Topology of Algebraic Varieties

(Notes for a course taught at the YMSC, Spring 2016)

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Contents

CHAPTER 1

Classical Lefschetz theory

1. The dual variety and Lefschetz pencils

1.1. The dual variety. Let $\mathbb{P}$ be a complex projective space of dimension N. We denote by $\check{\mathbb{P}}$ its dual, i.e., the projective space of hyperplanes of $\mathbb{P}$. For $\xi \in \check{\mathbb{P}}$ we write $H_\xi \subset \mathbb{P}$ for the hyperplane defined by it. Likewise, for $x \in \mathbb{P}$, $H_x \subset \check{\mathbb{P}}$ parametrizes the hyperplanes of $\mathbb{P}$ which contain x. Note that $H_x \subset \check{\mathbb{P}}$ is a hyperplane. The resulting map $\mathbb{P} \to \check{\check{\mathbb{P}}}$ is an isomorphism, and we shall use it to identify $\mathbb{P}$ with $\check{\check{\mathbb{P}}}$. The *incidence variety* $W \subset \mathbb{P} \times \check{\mathbb{P}}$ is the set of pairs $(x, \xi) \in \mathbb{P} \times \check{\mathbb{P}}$ with $x \in H_\xi$ or equivalently, $\xi \in H_x$. Via the identification above and exchanging factors, this is also the incidence variety for $\check{\mathbb{P}} \times \check{\check{\mathbb{P}}}$. A homogeneous coordinate system $[X_0 : \cdots : X_N]$ for $\mathbb{P}$ determines a homogeneous coordinate system $[\Xi_0 : \cdots : \Xi_N]$ for $\check{\mathbb{P}}$ such that W is given by $\sum_{\nu=0}^{N} X_\nu \Xi_\nu = 0$. So W is a hypersurface in $\mathbb{P} \times \check{\mathbb{P}}$ of bidegree $(1,1)$. The proof is of the following lemma is left as an exercise.

LEMMA 1.1. *The projections $p : W \to \mathbb{P}$ and $q : W \to \check{\mathbb{P}}$ are Zariski locally trivial with fiber a projective space of dimension $N-1$.*

Let $X \subset \mathbb{P}$ be a closed irreducible subset of dimension $n < N$. For $x \in X_{reg}$ we denote by $\hat{T}_x X \subset \mathbb{P}$ the projective completion of the tangent space $T_x X \subset T_x\mathbb{P}$. We say that $\xi \subset \check{\mathbb{P}}$ is tangent to X at x if $H_\xi \supset \hat{T}_x X$; this is equivalent to: $x \in H_\xi$ and $T_x X \subset T_x H_\xi$.

LEMMA 1.2. *The set $N_{X_{reg}}$ of $(x, \xi) \in X_{reg} \times \check{\mathbb{P}}$ for which ξ is tangent to X at x is a closed subvariety of $W_{X_{reg}} := q^{-1} X_{reg}$ and the projection $N_{X_{reg}} \xrightarrow{p} X_{reg}$ is locally trivial with fiber a projective space of dimension $N-n-1$. In particular, $N_{X_{reg}}$ is irreducible and nonsingular of dimension $N-1$.*

PROOF. Let $x \in X_{reg}$ and choose a linear subspace $K \subset \mathbb{P}$ of dimension $N-n-1$ which does not meet $\hat{T}_x X$. Let U be an open neighborhood of x in X_{reg} for which this property subsists: $\hat{T}_{x'} X \cap K = \emptyset$ for all $x' \in U$. Then every hyperplane of $\mathbb{P}$ containing $\hat{T}_{x'} X$ $(x' \in U)$ meets K in a hyperplane of K. This establishes an isomorphism $N_U \cong U \times \check{K}$ over U. $\square$

The variety $N_{X_{reg}}$ is sometimes referred to as the *projectivized conormal bundle* of X_{reg}. We write N_X for the closure of $N_{X_{reg}}$ in $\mathbb{P} \times \check{\mathbb{P}}$. So this is a closed irreducible subvariety of W of dimension $N-1$.

DEFINITION 1.3. The *dual of* X, denoted $X^\vee$, is the image of N_X under the projection $q : W \to \check{\mathbb{P}}$.

In more informal language: $\xi \in X^\vee$ means that H_ξ is tangent to X at a nonsingular point of X or represents a limit of such hyperplanes.

PROPOSITION 1.4. *With the above identifications, we have* $N_{X^\vee} = N_X$ *and hence* $(X^\vee)^\vee = X$. *Moreover, if* $X^\vee$ *is nonsingular of dimension* $N-1$ *at* $\alpha \in X^\vee$, *then* $N_X \to X^\vee$ *is a local isomorphism over* α, *the inverse being given by the Gauss map.*

We first prove:

LEMMA 1.5. *For* $(a, \alpha) \in W_{X_{reg}}$ *the following are equivalent:*

(i) $(a, \alpha) \in N_X$, *i.e.,* $H_\alpha \supset \hat{T}_a X$,
(ii) *the derivative of* $q|W_{X_{reg}}$ *at* (a, α) *is not surjective,*
(iii) *the derivative of* $q|W_{X_{reg}}$ *at* (a, α) *has image* $T_\alpha H_a$.

PROOF. Let $[X_0 : \cdots : X_N]$ be a homogeneous coordinate system for $\mathbb{P}$ such that $a = [1 : 0 : \cdots : 0]$ and α defines the hyperplane $X_N = 0$. In terms of the corresponding standard chart $\kappa : (x_1, \ldots, x_N) \in \mathbb{A}^N \mapsto [1 : x_1 : \cdots ; x_N] \in \mathbb{P}_{X_0}$ resp. $(\xi_0, \ldots, \xi_{N-1}) \in \mathbb{A}^N \mapsto [\xi_0 : \cdots ; \xi_{N-1} : 1] \in \check{\mathbb{P}}_{\Xi_N}$, $W \cap (\mathbb{P}_{X_0} \times \check{\mathbb{P}}_{\Xi_N})$ is parametrized by

$$(x; u) \in \mathbb{A}^N \times \mathbb{A}^{N-1} \mapsto ([1 : x_1, \ldots x_N], [-\sum_{\nu=1}^{N} x_\nu u_\nu - x_N : u_1 : \cdots : u_{N-1} : 1]),$$

with $(0, 0)$ mapping to (a, α). So q is there given by

$$(x; u) \mapsto (-\sum_{\nu=1}^{N} x_\nu u_\nu - x_N, u_1, \ldots, u_{N-1}).$$

The restriction to $(\kappa^{-1} X_{reg}) \times \mathbb{A}^{N-1}$ fails to be maximal rank at $(0; 0)$ precisely when $dx_N | T_0(\kappa^{-1} X_{reg})$ is zero. This last property is equivalent to $H_\alpha \supset \hat{T}_a X$ and so this proves (i)⇔(ii).

Applying dq to the inclusion $\{a\} \times H_a \subset W_X$ yields $T_\alpha H_a \subset dq(T_{(a,\alpha)} W_X)$. Since $\dim T_\alpha H_a = N-1$, we have $T_\alpha H_a = dq(T_{(a,\alpha)} W_X)$ if and only if $\dim dq(T_{(a,\alpha)} W_X < N$. This proves (ii)⇔(iii). □

PROOF OF PROPOSITION 1.4. Since X is the image under q of the irreducible N_X, we can find an open-dense subset U of $N_{X_{reg}}$ such that $q(U) \subset X^\vee_{reg}$ and $U \xrightarrow{q} X^\vee_{reg}$ is a submersion. By (i)⇔(iii) of Lemma 1.5, we have that for any $(a, \alpha) \in U$, $T_\alpha H_a = dq(T_{(a,\alpha)} W_X)$. In particular, $T_\alpha H_a \supset T_\alpha X^\vee$. This means that $(a, \alpha) \in N_{X^\vee}$. It follows that $U \subset N_{X^\vee}$ and after taking the closure, that $N_X \subset N_{X^\vee}$. Since N_X

and $N_{X^\vee}$ are closed and irreducible of dimension $N-1$, they must be equal.

The last assertion is clear. $\square$

COROLLARY 1.6. *When X is nonsingular (so that W_X is also nonsingular by Lemma 1.1), then $W_X \xrightarrow{q} \check{\mathbb{P}}$ has N_X as its set of critical points and $X^\vee$ as its set of critical values. In particular $q_X := q|W_X$ is C^∞-locally trivial over $\mathbb{P}^\vee \smallsetminus X^\vee$ (with typical fiber a nonsingular hyperplane section $X \cap H$ of X).*

PROOF. The first assertion follows from (i)⇔(ii) of Lemma 1.5. The second follows from the Ehresmann fibration theorem. $\square$

COROLLARY 1.7. *Any two nonsingular hypersurfaces in $\mathbb{P}^n$ of a fixed degree d are diffeomorphic.*

PROOF. Take for X the image of the d-fold Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$, where $N+1$ is the number of monomials of degree d in n variables. Its hyperplane sections in $\mathbb{P}^N$ pulled back to $\mathbb{P}^n$ then yield all the degree d hypersurfaces of $\mathbb{P}^n$. $\square$

REMARKS 1.8. Unless X is rather special there will exist hyperplanes in $\mathbb{P}$ that are tangent to X in a nonempty, but finite subset of X. This means that $N_X \to X^\vee$ has a finite fiber. But then a basic fact from algebraic geometry implies that $N_X \to X^\vee$ is a finite morphism over a nonempty open subset of $X^\vee$ so that $\dim X^\vee = \dim N_X = N-1$, meaning that $X^\vee$ is a hypersurface in $\mathbb{P}^\vee$. Since X is the dual of $X^\vee$, we see that $X^\vee$ must be rather special in case X itself is not a hypersurface (a projectively completed tangent hyperplane of $X^\vee$ will then be tangent to $X^\vee$ along a curve at least).

We note that when X is nonsingular, then $X^\vee$ is usually not. For example, when $X \subset \mathbb{P}^2$ is a nonsingular plane curve of degree $d \geq 4$, then X will have double tangents and such a double tangent produces a point of self-intersection of the dual curve $X^\vee \subset \check{\mathbb{P}}^2$. For $d \geq 3$, X has flex points and each of these yields a singular point (a cusp or worse) of $X^\vee$.

