Numerical Continuation and Normal Form Analysis of Limit Cycle Bifurcations without Computing Poincaré Maps

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References

Limit cycles of ODEs and their local bifurcations

Standard approach using Poincaré maps

Continuation and normal form analysis using BVPs

Open problems
1. References


2. Limit cycles of ODEs and their local bifurcations

\[ \frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m. \]

A limit cycle \( C_0 \) corresponds to a periodic solution \( x_0(t + T_0) = x_0(t) \) and has Floquet multipliers \( \{\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1\} = \sigma(M(T_0)) \), where

\[ \dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n. \]
2. Limit cycles of ODEs and their local bifurcations

\[
\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.
\]

A limit cycle $C_0$ corresponds to a periodic solution $x_0(t + T_0) = x_0(t)$ and has Floquet multipliers \(\{\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1\} = \sigma(M(T_0))\), where

\[
\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.
\]

- Fold (LPC): $\mu_1 = 1$;
2. Limit cycles of ODEs and their local bifurcations

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\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.
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\[
\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.
\]

- **Fold (LPC):** \( \mu_1 = 1 \);
- **Flip (PD):** \( \mu_1 = -1 \);
2. Limit cycles of ODEs and their local bifurcations

\[
\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.
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\[
M(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.
\]

- **Fold (LPC):** \(\mu_1 = 1\);
- **Flip (PD):** \(\mu_1 = -1\);
- **Torus (NS):** \(\mu_{1,2} = e^{\pm i\theta_0}\).
2.1. Generic LPC bifurcation: $\mu_1 = 1$
2.2. Generic PD bifurcation: $\mu_1 = -1$

\[
W^c_{\beta} < 0 \quad \beta(\alpha) < 0 \\
W^c_{0} \quad \beta(\alpha) = 0 \\
W^c_{\beta} \quad \beta(\alpha) > 0
\]
2.3. Generic NS bifurcation: $\mu_{1,2} = e^{\pm i\theta_0}$

- $\beta(\alpha) < 0$
- $\beta(\alpha) = 0$
- $\beta(\alpha) > 0$
3. Standard approach using Poincaré maps

- Poincaré map and its fixed points
- Continuation of PD and LP bifurcations of maps
- Continuation of NS bifurcation of maps
- Smooth normal forms of maps on center manifolds
- Difficulties using Poincaré maps

Reference on finite-dimensional bordering techniques:

3.1. Poincaré map and its fixed points

- Poincaré map $P : \mathbb{R}^{n-1} \times \mathbb{R}^m \to \mathbb{R}^{n-1}$
- A limit cycle $C_0$ corresponds to a fixed point:

$$P(y_0, \alpha) - y_0 = 0.$$  

- Write

$$P(y_0 + \eta, \alpha_0) = y_0 + A\eta + \frac{1}{2}B(\eta, \eta) + \frac{1}{6}C(\eta, \eta, \eta) + O(\|\eta\|^4),$$

where $A = P_y(y_0, \alpha_0)$ and, for $i = 1, 2, \ldots, n - 1$,

$$B_i(\eta, \zeta) = \sum_{j,k=1}^{n-1} \left. \frac{\partial^2 P_i(y, \alpha_0)}{\partial y_j \partial y_k} \right|_{y=y_0} \eta_j \zeta_k,$$

$$C_i(\eta, \zeta, \omega) = \sum_{j,k,l=1}^{n-1} \left. \frac{\partial^3 P_i(y, \alpha_0)}{\partial y_j \partial y_k \partial y_l} \right|_{y=y_0} \eta_j \zeta_k \omega_l.$$
3.2. Continuation of PD and LC bifurcations of maps

- Defining system: \( (y, \alpha) \in \mathbb{R}^{n-1} \times \mathbb{R}^2 \)

\[
\begin{align*}
\mathcal{P}(y, \alpha) - y &= 0, \\
g(y, \alpha) &= 0,
\end{align*}
\]

where \( g \) is defined by solving

\[
\begin{pmatrix}
\mathcal{P}_y(y, \alpha) \pm I_{n-1} & w_1 \\
v_1^T & 0
\end{pmatrix}
\begin{pmatrix}
v \\
g
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

