \left( d_0 + \frac{1}{2} b_0 (2 - a_0) \right), \quad d_3 = \frac{1}{4} (a_0 - 2), \quad d_4 = \frac{1}{4} b_0.\]
Fold-Flip: Critical Phase portraits

When $b_0 > 0$:
- If $a_0 < -2$, the system exhibits a fold bifurcation.
- If $-2 < a_0 < 0$, the system shows a flip bifurcation.
- If $a_0 > 0$, the system undergoes a hopf bifurcation.

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Analysis of maps
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Fold-Flip: Case $a_0, b_0 > 0$
Fold-Flip: Case $a_0 < 0 < b_0, 0$
Fold-Flip: Case $a_0 > 0 > b_0$
Fold-Flip: Case $a_0, b_0 < 0$