1.2. Lefschetz pencils. We choose a line $L \subset \mathbb{P}^\vee$. Then $A := L^\vee \subset \mathbb{P}$ is a codimension 2 linear subspace of $\mathbb{P}$ and the family of hyperplanes of $\mathbb{P}$ parametrized by L is the one-dimensional linear system on $\mathbb{P}$ of hyperplanes containing A; it is called a *pencil* with *axis* A.

In the remainder of this section we fix closed nonsingular and irreducible $X \subset \mathbb{P}$ of dimension n. We put $Y := X \cap A$, $\tilde{X} := \{(x,\xi) \in X \times L \,:\, x \in H_\xi\} = p^{-1}X \cap q^{-1}L$ and denote by

$$X \xleftarrow{\pi} \tilde{X} \xrightarrow{f} L$$

the projections. We note that $\pi^{-1}(Y) = Y \times L$; we will often write $\tilde{Y}$ for this closed subset. The projection $\tilde{X} \smallsetminus \tilde{Y} \xrightarrow{\pi} X \smallsetminus Y$ is an isomorphism whose inverse assigns to $x \in X \smallsetminus Y$ the pair (x, ξ), where H_ξ is the hyperplane containing A and x. For $\xi \in L$, the fiber $f^{-1}\xi$ is the hyperplane section $X \cap H_\xi$; we shall often denote it by X_ξ.

We now assume that L is *transversal* to $X^\vee$ in the sense that L meets $X^\vee$ only in its nonsingular part and for every $p \in L \cap X^\vee$, $T_pL + T_pX^\vee = T_p\check{\mathbb{P}}$ (so that $X^\vee$ must be of dimension $N-1$ at p). There are many such L:

LEMMA 1.9. *For every $\alpha \in \check{\mathbb{P}} \smallsetminus X^\vee$, the lines in $\check{\mathbb{P}}$ through α form a projective space $\mathbb{P}(\alpha)$ of dimension $N-1$ of which an open dense subset parametrizes $X^\vee$-transversal lines*

PROOF. Let $\phi : X^\vee \to \mathbb{P}(\alpha)$ be the map which assigns to $\xi \in X^\vee$ the line spanned by ξ and α. If $\dim X^\vee < N-1$, then $\mathbb{P}(\alpha) - \phi(X^\vee)$ is a nonempty open subset and hence as desired. Otherwise, $\xi \in (X^\vee)_{reg}$ is a critical point of ϕ if and only if the span of ξ and α is contained in $\hat{T}_\xi X^\vee$. So the set of lines in $\check{\mathbb{P}}$ passing through ξ that are not transversal to $X^\vee$ is the union of $\phi(X_{sing})$ and the closure of the set of critical values of $\phi|(X^\vee)_{reg}$ and hence contained in a proper subvariety. □

The number of points of $L \cap X^\vee$ is called the *class* of X, $\mathrm{class}(X)$. If $X^\vee$ is a hypersurface, then the class of X is just the degree of $X^\vee$; otherwise it is zero. The following proposition shows that when L is transversal to $\check{\mathbb{P}}$, nice things happen: f is going to look like the algebraic analogue of a Morse function with $\mathrm{class}(X)$ singular points. It will be an important tool to analyze the topology of X.

PROPOSITION 1.10. *When L is transversal to $X^\vee$, then*

(i) *A is transversal to X so that $Y = X \cap A$ is nonsingular in X and of codimension two, and $\pi : \tilde{X} \to X$ is the blowup of X along Y,*
(ii) *$W_X \xrightarrow{q} \check{\mathbb{P}}$ is transversal to L and hence $\tilde{X}$, being the preimage of L under this morphism, is nonsingular (but this also follows from (i)),*
(iii) *no critical point of f lies in $\tilde{Y}$ and f maps its critical set bijectively onto $L \cap X$ (and so will be $\mathrm{class}(X)$ in number),*
(iv) *each critical point $\tilde{x}$ of f is nondegenerate, i.e., the Hessian of f at such a point $\tilde{x}$ (which is intrinsically defined as a bilinear map $T_{\tilde{x}}\tilde{X} \times T_{\tilde{x}}\tilde{X} \to T_{f(x)}L$) is nondegenerate.*

PROOF. (i) Assume A is not transversal to X at $a \in X \cap A$. Then there is a hyperplane H containing $\hat{T}_aX \cup A$. So $H = H_\alpha$ for some $\alpha \in L$. Then $(a, \alpha) \in N_X$ and so $\alpha \in X^\vee$. Since L is transversal to $X^\vee$, we know that $X^\vee$ is nonsingular of dimension $N-1$ at α. By

Proposition 1.4 this implies that $\hat{T}_\alpha X^\vee = H_a$. We also have $L \subset H_a$ (because $a \in A$) and so then $\hat{T}_\alpha X + L = H_a$, which contradicts the fact that L is transversal to $X^\vee$.

Since $Y = A \cap X$ is of codimension two in X, $X \smallsetminus Y \cong \tilde{X} \smallsetminus \tilde{Y}$ is dense in $\tilde{X}$. So $\tilde{X}$ is the strict transform of X under the blowup of A in $\mathbb{P}$. In other words, $\pi : \tilde{X} \to X$ is the blowup of X along Y.

(ii) Let $(a, \alpha) \in \tilde{X}$. If $q_X : W_X \to \check{\mathbb{P}}$ is a submersion at (a, α) there is nothing to show, so let us suppose that this is not the case. Then $(a, \alpha) \in N_X$ and so by Lemma 1.5, $dq(T_{(a,\alpha)} W_X) = T_\alpha H_a \supset T_\alpha X^\vee$. Hence $dq(T_{(a,\alpha)} W_X) + T_\alpha L \supset T_\alpha X^\vee + T_\alpha L = T_\alpha \check{\mathbb{P}}$, since L is transversal to $X^\vee$ at α. This proves that $q_X : W_X \to \check{\mathbb{P}}$ is transversal to L at (a, α).

(iii) Let $(a, \alpha) \in \tilde{X}$ be a critical point of f. Since the restriction of f to $\tilde{Y} = Y \times L$ is the projection onto L, we must have $a \notin Y$. The transversality property proved in (ii) implies that $(a, \alpha) \in \tilde{X}$ is then also a critical point of $q_X : W_X \to \check{\mathbb{P}}$ and so $(a, \alpha) \in N_X$. This implies that $\alpha \in X^\vee$. But $\alpha \in L$ also, and so by our assumption, $X^\vee$ must be nonsingular of dimension $N - 1$ at α. According to Proposition 1.4, $N_X \to X^\vee$ is then an isomorphism over a nonsingular neighborhood of α in $X^\vee$. In particular, over α there is no other critical point than (a, α).

(iv) We continue with (a, α) as in (iii) so that $T_a X \subset T_a H_\alpha$. Since $a \notin A$, we can choose homogeneous coordinates $[X_0 : \cdots X_N]$ for $\mathbb{P}$ such that $a = [1 : 0 : \cdots : 0]$, A is given by $X_0 = X_N = 0$ and H_α is given by $X_N = 0$. Let

$$t \in U \mapsto [1 : x_1(t) : \cdots : x_N(t)]$$

be (the inverse of) an analytic chart of X at a, where U is neighborhood of $0 \in \mathbb{C}^n$ and with $x_\nu(0) = 0$ for all ν. Since H_α is tangent to X at a, $x_N(t)$ has order ≥ 2 at $t = 0$. We parametrize W_X near (a, α) as before:

$$(t, u) \in U \times \mathbb{A}^{N-1} \mapsto ([1 : x_1(t) : \cdots : x_N(t)], [g(t, u) : u_1 : \cdots : u_{N-1} : 1]),$$

where $g(t, u) = -\sum_{\nu=1}^{N-1} x_\nu(t) u_\nu - x_N(t)$. In terms of this parametrization, f is given as $t \in U \mapsto g(t, 0) = -x_N(t)$ and so we must show that the Hessian matrix $-(\partial^2 x_N / \partial t_i \partial t_j (0)_{i,j})$ is nonsingular. To this end we first observe that

$$\Big(\frac{\partial^2 g}{\partial t_i \partial u_\nu}(0, 0)\Big) = \Big(-\frac{\partial x_\nu}{\partial t_i}(0)\Big)$$

has rank n. This implies that the derivatives $\partial g / \partial t_1, \ldots, \partial g / \partial t_n$ are part of a system of local coordinates for W at (a, α). But notice that their common zero set is precisely the locus where the map $q_X : W_X \to \check{\mathbb{P}}$, given here as $(t, u) \in U \times \mathbb{A}^{N-1} \mapsto [g(t, u) : u_1 : \cdots : u_{N-1} : 1]$, is not of maximal rank. By Lemma 1.5 this is just N_U. Since $N_U \xrightarrow{q} \check{\mathbb{P}}$ has rank $N - 1$, it follows that the Jacobian matrix of $(t,u) \mapsto (\partial g/\partial t_1, \dots, \partial g/\partial t_n; g, u_1, \dots, u_{N-1})$ has rank $n+N-1$ at $(0,0)$. This matrix is

$$\begin{pmatrix} \partial^2 g/\partial t_i \partial t_j(0,0) & * \\ \partial g/\partial t_j(0,0) & -x_\nu(0) \\ 0 & I_{N-1} \end{pmatrix} = \begin{pmatrix} -\partial^2 x_N/\partial t_i \partial t_j(0) & * \\ 0 & 0 \\ 0 & I_{N-1} \end{pmatrix},$$

which shows that $-\partial^2 x_N/\partial t_i \partial t_j(0)$ must be nonsingular. □

The collection of hyperplane sections $\{X_\xi := X \cap H_\xi\}_{\xi \in L}$ is particular type of a one-dimensional linear system on X, called a general *Lefschetz pencil*. These sections are just the fibers of the map f which itself is called a *Lefschetz fibration*. The closed subset Y is its fixed point locus.

EXAMPLE 1.11. For a nonsingular hypersurface $X \subset \mathbb{P}$ of degree $d \geq 2$ its dual $X^\vee$ is the image of the Gauss map $p \in X \mapsto [\hat{T}_p X] \in \check{\mathbb{P}}$. The class of X is computed as follows. Choose homogeneous coordinates $[X_0 : \cdots : X_{n+1}]$ for $\mathbb{P}$ such that the line $L \subset \check{\mathbb{P}}$ spanned by X_0 and X_{n+1} is transversal to $X^\vee$. Then a point of $L \cap X^\vee$ is given by a hyperplane of the form $a_0 X_0 + a_{n+1} X_{n+1} = 0$ that is tangent to X. So if $F \in \mathbb{C}[X_0, \dots, X_{n+1}]$ is a defining equation for X, then this means that this point is a common zero of F and $\partial F/\partial X_1, \dots, \partial F/\partial X_n$. An intersection of $n+1$ hypersurfaces in $\mathbb{P}^{n+1}$ is nonempty, and in case this set is finite, Bézout's theorem tells us that it consists $d(d-1)^n$ of points, when each solution counted with multiplicity. In view of our transversality assumption, these multiplicities are all 1 and so $X^\vee$ is a hypersurface and $\mathrm{class}(X) = d(d-1)^n$.