- Vectors \( v_1, w_1 \in \mathbb{R}^{n-1} \) are adapted to make the linear system nonsingular.
- \( (g_y, g_\alpha) \) can be computed efficiently using the adjoint linear system.
3.3. Continuation of NS bifurcation of maps

- Defining system: \((y, \alpha, \kappa) \in \mathbb{R}^{n-1} \times \mathbb{R}^2 \times \mathbb{R}\)

\[
\begin{aligned}
P(y, \alpha) - y &= 0, \\
g_{11}(y, \alpha, \kappa) &= 0, \\
g_{22}(y, \alpha, \kappa) &= 0,
\end{aligned}
\]

where \(\kappa = \cos \theta_0\) and \(g_{ij}\) are defined by solving

\[
\begin{pmatrix}
I_{n-1} - 2\kappa P_y(y, \alpha) + [P_y(y, \alpha)]^2 & w_1 & w_2 \\
v_1^T & 0 & 0 \\
v_2^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
r & s \\
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

- Vectors \(v_{1,2}, w_{1,2} \in \mathbb{R}^{n-1}\) are adapted to ensure unique solvability.

- Efficient computation of derivatives of \(g_{ij}\).
### 3.4. Smooth normal forms of maps on center manifolds

<table>
<thead>
<tr>
<th>Eigenvectors</th>
<th>Normal form</th>
<th>Critical coefficients</th>
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<tbody>
<tr>
<td><strong>LP</strong></td>
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<tr>
<td>$Aq = q$</td>
<td>$\xi \mapsto \beta + \xi + \tilde{b}\xi^2$</td>
<td>$\tilde{b} = \frac{1}{2} \langle p, B(q, q) \rangle$</td>
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<tr>
<td>$A^T p = p$</td>
<td>$+ O(\xi^3), \xi \in \mathbb{R}$</td>
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<tr>
<td>$\langle p, q \rangle = 1$</td>
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<tr>
<td><strong>PD</strong></td>
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<tr>
<td>$Aq = -q$</td>
<td>$\xi \mapsto -(1 + \beta)\xi + \tilde{c}\xi^3$</td>
<td>$\tilde{c} = \frac{1}{6} \langle p, C(q, q, q) + 3B(q, h_2) \rangle$</td>
</tr>
<tr>
<td>$A^T p = -p$</td>
<td>$+ O(\xi^4), \xi \in \mathbb{R}$</td>
<td>$h_2 = (I_n - A)^{-1}B(q, q)$</td>
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<tr>
<td>$\langle p, q \rangle = 1$</td>
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<tr>
<td><strong>NS</strong></td>
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<tr>
<td>$Aq = e^{i\theta_0}q$</td>
<td>$\xi \mapsto \xi e^{i\theta} \left(1 + \beta + \tilde{d}</td>
<td>\xi</td>
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<tr>
<td>$A^T p = e^{-i\theta_0}p$</td>
<td>$+ O(</td>
<td>\xi</td>
</tr>
<tr>
<td>$e^{i\nu \theta_0} \neq 1$</td>
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<td>$+ B(\bar{q}, h_{20})$</td>
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<tr>
<td>$\nu = 1, 2, 3, 4$</td>
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<tr>
<td>$\langle p, q \rangle = 1$</td>
<td></td>
<td>$h_{11} = (I_n - A)^{-1}B(q, \bar{q})$</td>
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<tr>
<td></td>
<td></td>
<td>$h_{20} = (e^{2i\theta_0}I_n - A)^{-1}B(q, q)$</td>
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</table>
LP-coefficient $\tilde{b}$

Assume $y_0 = 0$ and write

$$P(H) = AH + \frac{1}{2}B(H, H) + O(\|H\|^3),$$

and locally represent the center manifold $\mathcal{W}_0^c$ as the graph of a function $H : \mathbb{R} \to \mathbb{R}^{n-1}$,

$$u = H(\xi) = \xi q + \frac{1}{2} h_2 \xi^2 + O(\xi^3), \quad \xi \in \mathbb{R}, \quad h_2 \in \mathbb{R}^{n-1}.$$

