K3 surface and related topics

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Abstract

We will run a seminar about K3 surface and its related topics in the autumn semester. First we will introduce the basis of K3 surface, including the definitions, some classical invariants, linear systems and Hodge structures. Next we will introduce the moduli space of polarized K3 surfaces, including the constructions (via GIT and Hodge theory), periods, and the global Torelli theorem. Then we will introduce the moduli space of vector bundles and coherent sheaves. After that we will introduce the derived category of the moduli space of coherent sheaves on K3 surfaces. After the basic discussion of each topic, we can discuss some further topics. We list some topics for references. Comments and supplements are welcomed!

Due to my lack of knowledge, only part of the seminar plan has been worked out, we will see how to complete it later.

- 1st week

  We will talk about the definition of K3 surfaces over any field and then introduce some basic examples. Then we will talk about the complex K3 surfaces and its additional properties and maybe more classical construction methods for K3 surfaces. (Chapter 1 in [Huy16])

- 2nd week

  We will talk about the linear systems on K3 surfaces. This will include the curves on K3 surfaces and a result by Saint-Donat. And we will talk about some vanishing and global generation result. We will also mention the existence of K3 surfaces. (Chapter 2 in [Huy16], [SD74])

- 3rd week

  We will talk about the Hodge structure of K3 surfaces. We may compare it to the Hodge structure of the complex torus. We may talk about the Kummer constructions. We may also talk about the endomorphism fields and the Mumford-Tate groups. (Chapter 3 in [Huy16])

The Kuga-Satake construction can be used to prove the Weil conjecture for K3 surfaces. There are two alternative proof. One was by Deligne ([Del72]), another was by Pjateckiĭ-Šapiro and Šafarevič ([PŠŠ73]). We may talk about this and this may need one more time. (Chapter 4 in [Huy16])

- 4th week

  We will talk about the constructions of moduli space of polarized K3 surfaces. We may talk about one of its approach: via Hilbert scheme. (Chapter 5 in [Huy16])

- 5th week

  We will continue to talk about the constructions of moduli space of polarized K3 surfaces. We will talk about another approach: via periods. We will also introduce the global Torelli theorem (Chapter 6, 7 in [Huy16]).

  There are some natural extensions to this topic, e.g. several important compactifications of moduli spaces (Baily-Borel [BB66], Toroidal [AMRT10], GIT, Looijenga [Loo03a, Loo03b]), also see the compactification given by Friedman for a new proof of the global Torelli theorem for K3 surfaces [Fri84]; there are lots of works on the geometry of the moduli spaces of K3, e.g. [DK07], [MSVZ09]; Verbitsky' work on Hodge theory on hyperkähler manifolds and its applications, e.g. some results on the cohomology of a hyperkähler manifold [Ver90, Ver96, Ver99], global Torelli theorem of hyperkähler manifolds (note that this may not be true) [Ver13, Huy12, Ver19]; etc.

- 6th week

  We may talk about the moduli space of sheaves on K3 surfaces (Chapter 9, 10 in [Huy16], [HL10]).

  There are some extensions of this topic, e.g. spherical bundles [Kul89, Kul90a]; the existence of stable bundles on K3 surfaces [Kul90b, Yos01]; the singular behavior of the moduli space of sheaves [KLS06, OY03]; etc.

- 7th week

  We will talk about the derived category of moduli space of sheaves on K3 surfaces (Chapter 16 in [Huy16]).

  There are some natural extensions, e.g. Bridgeland stability, see the original paper [Bri07], and recent works [YY14, BM14a, BM14b].

- 8th week and so on

  TBA

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