2. Intermezzo: some basic results from homotopy theory

We will need some results from homotopy theory. We just state them here and refer for proofs to a standard text such as G. W. Whitehead: *Elements of Homotopy theory,* [22]. For homotopy theory the most suitable class of spaces to work with are those which are compactly generated, that is spaces, that satisfy the Hausdorff axiom and have the property that a subset is closed if and only if its intersection with every compact subset is compact. We shall therefore assume that in this context all spaces belong to this class.

We say that a topological pair (X, A) is a *k-cellular extension* if X is obtained from A by simultaneously attaching a number of k-cells. Then every connected component $\mathring{e}$ of $X \smallsetminus A$ is homeomorphic to the open k-ball in $\mathbb{R}^k$ and called an *open k-cell*; its closure e in X is called a *closed k-cell* ([1]). Such a pair (X, A) is $(k-1)$-connected and its homology is only nonzero in degree k, in which case it is free abelian

[1]Formally, there exists for every connected component $\mathring{e}$ of $X \smallsetminus A$, a relative homeomorphism $(B^k, \partial B^k) \to (\mathring{e} \cup A, A)$ whose image is the closure e of $\mathring{e}$ (a closed cell). The topology of X is characterized by the fact that each closed cell has the

(with each connected component $\mathring{e}$ of $X \smallsetminus A$ yielding a generator after we give it an orientation).

A *relative cell complex of dimension* d is a filtered space

$$X_\bullet = (A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_d)$$

such that (X_k, X_{k-1}) is a k-cellular extension for $k = 0, \dots, d$ and each closed cell meets only finitely many open cells; (X, A) is then said to admit the structure of a *relative cell complex of dimension* d and X_k is called its k*-skeleton*. In case $A = \emptyset$, we omit the adjective *relative*. Any simplicial complex can be regarded as a CW complex whose closed k-cells are the k-simplices. A cell structure on (X, A) greatly facilitates the computation of its homology, for the differentials of the exact homology sequence of the triples $\{(X_{k+1}, X_k, X_{k-1})\}_{k\geq 0}$ produce a complex of free abelian groups

$$\cdots \to H_{k+1}(X_{k+1}, X_k) \xrightarrow{\partial_{k+1}} H_k(X_k, X_{k-1}) \xrightarrow{\partial_k} H_{k-1}(X_{k-1}, X_{k-2}) \to \cdots$$

whose homology is that of (X, A). Similar for cohomology.

We list a few properties of relative cell complexes.

PROPOSITION 2.1. *Let (X, A) admit the structure of a relative cell complex and assume that the inclusion $i : A \subset X$ is a homotopy equivalence. Then i is a deformation retract, i.e., there exists a continuous map $r : X \times [0, 1] \to X$ such $r|A \times [0, 1]$ is the projection onto A, $r_0 : X \to X$ is the identity and $r_1(X) = A$.*

If (X, A) admits the structure of a relative cell complex with $A = X_{k-1}$ (X is obtained from A by attaching cells of dimension $\geq k$ only), then iterated application of the long exact sequence for a homotopy triple shows that (X, A) is $(k-1)$-connected.

There is a homotopy converse which says that if (X, A) admits relative cell complex and is $(k-1)$-connected then (X, A) is homotopy equivalent relative to A to a pair (X', A) which admits the structure of a relative cell complex with only cells in dimension $\geq k$. This practically implies the following two propositions.

PROPOSITION 2.2. *Let $f : (X, A) \to (Y, B)$ be a relative homeomorphism of pairs, i.e., A is closed in X, B is closed in Y and f maps $X \smallsetminus A$ homeomorphically onto $Y \smallsetminus B$. If (X, A) is $(k-1)$-connected and admits the structure of a relative cell complex then the same is true for (Y, B).*

PROPOSITION 2.3. *Let (X, A) resp. (Y, B) be $(k-1)$-connected resp. $(l-1)$-connected and admit the structure of a relative cell complex. Then $(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y)$ is $(k+l-1)$-connected.*

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quotient topology of B^k and that a subset of X is closed if and only if its intersection with A and each closed cell is closed.

3. Topology of Lefschetz pencils

The central result here is the following:

THEOREM 3.1 (Weak Lefschetz theorem). *Let $X \subset \mathbb{P}$ be a closed and nonsingular of dimension n and let $X_\xi = X \cap H_\xi$ be a transversal hyperplane section. Then the pair (X, X_ξ) is $(n-1)$-connected and $X - X_\xi$ has the homotopy type of a finite cell complex of dimension $\leq n$.*

REMARKS 3.2. This implies that the inclusion $X_\xi \subset X$ is on homology an isomorphism in degrees $< n-1$ and is onto in degree $n-1$. The same is true for homotopy groups. For cohomology, replace *onto* by *into*.

The last statement is actually a special case of the more general fact that any affine variety of complex dimension $\leq n$ has the homotopy type of a finite cell complex of dimension $\leq n$.

COROLLARY 3.3. *If X is as in the previous theorem and $Z \subset \mathbb{P}$ is a linear subspace transversal to X. Then $(X, X \cap Z)$ is $\dim(X \cap Z)$-connected.*

PROOF. Use iterated application of the first assertion of the weak Lefschetz theorem and the long exact homotopy sequence for a triple. □

So if X is connected (equivalently, irreducible) of dimension ≥ 2, then any transversal linear section $X' \subset X$ of dimension ≥ 1 is still connected (equivalently, irreducible) and induces a surjection on fundamental groups (even an isomorphism if $\dim X' \geq 2$).

We prepare for the proof of the weak Lefschetz theorem by first recalling the complex-analytic version of the Morse lemma [17].

LEMMA 3.4 (Complex-analytic Morse lemma). *Let Z be a complex manifold of dimension n and $f : Z \to \mathbb{C}$ a holomorphic function. If f has a critical point at $x \in Z$ and the hessian of f is nondegenerate at x, then we can find local analytic coordinates $(z_1, \dots, z_n)$ at x such that f is there given by $\sum_{i=1}^n z_i^2$.*

3.1. A model for a quadratic singularity. This leads us to take a closer look at the topology of the function $f : \mathbb{C}^n \to \mathbb{C}$ defined by $f(z) = \sum_{i=1}^n z_i^2$. Choose $\varepsilon > 0$ and let $B_\varepsilon \subset \mathbb{C}^n$ be set of $z \in \mathbb{C}^n$ with $\|z\| \leq \varepsilon$.

First assume that $n > 1$. We then consider the map $z \in \mathbb{C}^n \mapsto (f(z), \|z\|^2)$. One verifies that a critical point is of the form λx, where $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$. Its value is $(\lambda^2\|x\|^2, |\lambda|^2\|x\|^2)$ and so this is a submersion wherever $|f(z)| < \|z\|^2$. So the fiber over $(0, \varepsilon^2)$, $K_\varepsilon := f^{-1}(0) \cap \partial B_\varepsilon$, is a submanifold of ∂B_ε and $f|\partial B_\varepsilon$ is a submersion at K_ε. Note that $f^{-1}(0) \cap B_\varepsilon$ is simply the cone over K_ε; this is why

K_ε is often called the *link* of the singular point 0 of $f^{-1}(0)$. It also follows from the preceding that if we take $0 < \eta < \varepsilon^2$, then $f|\partial B_\varepsilon$ is a submersion over the closed disk $D := \{w \in \mathbb{C} : \|w\| \leq \eta\}$. By the Ehresmann fibration theorem, $f|\partial B_\varepsilon$ is then locally trivial, and since D is contractible, even trivial. We put $B := B_\varepsilon \cap f^{-1}D$ and $\dot{B} := \partial B_\varepsilon \cap f^{-1}D$. Note that both spaces are compact. This makes also sense when $n = 1$, but then things greatly simplify: K_ε and $\dot{B}$ will be empty and $B = f^{-1}D = B_{\sqrt{\eta}}$.

We consider from now on f as a map $f : B \to D$ and write $\dot{f}$ for its restriction $\dot{B} \to D$. By construction $\dot{f}$ is a trivial fibration. We also note that $f : (B, \dot{B}) \to D$ is submersion of a pair except at 0 and hence, by the Ehresmann fibration theorem, locally trivial over $D - \{0\}$. Let us see what the fiber B_η is like. It consists of the $z = x + \sqrt{-1}y$ with the property that $\|z\| = \|x\| + \|y\|^2 \leq \varepsilon^2$ and $\eta = \sum_i z_i^2 = \|x\|^2 - \|y\|^2 + \sqrt{-1}\langle x, y\rangle$. This amounts to: $\|x\|^2 + \|y\|^2 \leq \varepsilon^2$, $\|x\|^2 = \eta + \|y\|^2$, $\langle x, y\rangle = 0$. So if we put

$$u = x/\|x\|, \quad v := y/\sqrt{(\varepsilon^2 - \eta)/2},$$

then (u, v) maps B_η diffeomorphically onto the the set of $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $\|u\| = 1$, $\|v\| \leq 1$ and $\langle u, v\rangle = 0$, in other words, onto the unit disk bundle of the tangent bundle $T^{\leq 1}S^{n-1}$ of the $(n-1)$-sphere. The zero section of this bundle (the copy of the $(n-1)$-sphere given by $v = 0$) is on B_η given by $y = 0$. So this is simply the real part of B_η (the sphere of radius η in $\mathbb{R}^n$) and this discussion shows that this is a deformation retract of B_η. We shall refer to it as a *vanishing sphere* for the pair (f, η). Note that this vanishing sphere bounds the closed ball of radius η in $\mathbb{R}^n$ and that this ball is contained in B. We denote that ball by e and refer to is as a *Lefschetz thimble* for (f, η).