The restriction of the Poincaré map to $\mathcal{W}_0^c$ is

$$\xi \mapsto \mathcal{G}(\xi) = \xi + \tilde{b}\xi^2 + O(\xi^3).$$

The invariance of the center manifold $\mathcal{W}_0^c$ means

$$P(H(\xi)) = H(\mathcal{G}(\xi)).$$
\[ A(\xi q + \frac{1}{2} h_2 \xi^2) + \frac{1}{2} B(\xi q, \xi q) + O(|\xi|^3) = (\xi + \tilde{b}\xi^2)q + \frac{1}{2} h_2 \xi^2 + O(|\xi|^3) \]

- The \( \xi \)-terms give the identity: \( Aq = q \).
- The \( \xi^2 \)-terms give the equation for \( h_2 \):

\[
(A - I_{n-1}) h_2 = -B(q, q) + 2\tilde{b}q.
\]

It is singular and its Fredholm solvability implies

\[
\tilde{b} = \frac{1}{2} \langle p, B(q, q) \rangle,
\]

where \( A^T p = p, \langle p, q \rangle = 1 \).
3.5. Difficulties using Poincaré maps

- The computation of $P$ can be unstable numerically near saddle or repelling limit cycles.
- Finite differences give low accuracy for $B$ and $C$ even if the cycle $C_0$ is stable.
- Simultaneous solving variational equations for the components of $A$, $B$, and $C$ is possible but costly and difficult to implement.
4. Continuation and normal form analysis using BVPs

- Continuation of limit cycles in one parameter
- Simple bifurcation points
- Continuation of bifurcations in two parameters
- Periodic normalization on center manifolds
4.1. Continuation of limit cycles in one parameter

- Defining system (BVP with IC) [Keller & Doedel, 1981]:

\[
\begin{align*}
\dot{u}(\tau) - T f(u(\tau), \alpha) &= 0, \quad \tau \in [0, 1], \\
\delta_0 - \delta_1 &= 0 \\
\int_0^1 \langle \dot{u}(\tau), u(\tau) \rangle \, d\tau &= 0,
\end{align*}
\]

where \(\hat{u}\) is a reference periodic solution.

- Linearization w.r.t. \((u, T, \alpha)\):

\[
\begin{bmatrix}
D - T f_x(u, \alpha) & -f(u, \alpha) & -T f_{\alpha}(u, \alpha) \\
\delta_0 - \delta_1 & 0 & 0 \\
\text{Int}_{\hat{u}} & 0 & 0
\end{bmatrix}
\]
Discretization via orthogonal collocation

- **Mesh points:** \( 0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1 \).

- **Basis points:**
  \[
  \tau_{i,j} = \tau_i + \frac{j}{m} (\tau_{i+1} - \tau_i), \quad i = 0, 1, \ldots, N - 1, \quad j = 0, 1, \ldots, m.
  \]

- **Approximation:**
  \[
  u^{(i)}(\tau) = \sum_{j=0}^{m} u^{i,j} l_{i,j}(\tau), \quad \tau \in [\tau_i, \tau_{i+1}],
  \]
  where \( l_{i,j}(\tau) \) are the Lagrange basis polynomials and \( u^{i,m} = u^{i+1,0} \).

- **Collocation** [de Boor & Swartz, 1973]:
  \[
  \left\{ \begin{array}{l}
  \left( \sum_{j=0}^{m} u^{i,j} l'_{i,j}(\zeta_{i,k}) \right) - T_f \left( \sum_{j=0}^{m} u^{i,j} l_{i,j}(\zeta_{i,k}), \alpha \right) = 0, \\
  u^{0,0} - u^{N-1,m} = 0, \\
  \sum_{i=0}^{N-1} \sum_{j=0}^{m} \sigma_{i,j} \langle \hat{u}^{i,j}, u^{i,j} \rangle = 0,
  \end{array} \right.
  \]
  where \( \zeta_{i,k}, \ k = 1, 2, \ldots, m, \) are the Gauss points.
Sparse Jacobian matrix

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<th>$u^{0,0}$</th>
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4.2. Simple bifurcation points

\[
\dot{\Phi}(\tau) - T f_x(u(\tau), \alpha_0) \Phi(\tau) = 0, \quad \Phi(0) = I_n,
\]
\[
\dot{\Psi}(\tau) + T f_x^T(u(\tau), \alpha_0) \Psi(\tau) = 0, \quad \Psi(0) = I_n.
\]

- **LPC:**
  \[(\Phi(1) - I_n)q_0 = 0, \ (\Phi(1) - I_n)q_1 = q_0, \ (\Psi(1) - I_n)p_0 = 0, \ (\Psi(1) - I_n)p_1 = p_0.\]

- **PD:**
  \[(\Phi(1) + I_n)q_2 = 0, \ (\Psi(1) + I_n)p_2 = 0.\]

- **NS:** \(\kappa = \cos \theta_0\)
  \[(\Phi(1) - e^{i\theta_0} I_n)(q_3 + iq_4) = 0, \ (\Psi(1) - e^{-i\theta_0} I_n)(p_3 + ip_4) = 0.\]

We have \((I_n - 2\kappa \Phi(1) + \Phi^2(1))q_{3,4} = 0.\)
4.3. Continuation of bifurcations in two parameters

- **PD and LPC:** \((u, T, \alpha) \in C^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2\)

\[
\begin{align*}
\dot{u}(\tau) - T f(u(\tau), \alpha) &= 0, \quad \tau \in [0, 1], \\
u(0) - u(1) &= 0, \\
\int_0^1 \langle \dot{u}(\tau), u(\tau) \rangle \, d\tau &= 0, \\
G[u, T, \alpha] &= 0.
\end{align*}
\]

- **NS:** \((u, T, \alpha, \kappa) \in C^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}\)

\[
\begin{align*}
\dot{u}(\tau) - T f(u(\tau), \alpha) &= 0, \quad \tau \in [0, 1], \\
u(0) - u(1) &= 0, \\
\int_0^1 \langle \dot{u}(\tau), u(\tau) \rangle \, d\tau &= 0, \\
G_{11}[u, T, \alpha, \kappa] &= 0, \\
G_{22}[u, T, \alpha, \kappa] &= 0.
\end{align*}
\]
There exist \( v_{01}, w_{01} \in C^0([0, 1], \mathbb{R}^n) \), and \( w_{02} \in \mathbb{R}^n \), such that

\[
N_1 : C^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \to C^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R},
\]

\[
N_1 = \begin{bmatrix}
D - T f_x (u, \alpha) & w_{01} \\
\delta_0 - \delta_1 & w_{02} \\
\text{Int}_{v_{01}} & 0
\end{bmatrix},
\]

is one-to-one and onto near a simple PD bifurcation point.

Define \( G \) by solving

\[
N_1 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
The BVP for \((v, G)\) can be written in the “classical form”

\[
\begin{align*}
\dot{v}(\tau) - T f_x(u(\tau), \alpha)v(\tau) + Gw_{01}(\tau) &= 0, \quad \tau \in [0, 1], \\
v(0) + v(1) + Gw_{02} &= 0, \\
\int_0^1 \langle v_{01}(\tau), v(\tau) \rangle d\tau - 1 &= 0.
\end{align*}
\]

One can take

\[w_{02} = 0\]

and

\[w_{01}(\tau) = \Psi(\tau)p_2, \quad v_{01}(\tau) = \Phi(\tau)q_2.\]
There exist $v_{01}, w_{01} \in C^0([0, 1], \mathbb{R}^n), w_{02} \in \mathbb{R}^n$, and $v_{02}, w_{03} \in \mathbb{R}$ such that $N_2 : C^1([0, 1], \mathbb{R}^n) \times \mathbb{R}^2 \rightarrow C^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

\[
N_2 = \begin{bmatrix}
D - T f_x(u, \alpha) & -f(u, \alpha) & w_{01} \\
\delta_0 - \delta_1 & 0 & w_{02} \\
\text{Int}_f(u, \alpha) & 0 & w_{03} \\
\text{Int}_{v_{01}} & v_{02} & 0
\end{bmatrix},
\]

is one-to-one and onto near a simple LPC bifurcation point.