LEMMA 3.5. *Each of the inclusions $e \subset B_\eta \cup e \subset \dot{B} \cup B_\eta \cup e \subset B$ is a deformation retract.*

PROOF. Since $B_\eta \cap e$ is the vanishing sphere (the 'zero section' of B_η), it is a deformation retract of B_η and hence e is a deformation retract of $B_\eta \cup e$. Since $\dot{B} \to D$ is trivial, it has $\dot{B}_\eta$ as a deformation retract and since $\dot{B}_\eta = \dot{B} \cap (B_\eta \cup e)$ it follows that the second inclusion is a deformation retract. That the last inclusion is also one can probably be shown directly, but tedious work can be avoided by observing that B is starlike and hence contractible. Since e is contractible so is $\dot{B} \cup B_\eta \cup e$ and hence the last inclusion is a homotopy equivalence. Now apply Proposition 2.1. $\square$

We apply this lemma to the case when f is part of a more global situation:

LEMMA 3.6. *Let Z be a compact $2n$-manifold with boundary and $F : Z \to D$ a smooth mapping such that $\partial Z = F^{-1}\partial D$. Suppose we are* *given an embedding $j : B \hookrightarrow Z$ such that F extends f: $f = Fj$, and assume that F is a submersion away from $j(0)$. Then $j(e) \cup Z_\eta \subset Z$ is a deformation retract. In particular, Z is obtained from Z_η up to homotopy by attaching an n-cell over the vanishing sphere $j(\partial e)$.*

If we are further given an open neighborhood W of $j(B)$ whose topological boundary is C^∞-smooth such that F restricted to this boundary is a submersion, then $j(e) \cup W_\eta \subset W$ is also a deformation retract.

PROOF. $\dot{Z} := Z \smallsetminus j(B \smallsetminus \dot{B})$. Since F has no critical points except $j(0)$, the Ehresmann fibration theorem implies that $\dot{Z} \to D$ is locally trivial, hence trivial (because D is contractible) and so $\dot{Z}_\eta \subset W$ is a deformation retract.

Lemma 3.6 shows that $j(e) \cup Z_\eta \cup \dot{Z} \subset Z$ is a deformation retract. Since $\dot{Z}_\eta \subset \dot{Z}$ is a deformation retract, $j(e) \cup Z_\eta \subset j(e) \cup Z_\eta \cup \dot{Z}$ is also one.

The proof of the last assertion involves a minor modification of the preceding argument: our hypotheses and the Ehresmann fibration theorem imply that the pair $(\dot{Z}, Z \smallsetminus W)$ is (locally) trivial over D so that in particular $W \cap \dot{Z}$ is trivial over D. Then proceed as before. $\square$

3.2. Geometry of the weak Lefschetz theorem. We now turn to the situation of the weak Lefschetz theorem. Without loss of generality we may assume that X is connected, or equivalently, irreducible. According to Lemma 1.9, there exists a line $L \subset \check{\mathbb{P}}$ passing through ξ which is transversal to $X^\vee$. This defines a general Lefschetz pencil on $X^\vee$. We use the notation of the previous lecture. So we have a Lefschetz fibration $f : \tilde{X} \to X$ satisfying the properties listed in Proposition 1.10. Write r for the class of X so that $L \cap X$ consists of r distinct points $\xi_1, \dots, \xi_r$. We put $\mathring{X} := X \smallsetminus Y = \tilde{X} \smallsetminus \tilde{Y}$ and write $\mathring{f} : \mathring{X} \to L$ for the restriction of f.

We choose an affine coordinate w on L such that $w(\xi) = 1$ and $|w(\xi_\rho)| < 1$ for all $\rho = 1, \dots, r$. This decomposes L in two disks: L_0 defined by $|w| \leq 1$ (whose interior contains the critical values of f) and L_∞ $|w| \geq 1$ (including $w = \infty$), over which f is C^∞-trivial. For $\rho = 1, \dots, r$, we choose a small disk $D_\rho \subset L$ defined by $|w - w(\xi_\rho)| \leq \varepsilon_\rho$ so that

(i) the disks $D_1, \dots, D_r$ are pairwise disjoint and contained in the interior of L_0 and
(ii) over each $f^{-1}D_\rho$ we have the situation of Lemma 3.6: if ξ'_ρ is defined by $w(\xi'_\rho) = w(\xi_\rho) + \varepsilon_\rho$, then we have defined a Lefschetz thimble $e_\rho \subset f^{-1}D_\rho$ with boundary on $X_{\xi'_\rho}$ such that $e_\rho \cup X_{\xi'_\rho} \subset f^{-1}D_\rho$ is a deformation retract. With $\mathring{f}^{-1}D_\rho$ taking the role of W in 3.6, we also see that $e_\rho \cup \mathring{X}_{\xi'_\rho} \subset \mathring{f}^{-1}D_\rho$ is a deformation retract.

Choose an arc (i.e., a smooth copy of the unit interval) γ_ρ in L_0 with end points ξ and ξ'_ρ whose relative interior does not meet L_∞, any D_σ ($\sigma = 1, \dots, r$) or any other γ_σ with $\sigma \neq \rho$. A trivialization of $f : (\tilde{X}, \tilde{Y}) \to L$ over γ_ρ, $(f^{-1}\gamma_\rho, Y \times \gamma_\rho) \cong (X_\xi, Y) \times \gamma_\rho$, enables us to pass from the deformation retract in (ii) above to a deformation retract

$$\tilde{e}_\rho \cup X_\xi \subset f^{-1}(D_\rho \cup \gamma_\rho), \quad \tilde{e}_\rho \cup \mathring{X}_\xi \subset \mathring{f}^{-1}(D_\rho \cup \gamma_\rho),$$

where $\tilde{e}_\rho$ is an elongated Lefschetz thimble whose boundary lies in $\mathring{X}_\xi$: the union of e_ρ and a copy of a cylinder $\gamma \times \partial e_\rho$ over γ_ρ, and so still a topological n-disk. So if we put $E := \cup_{\rho=1}^r D_\rho \cup \gamma_\rho$, then these deformation retracts together produce deformation retracts

$$\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi \subset f^{-1}E, \quad \cup_{\rho=1}^r \tilde{e}_\rho \cup \mathring{X}_\xi \subset \mathring{f}^{-1}E$$

COROLLARY 3.7. *The inclusions $\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi \subset f^{-1}L_0$ and $\cup_{\rho=1}^r \tilde{e}_\rho \cup \mathring{X}_\xi \subset \mathring{f}^{-1}L_0$ are deformation retracts and $(\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi, X_\xi)$ and $(\cup_{\rho=1}^r \tilde{e}_\rho \cup \mathring{X}_\xi, \mathring{X}_\xi)$ are n-cellular extensions. In particular, $(f^{-1}L_0, X_\xi)$ and $(\mathring{f}^{-1}L_0, \mathring{X}_\xi)$ have the homotopy type of an n-cellular extension.*

PROOF. The inclusion $E \subset L_0$ is a deformation retract. This can be shown by constructing one (not hard, if the $\gamma'_\rho s$ haven been chosen nicely), or we may observe that the pair (L_0, E) can be given the structure of a CW complex and since both items are contractible, one may then invoke Proposition 2.1. Since f is locally trivial over $L_0 - \{\xi_1, \dots, \xi_r\}$, this deformation retraction lifts to the f-preimages: $f^{-1}E \subset f^{-1}L_0$ is a deformation retraction retract. Composing this with the deformation retraction retract $\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi \subset f^{-1}E$ found above completes the proof for X_ξ. The proof for $\mathring{X}_\xi$ is similar. □

We are now ready to prove the weak Lefschetz theorem:

PROOF OF THEOREM 3.1. We proceed with induction on n. Since Y is a nonsingular hyperplane section of X_ξ, we may then assume that (X_ξ, Y) is $(n-2)$-connected (for $n = 1$, this is an empty property) and that $\mathring{X}_\xi$ has the homotopy type a finite CW complex of dimension $\leq n - 1$.

We first note that we have a relative homeomorphism $(\tilde{X}, X_\xi \cup \tilde{Y}) \xrightarrow{\pi} (X, X_\xi)$ and so by Proposition 2.2, the first assertion of Theorem 3.1 will follow if we show that $(\tilde{X}, X_\xi \cup \tilde{Y})$ is $(n-1)$-connected. To this end we consider the triple

$$(\tilde{X}, f^{-1}L_0 \cup \tilde{Y}, X_\xi \cup \tilde{Y}).$$

In view of the long exact homotopy sequence of the above triple, it then suffices to show that its sub-pairs $(f^{-1}L_0 \cup \tilde{Y}, X_\xi \cup \tilde{Y})$ and $(\tilde{X}, f^{-1}L_0 \cup \tilde{Y})$ and are $(n-1)$-connected.

The inclusion $(f^{-1}L_0\cup\tilde{Y}, X_\xi\cup\tilde{Y})\supset(f^{-1}L_0, X_\xi)$ is a relative homeomorphism and since $(f^{-1}L_0, X_\xi)$ is $(n-1)$-connected by Corollary 3.7, it follows from Proposition 2.2 that $(f^{-1}L_0\cup\tilde{Y}, X_\xi\cup\tilde{Y})$ is $(n-1)$-connected.

We also have a relative homeomorphism

$$(\tilde{X}, f^{-1}L_0\cup\tilde{Y})\supset(f^{-1}L_\infty, f^{-1}\partial L_\infty\cup(Y\times L_\infty)).$$

Since $f:(\tilde{X},\tilde{Y})\to L$ is trivial over L_∞,

$$(f^{-1}L_\infty, f^{-1}\partial L_\infty\cup(Y\times L_\infty))\cong(X_\xi, Y)\times(L_\infty,\partial L_\infty).$$

By induction hypothesis (X_ξ, Y) is $(n-2)$-connected, and clearly, $(L_\infty,\partial L_\infty)$ is 1-connected. So it follows from Proposition 2.3 that $(X_\xi, Y)\times(L_\infty,\partial L_\infty)$ is n-connected. So by Proposition 2.2, $(\tilde{X}, f^{-1}L_0\cup\tilde{Y})$ is also n-connected.