Define $G$ by solving $\begin{bmatrix} v \\ s \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. 
NS-continuation

- There exist \( v_{01}, v_{02}, w_{11}, w_{12} \in C^0([0, 2], \mathbb{R}^n) \), and \( w_{21}, w_{22} \in \mathbb{R}^n \), such that \( N_3 : C^1([0, 2], \mathbb{R}^n) \times \mathbb{R}^2 \rightarrow C^0([0, 2], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2 \),

\[
N_3 = \begin{bmatrix}
D - T f_x(u, \alpha) & w_{11} & w_{12} \\
\delta_0 - 2\kappa \delta_1 + \delta_2 & w_{21} & w_{22} \\
\text{Int}_{v_{01}} & 0 & 0 \\
\text{Int}_{v_{02}} & 0 & 0
\end{bmatrix},
\]

is one-to-one and onto near a simple NS bifurcation point.

- Define \( G_{jk} \) by solving

\[
N_3 \begin{pmatrix} r \\ s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Remarks on continuation of bifurcations

- After discretization via orthogonal collocation, all linear BVPs for $G$’s have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.

- For each defining system holds: *Simplicity of the bifurcation* + *Transversality* $\Rightarrow$ *Regularity of the defining BVP*.

- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.

- Border adaptation using solutions of the adjoint linear BVPs.

- Actually implemented in MATCONT, also with compiled C-codes for the Jacobian matrices.
4.4. Periodic normalization on center manifolds

- Parameter-dependent periodic normal forms for LPC, PD, and NS
- Critical normal form coefficients

References on periodic normal forms:

$T_0$-periodic normal form on $W^c_{\alpha}$:

\[
\begin{align*}
\frac{d\tau}{dt} &= 1 + \nu(\alpha) - \xi + a(\alpha)\xi^2 + O(\xi^3), \\
\frac{d\xi}{dt} &= \beta(\alpha) + b(\alpha)\xi^2 + O(\xi^3),
\end{align*}
\]

where $a, b \in \mathbb{R}$, $\text{sign}(b) = \text{sign}(\tilde{b})$. 

\[\beta(\alpha) < 0 \quad \beta(\alpha) = 0 \quad \beta(\alpha) > 0\]
2\(T_0\)-periodic normal form on \(W^c_{\alpha}\):

\[
\begin{align*}
\frac{d\tau}{dt} &= 1 + \nu(\alpha) + a(\alpha)\xi^2 + O(\xi^4), \\
\frac{d\xi}{dt} &= \beta(\alpha)\xi + c(\alpha)\xi^3 + O(\xi^4),
\end{align*}
\]

where \(a, c \in \mathbb{R}, \text{sign}(c) = -\text{sign}(\tilde{c}).\)
$T_0$-periodic normal form on $W^c_{\alpha}$:

\[
\begin{align*}
\frac{d\tau}{dt} &= 1 + \nu(\alpha) + a(\alpha)|\xi|^2 + O(|\xi|^4), \\
\frac{d\xi}{dt} &= \left(\beta(\alpha) + \frac{i\theta(\alpha)}{T(\alpha)}\right)\xi + d(\alpha)|\xi|^2 + O(|\xi|^4),
\end{align*}
\]

where $a \in \mathbb{R}$, $d \in \mathbb{C}$, $\text{sign}(\text{Re}(d)) = \text{sign}(\text{Re}(\tilde{d}))$. 

\[\beta(\alpha) < 0 \quad \beta(\alpha) = 0 \quad \beta(\alpha) > 0\]
Critical normal form coefficients