As to the second assertion, the local triviality of $\mathring{f}$ over L_∞ implies that the all maps in the diagram

$$X\smallsetminus X_\xi\overset{\cong}{\leftarrow}\tilde{X}\smallsetminus(X_\xi\cup\tilde{Y})\supset\tilde{X}\smallsetminus(f^{-1}L_\infty\cup\tilde{Y})\subset\mathring{f}^{-1}L_0$$

are homotopy equivalences and so it suffices to show that $\mathring{f}^{-1}L_0$ has the homotopy type a finite CW complex of dimension $\le n$. This follows from the fact that $(\mathring{f}^{-1}L_0,\mathring{X}_\xi)$ has the homotopy type of an n-cellular extension and our inductive assumption that $\mathring{X}_\xi$ has the homotopy type a finite CW complex of dimension $\le n-1$. □

The above argument proved that $(\tilde{X}, f^{-1}L_0\cup\tilde{Y})$ is n-connected, but only used its $(n-1)$-connectivity. We can exploit this extra bit of information by looking at the exact homology sequence of our triple. We have $H_k(X, X_\xi)\cong H_k(\tilde{X}, X_\xi\cup\tilde{Y})$ and

$$H_k(\tilde{X}, f^{-1}L_0\cup\tilde{Y})\cong H_k((X_\xi, Y)\times(L_\infty,\partial L_\infty))=H_{k-2}(X_\xi, Y)$$

by the Künneth theorem (for $H_l(L_\infty,\partial L_\infty)$ is naturally identified with $\mathbb{Z}$ for $l=2$ and is zero otherwise) so that the inclusion $(\tilde{X}, X_\xi\cup\tilde{Y})\subset(\tilde{X}, f^{-1}L_0\cup\tilde{Y})$ defines a map

$$\mathcal{L}: H_k(X, X_\xi)\to H_{k-2}(X_\xi, Y).$$

This map can be understood as follows: let $\xi'\in L$ be characterized by $w(\xi')=\infty$ and let $i: X_\xi\smallsetminus Y\subset X\smallsetminus X_{\xi'}$ be the inclusion. Then $\mathcal{L}$ is the composite of the identification $H_k(X, X_\xi)\cong H_k(X, X_{\xi'})$ and the Gysin map

$$i^!: H_k(X, X_{\xi'})\cong H^{2n-k}(X\smallsetminus X_{\xi'})\xrightarrow{i^*}H^{2n-k}(X_\xi\smallsetminus Y)\cong H_{k-2}(X_\xi, Y),$$

where we used on both sides we used Alexander duality. Geometrically it is given as follows: the hyperplane H_ξ meets $(X, X_{\xi'})$ transversally with intersection (X_ξ, Y) and intersecting a relative k-cycle on $(X, X_{\xi'})$ that is in general position with respect to H_ξ then yields a relative $(k-2)$-cycle on (X, X_ξ) and this induces $i^!$.

COROLLARY 3.8. *The map $\mathcal{L} : H_k(X, X_\xi) \to H_{k-2}(X_\xi, Y)$ is an isomorphism for $k \neq n, n+1$ (but for $k < n$ we already knew this, as both sides are zero by the weak Lefschetz theorem) and we have an exact sequence*

$$0 \to H_{n+1}(X, X_\xi) \xrightarrow{\mathcal{L}} H_{n-1}(X_\xi, Y) \to \oplus_{\rho=1}^r H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_n(X, X_\xi) \to 0,$$

where the direct sum can be identified with $\mathbb{Z}^r$ after we have given each Lefschetz thimble $\tilde{e}_\rho$ an orientation.

Moreover, we have the following identity of topological Euler characteristics:

$$e(X) + e(Y) - 2e(X_\xi) = e(X, X_\xi) - e(X_\xi, Y) = (-1)^n \operatorname{class}(X).$$

PROOF. The map

$$(\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi, X_\xi) \subset (f^{-1}L_0, X_\xi) \subset (f^{-1}L_0 \cup \tilde{Y}, X_\xi \cup \tilde{Y})$$

induces an isomorphism on homology and excision shows that the homology of the left hand side is zero except in degree n, for which the natural map $\oplus_{\rho=1}^r H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_n(\cup_{\rho=1}^r \tilde{e}_\rho \cup X_\xi, X_\xi)$ is an isomorphism. The first part of the corollary then follows from the long exact sequence

$$\to \oplus_\rho H_k(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_k(X, X_\xi) \xrightarrow{\mathcal{L}} H_{k-2}(X_\xi, Y) \to \oplus_\rho H_{k-1}(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to$$

The last assertion follows from the additivity of Euler characteristics for (finite) long exact sequences. □

REMARK 3.9. Dually, we have $\mathcal{L}^\vee : H^{k-2}(X_\xi, Y) \to H^k(X, X_\xi)$ (which is just the composite $H^{k-2}(X_\xi, Y) \xrightarrow{i_!} H^k(X, X_{\xi'}) \cong H^k(X, X_\xi)$, see Section 5) for which the analogue of Corollary 3.8 holds (with a similar proof): it is an isomorphism for $k \neq n, n+1$ and we have an exact sequence

$$0 \to H^n(X, X_\xi) \to \oplus_{\rho=1}^r H^n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H^{n-1}(X_\xi, Y) \xrightarrow{\mathcal{L}^\vee} H^{n+1}(X, X_\xi) \to 0.$$

DEFINITION 3.10. The *vanishing homology* of X_ξ in X, $H_{n-1}^{van}(X_\xi)$, is the kernel of $i_* : H_{n-1}(X_\xi) \to H_{n-1}(X)$ (this is also the image of $\partial : H_n(X, X_\xi) \to H_{n-1}(X_\xi)$). Dually, the *vanishing cohomology* $H^n_{van}(X_\xi)$ of X_ξ in X is the cokernel of $H^{n-1}(X) \to H^{n-1}(X_\xi)$ (which may be identified with the kernel of the map $H^n(X, X_\xi) \to H^n(X)$).

We call this so because $H_{n-1}^{van}(X_\xi)$ is the part of $H_{n-1}(X_\xi)$ that 'vanishes' in $H_{n-1}(X)$ (but $H^n_{van}(X_\xi)$ might have been better termed *covanishing* cohomology).

COROLLARY 3.11. *The vanishing homology $H_{n-1}^{van}(X_\xi)$ is generated by the classes of the vanishing spheres $\partial\tilde{e}_1, \dots, \partial\tilde{e}_1$ (after orientation) and $H_{n-1}(X)$ may be identified with $H_{n-1}(X_\xi)/H_{n-1}^{van}(X_\xi)$.*

PROOF. The composite map

$$\oplus_{\rho=1}^{r} H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_n(X, X_\xi) \xrightarrow{\partial} H_{n-1}(X_\xi)$$

is, when restricted to the summand $H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho)$, equal to the composite $H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_{n-1}(\partial\tilde{e}_\rho) \to H_{n-1}(X_\xi)$ and hence takes the class of $\partial\tilde{e}_\rho$ (after it has been oriented) to its image in $H_{n-1}(X_\xi)$. Since $\oplus_{\rho=1}^{r} H_n(\tilde{e}_\rho, \partial\tilde{e}_\rho) \to H_n(X, X_\xi)$ is onto, the first assertion follows. Since $H_{n-1}(X, X_\xi) = 0$, $i_* : H_{n-1}(X_\xi) \to H_{n-1}(X)$ is onto. This implies the second assertion. □

EXAMPLE 3.12. We take for $X \subset \mathbb{P}^2$ a nonsingular plane curve of degree $d > 1$. Then X_ξ is a finite set of d points, Y is empty and by Example 1.11 $\mathrm{class}(X) = d(d-1)$. So $e(X) = 2e(Y) - \mathrm{class}(X) = 2d - d(d-1) = -d(d-3)$. Since $e(X) = 2 - g(X)$, where $g(X)$ is the genus of X, we recover the familiar formula $g(X) = \frac{1}{2}(d-1)(d-2)$.

EXAMPLE 3.13. More generally, let $X = X_d^n \subset \mathbb{P}$ be a nonsingular hypersurface of dimension n and of degree $d > 1$ (so $\dim \mathbb{P} = n+1$). A transversal hyperplane section of X is a nonsingular hypersurface of degree d of dimension $n-1$, and so let us denote by X_d^{n-1}. Proceeding in this manner we find a chain $X = X_d^n \supset X_d^{n-1} \supset \cdots X_d^{-1} = \emptyset$. If we combine Corollary 3.8 and Example 1.11 we find that $e(X_d^r, X_d^{r-1}) = e(X_d^{r-1}, X_d^{r-2}) + (-1)^r d(d-1)^r$. This enables us to compute $e(X_d^r)$ inductively, yielding the formula

$$e(X_d^r) = n + 1 + (d-1)(1 - (1-d)^{n+1})/d.$$

Now recall that $H_k(\mathbb{P}^{n+1})$ can be identified with $\mathbb{Z}$ for $k = 2r$, $0 \le r \le n+1$, where we can takes for generator the fundamental class of $\mathbb{P}^r \subset \mathbb{P}^{n+1}$, and is zero otherwise. The same is true for $H^k(\mathbb{P}^{n+1})$ (a generator of $H^{2r}(\mathbb{P}^{n+1})$ is the the Poincaré dual of the fundamental class of $\mathbb{P}^{n+1-r} \subset \mathbb{P}^{n+1}$). If we regard X as a hyperplane section of $\mathbb{P}^{n+1}$ with respect to the d-fold Veronese embedding (so that $(\mathbb{P}^{n+1}, X)$ takes the role of (X, X_ξ)), then the weak Lefschetz theorem tells us that for $k < n$ $H_k(X) \xrightarrow{\cong} H_k(\mathbb{P}^{n+1})$ and $H^k(X) \xleftarrow{\cong} H^k(\mathbb{P}^{n+1})$. By Poincaré duality, we have then for $k > n$, $H_k(X) \cong H^{2n-k}(X) \cong H^{2n-k}(\mathbb{P}^n)$ and $H^k(X) \cong H_{2n-k}(X) \cong H_{2n-k}(\mathbb{P}^n)$. So the only (co)homology of X that is unaccounted for sits in the middle dimension n and is the vanishing homology. Since we have $e(X) = n + 1 + (-1)^n \mathrm{rk}(H_n^{van}(X))$, we obtain

$$\mathrm{rk}\, H_n^{van}(X) = (d-1)\big((d-1)^{n+1} - (-1)^{n+1}\big)/d.$$

4. The Picard-Lefschetz formula and vanishing lattices

4.1. Monodromy and the variation homomorphism. Suppose we are given a locally trivial map of topological spaces $f : E \to S$. If $\gamma : [0,1] \to S$ is a path from $p \in S$ to $q \in S$ then the pull back

$\gamma^*E \to [0,1]$ is locally trivial, and hence trivial. A trivialization defines a homeomorphism $h_\gamma : E_p \xrightarrow{\cong} E_q$ that is unique up isotopy and thus yields a well-defined isotopy class $[h_\gamma] \in \pi_0(\mathrm{Homeo}(E_p, E_q))$. This isotopy class only depends on the homotopy class $[\gamma]$ of γ and composition of homotopy classes of paths corresponds to composition of isotopy classes. An efficient way to express these observations is to say that $[\gamma] \mapsto [h_\gamma]$ defines a functor from the fundamental groupoid of S (whose objects are points of S and morphisms are homotopy classes of paths) to the groupoid whose objects are also points of S, but for which a morphism $p \to q$ is given by an element of $\pi_0(\mathrm{Homeo}(E_p, E_q))$. Here we adopt the convention that if γ_1 is a path from p_0 to p_1 and γ_2 one from p_1 to p_2, then the composite path is written $\gamma_2\gamma_1$. In particular, we get a group homomorphism $\pi_1(S,p) \to \pi_0(\mathrm{Homeo}(E_p))$. This is often called the *geometric monodromy* of f. It determines a (monodromy) representation of $\pi_1(S,p)$ on $H_\bullet(E_p)$ (given by $[\gamma] \mapsto h_{\gamma*}$) and on $H^\bullet(E_p)$ (given by $[\gamma] \mapsto (h^*_\gamma)^{-1}$). When f is a submersion of oriented manifolds and the fiber $E - p$ has dimension $2m$, then h_γ can be represented by an orientation preserving diffeomorphism and hence will preserve the intersection pairing on $H_m(E_p)$ (which is $(-1)^m$-symmetric).

We also note that if we are given a subspace $\dot{E} \subset E$ such that the pair $(E, \dot{E})$ is locally trivial, then the geometric monodromy takes values $\pi_0(\mathrm{Homeo}(E_p, \dot{E}_p))$ so that we get representations of $\pi_1(S,p)$ on the long exact (co)homology sequence of the pair $(E, \dot{E})$. Of special interest is the case when we are given a trivialization of $\dot{E}$, or rather, an isotopy class of trivialization of $\dot{E}$, and we take our local trivializations compatible with this: then h_γ is a homeomorphism that is the identity on $\dot{E}_p$. The corresponding isotopy class (the connected component of homeomorphisms of E_p that are the identity on $\dot{E}_p$) is well-defined. If $w \in H_k(E_p, \dot{E}_p)$ is represented by a chain W on E_p whose boundary has support in $\dot{E}_p$, then $h_{\gamma*}W$ and W have the same boundary and so their difference $h_{\gamma*}W - W$ is then a k-cycle on E_p. The class of this cycle only depends of the isotopy class of h_γ and so we get a well-defined element of $H_k(E_p)$. The resulting map is called the *variation map*:

$$\mathrm{var}_{[\gamma]} : H_k(E_p, \dot{E}_p) \to H_k(E_p)$$

It determines the action of h_γ on both $H_k(E_p)$ and $(E_p, \dot{E}_p)$, for if $j : (E_p, \emptyset) \subset (E_p, \dot{E}_p)$ denotes the inclusion, then we have factorizations

$$h_{\gamma*} - 1 : H_k(E_p) \xrightarrow{j_*} H_k(E_p, \dot{E}_p) \xrightarrow{\mathrm{var}_{[\gamma]}} H_k(E_p),$$
$$h_{\gamma*} - 1 : H_k(E_p, \dot{E}_p) \xrightarrow{\mathrm{var}_{[\gamma]}} H_k(E_p) \xrightarrow{j_*} H_k(E_p, \dot{E}_p).$$

and so $\mathrm{var}_{[\gamma]}$ determines the action of $h_{\gamma*}$ on the exact sequence of the pair $(E_p\dot{E}_p)$.

4.2. The fibration near a quadratic singularity. A Lefschetz pencil $f : \tilde{X} \to L$ (situation and notation as before), $(\tilde{X}, \tilde{Y})$ is locally trivial over $L \smallsetminus \check{X}$ and so we have a monodromy representation of $\pi_1(L \smallsetminus \check{X}, \xi)$ on $H_\bullet(X_\xi, Y))$ and $H_\bullet(X_\xi)$. If we traverse the arc γ_ρ from ξ to ξ'_ρ and subsequently the circle ∂D_ρ in the counterclockwise direction and then return via γ_ρ, we get a homotopy class $\alpha_\rho \in \pi_1(L \smallsetminus \check{X}, \xi)$. These generate $\pi_1(L \smallsetminus \check{X}, \xi)$ and so the monodromy representation is completely determined by its values on $\alpha_1, \dots, \alpha_r$. We will therefore need to understand the monodromy of f over D_ρ. (We shall later assume that we have chosen our arcs $\gamma_1, \dots, \gamma_r$ in such a manner that they have distinct rays of departure in $T_\xi L$ such that these rays have the obvious cyclic (counterclockwise) ordering modulo r; then $\alpha_r \alpha_{r-1} \cdots \alpha_1$ is homotopic to the counterclockwise loop defined by the boundary of L_0 and hence trivial.)

4.3. The Picard-Lefschetz formula. We apply the preceding to the situation of Lemma 3.6. Here $F : W \to D$ is a fibration over $D \smallsetminus \{0\}$ and is trivial when restricted to $\dot{W} = W \smallsetminus j(B \smallsetminus \dot{B})$. This implies that the geometric monodromy of h_F over the simple loop which traverses ∂D counterclockwise is a homeomorphism H of W_η which is the identity on $\dot{W}_\eta$. Such geometric monodromy pulls back via $j : B \to W$ to a homeomorphism h of B_η which is the identity on $\dot{B}_\eta$. The variation of h_F is then determined by the variation of h, because it is composite

$$\mathrm{var}(h_{F*}) : H_k(W_\eta, \dot{W}_\eta) \xleftarrow{\cong} H_k(B_\eta, \dot{B}_\eta) \xrightarrow{\mathrm{var}(h_*)} H_k(B_\eta) \xrightarrow{j_{\eta*}} H_k(W_\eta),$$

where the first map is an isomorphism by excision. We will therefore want to know $\mathrm{var}(h_*)$. Since the focus is now on the fiber rather than on the total space, we express our results in terms of $m := n - 1$.

Since $(B_\eta, \dot{B}_\eta)$ is diffeomorphic to the unit disk bundle of the m-sphere, Alexander duality (where we use the complex orientation) yields $H_k(B_\eta, \dot{B}_\eta) \cong H^{2m-k}(B_\eta) \cong \mathrm{Hom}(H_{2m-k}(B_\eta), \mathbb{Z})$, which is zero unless $k = m$ or $k = 2m$. We orient the vanishing sphere ∂e in B_η and then get a generator for $H_m(B_\eta)$ that we shall call a *vanishing cycle* for f and denote by δ. We denote by δ^* the generator of $H_m(B_\eta, \dot{B}_\eta) \cong \mathrm{Hom}(H_m(B_\eta), \mathbb{Z})$ that takes the value 1 on δ. The *self-intersection* $\delta \cdot \delta \in \mathbb{Z}$ may be defined by the formula $i_*\delta = (\delta \cdot \delta)\delta^*$, where $i : (B_\eta, \emptyset) \subset (B_\eta, \dot{B}_\eta)$ is the inclusion. To compute this, let us recall that the tangent bundle of a compact oriented manifold inherits from that manifold an orientation and for *this orientation* the self-intersection of the zero section is the Euler characteristic of that manifold. So for S^m this gives $1 + (-1)^m$. However, the orientation of B_η that we use is that of its complex structure. The sign by which this differs from the tangent bundle of S^m is that of the permutation $(u_1, \dots, u_m, v_1, \dots, v_m) \mapsto (u_1, v_1, \dots, u_m, v_m)$, which is $(-1)^{m(m-1)/2}$ and so we find

$$\delta \cdot \delta = (-1)^{m(m-1)/2} + (-1)^{m(m+1)/2}.$$

For odd m, we get $\delta \cdot \delta = 0$, which was a priori clear as the intersection pairing on $H_m(B_\eta)$ is then antisymmetric. For even m the intersection pairing on $H_m(B_\eta)$ is symmetric and we see that $\delta \cdot \delta = 2$ when $m \equiv 0 \pmod 4$ and $\delta \cdot \delta = -2$ when $m \equiv 2 \pmod 4$.

THEOREM 4.1 (Picard-Lefschetz formula). *We have*

$$\mathrm{var}_*(h)(\delta^*) = -(-1)^{m(m-1)/2}\delta$$

The proof of Theorem 4.1 rests on a careful geometric analysis. It is straightforward, but a bit tedious. We omit it and refer to [16].

For $m = 1$, h is represented by a *Dehn twist* (a cylinder whose interior is being twisted) and for other odd m, it is a generalization thereof.

COROLLARY 4.2. *Let $F : W \to D$ and h_F be as above and denote by $k : B_\eta \subset W_\eta$ the inclusion. Then the monodromy $h_{F*} : H_m(W_\eta) \to H_m(W_\eta)$, the self-intersection of the vanishing cycle and the monodromy action on the vanishing cycle is given by the table below.*

$m \pmod 4$	*monodromy on* $H_m(W_\eta)$	$k_*\delta \cdot k_*\delta$	$h_{F*}(k_*\delta)$
0	$v - (v \cdot k_*\delta)k_*\delta$	2	$-k_*\delta$
1	$v - (v \cdot k_*\delta)k_*\delta$	0	$k_*\delta$
2	$v + (v \cdot k_*\delta)k_*\delta$	-2	$-k_*\delta$
3	$v + (v \cdot k_*\delta)k_*\delta$	0	$k_*\delta$

In particular, for m even, $k_*\delta \neq 0$ and h_{F*} is an orthogonal reflection with respect to $k_*\delta$. For m odd, the intersection pairing on $H_m(Z_\eta)$ is symplectic, and h_{F*} is a symplectic transvection .