At a codimension-one point write

\[ f(x_0(t)+v, \alpha_0) = f(x_0(t), \alpha_0) + A(t)v + \frac{1}{2}B(t; v, v) + \frac{1}{6}C(t; v, v, v) + O(||v||^4), \]

where \( A(t) = f_x(x_0(t), \alpha_0) \) and the components of the multilinear functions \( B \) and \( C \) are given by

\[ B_i(t; u, v) = \sum_{j,k=1}^{n} \left. \frac{\partial^2 f_i(x, \alpha_0)}{\partial x_j \partial x_k} \right|_{x=x_0(t)} u_j v_k \]

and

\[ C_i(t; u, v, w) = \sum_{j,k,l=1}^{n} \left. \frac{\partial^3 f_i(x, \alpha_0)}{\partial x_j \partial x_k \partial x_l} \right|_{x=x_0(t)} u_j v_k w_l, \]

for \( i = 1, 2, \ldots, n \). These are \( T_0 \)-periodic in \( t \).
Fold (LPC): $\mu_1 = 1$

- Critical center manifold $W^c_0 : \tau \in [0, T_0], \, \xi \in \mathbb{R}$
  \[
x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),
\]
  where $H(T_0, \xi) = H(0, \xi)$,
  \[
  H(\tau, \xi) = \frac{1}{2} h_2(\tau) \xi^2 + O(\xi^3)
  \]
- Critical periodic normal form on $W^c_0$:
  \[
  \begin{align*}
  \frac{d\tau}{dt} &= 1 - \xi + a\xi^2 + O(\xi^3), \\
  \frac{d\xi}{dt} &= b\xi^2 + O(\xi^3),
  \end{align*}
  \]
  where $a, b \in \mathbb{R}$, while the $O(\xi^3)$-terms are $T_0$-periodic in $\tau$. 
**LPC: Eigenfunctions**

\[
\begin{aligned}
\dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau), \alpha_0) &= 0, \quad \tau \in [0, T_0], \\
v(0) - v(T_0) &= 0, \\
\int_0^{T_0} \langle v(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau &= 0,
\end{aligned}
\]

implying

\[
\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle \, d\tau = 0,
\]

where \( \varphi^* \) satisfies

\[
\begin{aligned}
\dot{\varphi}^*(\tau) + A^T(\tau)\varphi^*(\tau) &= 0, \quad \tau \in [0, T_0], \\
\varphi^*(0) - \varphi^*(T_0) &= 0, \\
\int_0^{T_0} \langle \varphi^*(\tau), v(\tau) \rangle d\tau - 1 &= 0.
\end{aligned}
\]
LPC: Computation of $b$

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial x}{\partial \tau} \frac{d\tau}{dt}$$

- Collect

$$\xi^0 : \dot{x}_0 = f(x_0, \alpha_0),$$

$$\xi^1 : \dot{v} - A(\tau)v = \dot{x}_0,$$

$$\xi^2 : \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + 2\dot{v} - 2bv.$$  

- Fredholm solvability condition

$$b = \frac{1}{2} \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau) \rangle \, d\tau.$$
LPC: Example in MATCONT (ABC-reaction model)

\[
\begin{align*}
\dot{u}_1 &= -u_1 + p_1(1 - u_1)e^{u_3}, \\
\dot{u}_2 &= -u_2 + p_1(1 - u_1 - p_5u_2)e^{u_3}, \\
\dot{u}_3 &= -u_3 - p_3u_3 + p_1p_4(1 - u_1 + p_2p_5u_2)e^{u_3}.
\end{align*}
\]

\[p_3 = 1.5\]
\[p_4 = 8.0\]
\[p_5 = 0.04\]

\[b = 0\] at CPC (cusp)
Flip (PD): $\mu_1 = -1$

- Critical center manifold $W^c_0 : \tau \in [0, 2T_0], \xi \in \mathbb{R}$

\[ x = x_0(\tau) + \xi w(\tau) + H(\tau, \xi), \]

where $H(2T_0, \xi) = H(0, \xi),$ 

\[ H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + \frac{1}{6} h_3(\tau)\xi^3 + O(\xi^4) \]

- Critical periodic normal form on $W^c_0$:

\[
\begin{aligned}
\frac{d\tau}{dt} &= 1 + a\xi^2 + O(\xi^4), \\
\frac{d\xi}{dt} &= c\xi^3 + O(\xi^4),
\end{aligned}
\]

where $a, c \in \mathbb{R},$ while the $O(\xi^4)$-terms are $2T_0$-periodic in $\tau.$
\[ w(\tau) = \begin{cases} 
 v(\tau), & \tau \in [0, T_0], \\
 -v(\tau - T_0), & \tau \in [T_0, 2T_0], 
\end{cases} \]

with

\[
\begin{align*}
\dot{v}(\tau) - A(\tau)v(\tau) &= 0, & \tau \in [0, T_0], \\
v(0) + v(T_0) &= 0, \\
\int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 &= 0,
\end{align*}
\]

\[
\begin{align*}
\dot{v}^*(\tau) + A^T(\tau)v^*(\tau) &= 0, & \tau \in [0, T_0], \\
v^*(0) + v^*(T_0) &= 0, \\
\int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1/2 &= 0.
\end{align*}
\]
\[ \xi^2 : \dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{x}_0, \quad \tau \in [0, 2T_0]. \]

Since \( \text{Ker} \left( \frac{d}{d\tau} - A(\tau) \right) = \text{span}\{w, \psi = \dot{x}_0\} \), we must have

\[
\begin{aligned}
\int_0^{2T_0} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau) \rangle \, d\tau &= 0, \\
\int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau) \rangle \, d\tau &= 0,
\end{aligned}
\]

where \( \psi^* \) satisfies

\[
\begin{aligned}
\dot{\psi}^*(\tau) + A^T(\tau)\psi^*(\tau) &= 0, \quad \tau \in [0, T_0], \\
\psi^*(0) - \psi^*(T_0) &= 0, \\
\int_0^{T_0} \langle \psi^*(\tau), f(x_0(\tau), \alpha_0) \rangle \, d\tau - 1/2 &= 0,
\end{aligned}
\]

and is extended to \([T_0, 2T_0]\) by periodicity.
PD: Computation of $a$ and $h_2$

- The first Fredholm condition holds identically for all $a$, while the second gives

$$a = \frac{1}{2} \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) \rangle d\tau = \int_0^{T_0} \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)) \rangle d\tau.$$

- Define $h_2$ on $[0, T_0]$ as the unique solution to

$$\begin{cases} 
    \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), \alpha_0) &= 0, \\
    h_2(0) - h_2(T_0) &= 0, \\
    \int_0^{T_0} \langle \psi^*(\tau), h_2(\tau) \rangle d\tau &= 0,
\end{cases}$$

and extend it by periodicity to $[T_0, 2T_0]$. 
PD: Computation of $c$

Cubic terms:

$$\xi^3 : \dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6aw - 6cw.$$ 

The Fredholm solvability condition implies

$$6c = \int_0^{2T_0} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + 3B(\tau; w(\tau), h_2(\tau)) \rangle d\tau$$

$$- \int_0^{2T_0} \langle w^*(\tau), 6aA(\tau)w(\tau) \rangle d\tau$$

or

$$c = \frac{1}{3} \int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau) \rangle d\tau$$
Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$ with $e^{i\nu\theta_0} \neq 1, \nu = 1, 2, 3, 4$

- Critical center manifold $W^c_0: \tau \in [0, T_0], \xi \in \mathbb{C}$

  \[ x = x_0(\tau) + \xi v(\tau) + \bar{\xi} \bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad H(T_0, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi}), \]

  \[ H(\tau, \xi, \bar{\xi}) = \frac{1}{2} h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2} h_{02}(\tau)\bar{\xi}^2 \]

  \[ + \frac{1}{6} h_{30}(\tau)\xi^3 + \frac{1}{2} h_{21}(\tau)\xi^2 \bar{\xi} + \frac{1}{2} h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6} h_{03}(\tau)\bar{\xi}^3 + O(|\xi|^4). \]