REMARK 4.3. It is not difficult to see that the central fiber $W_0 \subset W$ is a deformation retract. We also know that $e \cup W_\eta \subset W$ is a deformation retract and so attaching the $(m+1)$-cell to W_η yields W_0 up to homotopy. This proves that $H_k(W_\eta) \cong H_k(W_0)$ for $k \neq m, m+1$ and that we have an exact sequence

$$0 \to H_{m+1}(W_\eta) \to H_{m+1}(W_0) \to \mathbb{Z}\delta \to H_m(W_\eta) \to H_m(W_0) \to 0,$$

where δ is mapped to $k_*\delta \in H_m(W_\eta)$. The monodromy acts trivially on all the terms except possibly $\mathbb{Z}\delta$ and $H_m(W_\eta)$. Since the monodromy action on $H_m(W_\eta)$ is given by $a \mapsto a \pm (a \cdot k_*\delta)\delta$, the monodromy action on $H_m(W_\eta)$ is nontrivial if and only if taking the intersection product with δ defines zero linear form on $H_m(W_\eta)$ (which is always the case when m is even). In that case $H_m(W_0)$ appears as its group of co-invariants, that is, as the largest quotient $H_m(W_\eta)_{h_{F*}}$ of $H_m(W_\eta)$ on which h_{F*} acts trivially.

Similarly for cohomology: we have an exact sequence

$$0 \to H^m(W_0) \to H^m(W_\eta) \to \mathbb{Z}\delta^* \to H^{m+1}(W_0) \to H^{m+1}(W_\eta) \to 0,$$

where the monodromy action on $H^m(W_\eta)$ is given by $\alpha \mapsto \alpha \pm \langle\alpha|\delta\rangle\delta^*$. So when δ defines a nonzero linear form on $H^m(W_\eta)$, the monodromy action on $H^m(W_\eta)$ is nontrivial and has $H^m(W_0)$ as its subgroup of invariants.

4.4. The monodromy representation defined by a Lefschetz pencil. We return to the situation where $X \subset \mathbb{P}$ is a nonsingular *irreducible* (Zariski) closed subset of dimension $n = m + 1$ of class r, X_ξ is a transversal hyperplane section (so of dimension m) and $\xi \in L \subset \check{\mathbb{P}}$ a Lefschetz pencil. Recall that the projection $q_X : W_X \to \check{\mathbb{P}}$ has critical set N_X whose image in $\check{\mathbb{P}}$ is $X^\vee$. In particular $\xi \in \check{\mathbb{P}} \smallsetminus X^\vee$ and q_X is C^∞-locally trivial over $\check{\mathbb{P}} \smallsetminus X^\vee$. We therefore have a monodromy representation of $\pi_1(\check{\mathbb{P}} \smallsetminus X^\vee, \xi)$ on $H_m(X_\xi)$. This extends the monodromy representation of $\pi_1(L \smallsetminus X^\vee, \xi)$ on $H_m(X_\xi)$. The image is however the same:

PROPOSITION 4.4 (Zariski). *Let $\mathbb{P}$ be a projective space, $V \subset \mathbb{P}$ a proper closed subset, $p \in \mathbb{P} \smallsetminus V$ and $L \subset \mathbb{P}$ a line through p transversal to V. Then the inclusion $L \smallsetminus V \subset \mathbb{P} \smallsetminus V$ induces a surjection on fundamental groups. Moreover, if V is irreducible, then the elements of $\pi_1(\mathbb{P} \smallsetminus V, p)$ defined by simple loops in $L \smallsetminus V$ lie in the same conjugacy class.*

PROOF. Denote by $\mathbb{P}(p)$ the projective space of lines in $\mathbb{P}$ that pass through p. Then the space set of pairs $(p', [L']) \in \mathbb{P} \times \mathbb{P}(p)$ with $p' \in L'$ is a closed subset that we shall denote by $\mathrm{Bl}_p \mathbb{P}$ (because it is just the blowup of p in $\mathbb{P}$). The projection $\mathrm{Bl}_p \mathbb{P} \to \mathbb{P}(p)$ is a $\mathbb{P}^1$-bundle which comes with a section $\sigma : [L'] \in \mathbb{P}(p) \mapsto (p, [L']) \in \mathrm{Bl}_p \mathbb{P}$. We identify V with its image in $\mathrm{Bl}_p \mathbb{P}$. Note that the projection $\mathrm{Bl}_p \mathbb{P} \smallsetminus V \to \mathbb{P} \smallsetminus V$ induces an isomorphism on fundamental groups.

The projection $V \to \mathbb{P}(p)$ is finite and so there exists a Zariski-open neighborhood U of $[L]$ in $\mathbb{P}(p)$ such that $V \to \mathbb{P}(p)$ is an unramified covering over U or is empty (this happens when V is of codimension ≥ 2). Then $\mathrm{Bl}_p \mathbb{P} \smallsetminus V$ is C^∞-locally trivial over U. Since $(\mathrm{Bl}_p \mathbb{P} \smallsetminus V)_U \subset \mathrm{Bl}_p \mathbb{P} \smallsetminus V$ is Zariski open-dense, this inclusion is surjective on fundamental groups. The fiber bundle $(\mathrm{Bl}_p \mathbb{P} \smallsetminus V)_U \to U$ comes with a section, and so we have a semi-direct product of fundamental groups

$$\pi_1((\mathrm{Bl}_p \mathbb{P} \smallsetminus V)_U, \sigma([L])) \cong \pi_1(U, [L]) \ltimes \pi_1(L \smallsetminus V, p).$$

But under the inclusion $(\mathrm{Bl}_p \mathbb{P} \smallsetminus V)_U \subset \mathrm{Bl}_p \mathbb{P} \smallsetminus V$, the factor $\pi_1(U, [L])$ dies under the inclusion $\sigma(U) \subset \sigma(\mathbb{P}(p)) \subset \mathrm{Bl}_p \mathbb{P} \smallsetminus V$ (for $\pi_1(\mathbb{P}(p), [L]) =$

$\{1\})$. This proves that $L \smallsetminus V \subset \mathrm{Bl}_p \mathbb{P} \smallsetminus V$ induces a surjection on fundamental groups and the first assertion follows.

Assume now V irreducible. Suppose $\beta, \beta' \in \pi_1(\mathbb{P} \smallsetminus V, p)$ are represented by simple loops in $\mathbb{P} \smallsetminus V$ which encircle in the positive direction a regular point q resp. q' of $\mathbb{P} \smallsetminus V$. Since V is irreducible, V_{reg} is connected and so there is a path in V_{reg} from q to q'. We can use that path to arrange that $q' = q$. Then the two simple loops differ by conjugation by a loop (which comes near q) and so β and β' belong to the same conjugacy class. □

We apply this to $L \subset \check{\mathbb{P}} \supset X^\vee$ and $\xi \in L \smallsetminus X^\vee$. Since X is irreducible, so is $X^\vee$, and hence we find:

COROLLARY 4.5. *The images of $\alpha_1, \dots, \alpha_r$ in $\pi_1(\check{\mathbb{P}} \smallsetminus X^\vee, \xi)$ generate the latter and lie in a single conjugacy class. In particular, the monodromy representation of $\pi_1(\check{\mathbb{P}} \smallsetminus X^\vee, \xi)$ on $H_m^{van}(X_\xi)$ is generated by the Picard-Lefschetz transformations defined by the $\alpha_1, \dots, \alpha_r$. The associated vanishing spheres, when suitably oriented define vanishing cycles $\delta_1, \dots, \delta_r$ in $H_m^{van}(X_\xi)$ that lie in a single $\pi_1(\check{\mathbb{P}} \smallsetminus X^\vee, \xi)$-orbit.*

This suggests that we should consider the orbit $\Delta_\xi \subset H_m^{van}(X_\xi)$ of $\{\pm\delta\}$, where δ is some vanishing cycle. The situation we then end up with is of sufficient interest to be codified in the following notion.

DEFINITION 4.6. A *vanishing lattice* of weight $w \in \mathbb{Z}/4$ consists of a free abelian group V of finite rank, a $(-1)^w$-symmetric bilinear form $V \times V \xrightarrow{\cdot} \mathbb{Z}$ and a subset $\Delta \subset V$ such that

(i) Δ generates V and $-\Delta = \Delta$,
(ii) if w is even, then for every $\delta \in \Delta$ we have $\delta \cdot \delta = (-1)^{w/2} 2$,
(iii) if for $\delta \in \Delta$, we let $s_\delta \in \mathrm{Aut}(V)$ be defined by $s_\delta(v) = v - (-1)^{w(w-1)/2}(v \cdot \delta)\delta$, then the group Γ_Δ generated by these transformations acts transitively on Δ up to sign.

So part of our discussion is now summed up by saying that $H_m^{van}(X_\xi)$ comes naturally with the structure of a vanishing lattice.

Notice that each s_δ respects the form $(\ \cdot\ )$: for w even it is an orthogonal reflection, whereas for w odd it is a symplectic transvection. The group Γ leaves pointwise fixed the *radical* of V, that is, the subgroup V_0 of $v \in V$ with $v \cdot v' = 0$ for all $v' \in V$. Notice that the image of Δ in V/V_0 also defines a vanishing lattice.

LEMMA 4.7. *Every proper Γ-invariant subspace of $V_\mathbb{C}$ is contained $V_{0\mathbb{C}}$ and (hence) Γ acts reductively on $V_\mathbb{C}$ if and only if $V_0 = 0$ or V is of rank one and of odd weight.*

PROOF. Let $W \subset V_\mathbb{C}$ be a Γ-invariant subspace not contained in $V_{0\mathbb{C}}$. Then there exist $\delta \in \Delta$ and $z \in W$ such that $z \cdot \delta \neq 0$ (we complexified the bilinear form). Since $s_\delta(z) = z \pm (z \cdot \delta)\delta \in W$, it follows that $\delta \in W$. As $\Delta = \Gamma.\{\pm\delta\}$, it follows that $\Delta \subset W$ and so we must have $V_{\mathbb{C}} = W$. If the action of $V_{\mathbb{C}}$ is reductive, then $V_{0\mathbb{C}}$ has a Γ-invariant supplement and by the preceding, this can only happen if $V_0 = 0$ or $V_0 = V$. The last case implies that V is of rank one and of odd weight. □

DEFINITION 4.8. The *invariant homology* $H_m^{inv}(X_\xi)$ is the subgroup of $H_m(X_\xi)$ of $\pi_1(\check{\mathbb{P}} \smallsetminus X^\vee, \xi)$-invariants.

In view of the above argument, this is also the set $v \in H_m^{inv}(X_\xi)$ with $v \cdot \delta_\rho = 0$ for $\rho = 1, \ldots, r$, or equivalently, for which $v \cdot v' = 0$ for all $v' \in H_m^{van}(X_\xi)$. So $H_m^{van}(X_\xi) \cap H_m^{inv}(X_\xi)$ is the radical of $H_m^{van}(X_\xi)$.