- Critical periodic normal form on $W^c_0$:

  \[
  \left\{ \begin{array}{l}
  \frac{d\tau}{dt} = 1 + a|\xi|^2 + O(|\xi|^4), \\
  \frac{d\xi}{dt} = \frac{i\theta_0}{T_0} \xi + d|\xi|^2 + O(|\xi|^4), 
  \end{array} \right. 
  \]

  where $a \in \mathbb{R}$, $d \in \mathbb{C}$, and the $O(|\xi|^4)$-terms are $T_0$-periodic in $\tau$. 
NS: Complex eigenfunctions

\[
\begin{align*}
\dot{v}(\tau) - A(\tau)v(\tau) + \frac{i\theta_0}{T_0}v(\tau) &= 0, \quad \tau \in [0, T_0], \\
v(0) - v(T_0) &= 0, \\
\int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 &= 0.
\end{align*}
\]

and

\[
\begin{align*}
\dot{v}^*(\tau) + A^T(\tau)v^*(\tau) - \frac{i\theta_0}{T_0}v^*(\tau) &= 0, \quad \tau \in [0, T_0], \\
v^*(0) - v^*(T_0) &= 0, \\
\int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1 &= 0.
\end{align*}
\]
NS: Quadratic terms

- $\xi^2 \tilde{x}^0 : \dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0} h_{20} = B(\tau; v, v)$

  Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

  \[
  \begin{cases}
  \dot{h}_{20}(\tau) - A(\tau)h_{20}(\tau) + \frac{2i\theta_0}{T_0} h_{20}(\tau) - B(\tau; v(\tau), v(\tau)) = 0, \\
  h_{20}(0) - h_{20}(T_0) = 0.
  \end{cases}
  \]

  has a unique solution on $[0, T_0]$.

- $|\xi|^2 : \dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$

  Here

  \[\text{Ker} \left( \frac{d}{d\tau} - A(\tau) \right) = \text{span}(\varphi = \dot{x}_0).\]
**NS: Computation of \( a \) and \( h_{11} \)**

- Define \( \varphi^* \) as the unique solution of

\[
\begin{aligned}
\dot{\varphi}^*(\tau) + A^T(\tau)\varphi^*(\tau) &= 0, \quad \tau \in [0, T_0], \\
\varphi^*(0) - \varphi^*(T_0) &= 0, \\
\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau - 1 &= 0.
\end{aligned}
\]

- **Fredholm solvability:** \( a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)) \rangle d\tau. \)

- Then find \( h_{11} \) on \([0, T_0]\) from the BVP

\[
\begin{aligned}
\dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) - B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) &= 0, \\
h_{11}(0) - h_{11}(T_0) &= 0, \\
\int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau) \rangle d\tau &= 0.
\end{aligned}
\]
**NS: Computation of $d$**

- **Cubic terms:**

\[
\xi^2 \bar{\xi} : \dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0} h_{21} = 2B(\tau; h_{11}, v) + B(\tau; h_{20}, \bar{v}) + C(\tau; v, v, \bar{v}) - 2a\dot{v} - 2dv.
\]

- **Fredholm solvability condition:**

\[
d = \frac{1}{2} \int_0^{T_0} \langle v^* (\tau), C(\tau; v(\tau), v(\tau), \bar{v}(\tau)) \rangle \, d\tau
\]
\[
+ \frac{1}{2} \int_0^{T_0} \langle v^* (\tau), B(\tau; h_{11}(\tau), v(\tau)) + B(\tau; h_{20}(\tau), \bar{v}(\tau)) \rangle \, d\tau
\]
\[
- a \int_0^{T_0} \langle v^* (\tau), A(\tau)v(\tau) \rangle \, d\tau + \frac{ia\theta_0}{T_0}.
\]
Remarks on numerical periodic normalization

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map $P(y, \alpha_0)$.

- Detection of codim 2 points is easy.

- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.

- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.

- Actually implemented in MATCONT.
5. Open problems

- Automatic differentiation of the Poincaré map vs. BVPs.
- Periodic normal forms for codim 2 bifurcations of limit cycles.
- Branch switching at codim 2 points to limit cycle codim 1 continuation.