REMARKS 4.9. To some extent vanishing systems have been classified. For example in the case of odd weight with nondegenerate form, W. Janssen [14] has shown that the group Γ_Δ contains the the kernel of $\mathrm{Sp}(V) \to \mathrm{Sp}(V/2V)$ provided there exist $\delta, \delta' \in \Delta$ with $\delta \cdot \delta' = 1$. This last condition is satisfied in our case when there a hyperplane section of X which has a singular point of type A_2, i.e., that is in local analytic coordinates given as $z_1^3 + z_2^2 + \cdots + z_n^2$ (this is almost always the case).

When we have even weight and the form is definite, then Δ is a root system with the property that the Weyl group acts transitively on the roots, i.e., one of type $A_{k\geq 1}$, $D_{k\geq 4}$, E_6, E_7 and E_8. In some sense all of these occur in algebraic geometry. For instance, the E_6-case is realized as the vanishing homology of a cubic surface. In case we only know that the form is nondegenerate, W. Ebeling [11] has given sufficient conditions in order that Γ_Δ is of finite index in $O(V)$.

5. The Hard Lefschetz theorem

5.1. Review of the Gysin map. For a space X, the cup product gives its total cohomology $H^\bullet(X)$ the structure of a graded commutative ring with unit (but $X = \emptyset$ gives the zero ring); the cap product makes the total homology $H_\bullet(X)$ a graded unital $H^\bullet(X)$-module: $1 \cap a = 1$ and $(\alpha \cup \beta)\cap = \alpha \cap (\beta \cap a)$ (we adopt the convention to denote homology classes by Latin letters and cohomology classes by Greek ones). The singleton space pt is the final object of the category of topological spaces and we can use the map $\mathrm{pt}_X : X \to \mathrm{pt}$ to define the pairing between cohomology and homology:

$$H^\bullet(X) \times H_\bullet(X) \xrightarrow{\cap} H_\bullet(X) \xrightarrow{\mathrm{pt}_{X*}} H_\bullet(\mathrm{pt}) = H_0(\mathrm{pt}) = \mathbb{Z},$$

denoted by $(\alpha, a) \mapsto \langle \alpha | a \rangle$ (sometimes also suggestively by $\int_a \alpha$). The universal coefficient theorem for cohomology implies that this pairing is perfect if we divide out the torsion. All of this is functorial: if $f : X \to Y$ is a continuous map, then $f^* : H^\bullet(Y) \to H^\bullet(X)$ is a unital ring homomorphism (so $\mathrm{pt}_X^*(1)$ is the unit of $H^\bullet(X)$) and $f_* : H_\bullet(X) \to H_\bullet(Y)$ is a homomorphism of $H^\bullet(Y)$-modules.

Now let M be an oriented manifold without boundary of dimension m. We do not assume that M is compact, but assume that M admits a topological compactification $M \subset \overline{M}$ which is reasonable in the sense that $\partial M := \overline{M} \smallsetminus M$ is closed in $\overline{M}$ and $(\overline{M}, \partial\overline{M})$ has the homotopy type of a finite relative CW complex (this is so when $\overline{M}$ is a compact manifold with boundary). Then $H_\bullet(\overline{M}, \partial\overline{M})$ can be expressed in terms of M as its homology *with closed support*: $H^{cl}_\bullet(M)$([2]). In particular, we have a fundamental class $[M] \in H^{cl}_m(M)$. Likewise, we can identify $H^\bullet(\overline{M}, \partial\overline{M})$ with the cohomology of M *with compact support*, $H^\bullet_c(M)$. The duality map $\alpha \mapsto \alpha \cap [M]$ defines a commutative square with horizontal isomorphisms and obvious vertical maps:

$$\begin{array}{ccc} H^k_c(M) & \xrightarrow[\cong]{\cap [M]} & H_{m-k}(M) \\ \Big\downarrow & & \Big\downarrow \\ H^k(M) & \xrightarrow[\cong]{\cap [M]} & H^{cl}_{m-k}(M) \end{array},$$

where the bottom map takes the unit in $H^0(M)$ to $[M]$. If we combine the isomorphism at the bottom with the pairing between cohomology and homology, then we get the *intersection pairing*

$$(a, b) \in H^{cl}_{m-k}(M) \times H_k(M) \mapsto a \cdot b \in \mathbb{Z}.$$

This is still a perfect pairing if we divide out the torsion.

Let P be another oriented manifold that admits a reasonable topological compactification and let $f : M \to P$ be continuous *proper* map. Then f induces maps $H^{cl}_\bullet(M) \to H^{cl}_\bullet(P)$ and $H^\bullet_c(P) \to H^\bullet_c(M)$ and we define the *Gysin map*

$$f_! : H^k(M) \cong H^{cl}_{m-k}(M) \xrightarrow{f_*} H^{cl}_{m-k}(P) \cong H^{k-m+p}(P) = H^{k+\operatorname{codim} f}(P),$$

where $p := \dim P$ and $\operatorname{codim} f := p - m$ is the *virtual codimension* of f. It is a homomorphism of $H^\bullet(P)$ modules: $f_!(f^*\omega \cup \alpha) = \omega \cup f_!\alpha$. In particular, $f_! f^*\omega = \omega \cup f_!(1)$. Since $f_!$ increases the degree by $\operatorname{codim} f$, this is zero when $\operatorname{codim} f < 0$. In case f is an embedding, $f_!(1)$ is just the cohomology class defined by P as a cycle. There is similarly defined a map on cohomology with compact support. When f is a fiber bundle with compact fibers, $f_!$ is also referred to as *integration along the fibers*. We have a similarly defined homomorphism of $H^\bullet(M)$-modules $f_! : H^k_c(M) \to H^{k+\operatorname{codim} f}_c(P)$.

Dually we let

$$f^! : H_k(P) \cong H^{p-k}_c(P) \xrightarrow{f^*} H^{p-k}_c(M) \cong H_{m-p+k}(P) = H_{k+\dim f}(P),$$

[2]One may use this here as a definition, but the point is that this notion is intrinsic: it is independent of the boundary added to M. Besides, there exist manifolds of finite type that are not the interior of a compact manifold with boundary.

where$\dim f := m-p$ is the *virtual fiber dimension* of f, and a similarly defined map for homology with closed support. It satisfies $f_*(\alpha \cap f^! z) = f_!(\alpha) \cap z$. In particular, $f_* f^!(z) = f_!(1) \cap z$. For a fiber bundle with compact fibers it is geometrically given by assigning to a cycle on P its preimage in M, suitably interpreted as a cycle.

These maps have various functorial properties which we will not list.

When M is compact the Künneth theorem combined with Poincaré duality identifies $\mathrm{End}_{\mathbb{Q}} H^\bullet(M;\mathbb{Q}) \cong H^\bullet(M;\mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(M;\mathbb{Q})^\vee$ with $H^\bullet(M;\mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(M;\mathbb{Q}) \cong H^\bullet(M \times M;\mathbb{Q})$. Thus any endomorphism T of $H^\bullet(M;\mathbb{Q})$ of degree d corresponds to a class $\gamma(T) \in H^{m+d}(X \times X;\mathbb{Q})$ such that $T(\alpha) = \pi_{2!}(\pi_1^*(\alpha) \cup \gamma(T))$ and vice versa.

This is of special interest when M is a compact complex-algebraic manifold. Let us first mention that every irreducible complex algebraic variety Z of dimension d, even when singular, carries a fundamental class $[Z] \in H^{cl}_{2d}(Z)$, which in fact generates $H^{cl}_{2d}(Z)$. So if $i : Z \to M$ is a proper map (for instance, a closed embedding), then, $i_*[Z] \in H_{2d}(M)$ is defined. Although we have not defined $i_!$ in this setting, we denote its Poincaré dual by $i_!(1) \in H^{2\dim_{\mathbb{C}} M - 2d}(M)$. The part of $H^\bullet(M;\mathbb{Q})$ generated by such classes is called the *algebraic cohomology* of M.

We say that an endomorphism T of $H^\bullet(M;\mathbb{Q})$ is algebraic, when $\gamma(T)$ is. For example, the identity of $H^\bullet(M;\mathbb{Q})$ is defined by the diagonal map $\Delta_X : X \to X \times X$. It is a fun exercise to check that the algebraic endomorphisms make up graded $\mathbb{Q}$-subalgebra of $\mathrm{End}_{\mathbb{Q}}\big(H^\bullet(M;\mathbb{Q})\big)$.

EXAMPLE 5.1. If $i_H : H \subset \mathbb{P}^n$ is a hyperplane then $\eta := i_{H!}(1) \in H^2(\mathbb{P}^n)$ is independent of H and we have $H^\bullet(\mathbb{P}^n) = \mathbb{Z}[\eta]/(\eta^{n+1})$. If $Z \subset \mathbb{P}^n$ is an irreducible Zariski closed subset of pure codimension d, then $i_{Z!}(1)$ is a constant times η^d; this coefficient is (or can be defined as) the degree of Z. Equivalently, $\deg(Z) = \langle \eta^{n-d} | i_{Z*}[Z]\rangle = \langle i_Z^*\eta^{n-d} | [Z]\rangle$. If $W \subset \mathbb{P}^n$ is another irreducible closed subset, of dimension e say, so that $i_{W!}(1) = \deg(W)\eta^e$, then $i_{Z!}(1) \cup i_{W!}(1) = \deg(Z)\deg(W)\eta^{d+e}$ and this represents the degree of the intersection of Z and W (counted with multiplicities of the irreducible components) if they meet each other in a closed subset of pure codimension $d+e$. (This is the algebraic topology that underlies the Bézout theorem.)

5.2. Properties equivalent to the hard Lefschetz theorem. Here $X \subset \mathbb{P}$ is a projective complex manifold of dimension $m+1$ and $X_\xi = X \cap H_\xi$ a transversal hyperplane section. We begin with a simple characterization of the invariant homology. Let $i : X_\xi \subset X$ be the inclusion.

LEMMA 5.2. *The map* $i^! : H_{m+2}(X) \to H_m(X_\xi)$ *is injective with image* $H_m^{inv}(X_\xi)$.

PROOF. By the weak Lefschetz theorem, $H^{m-1}(X, X_\xi) = 0$ and so $i^* : H^m(X) \to H^m(X_\xi)$ is injective. Hence so is

$$i^! : H_{m+2}(X) \cong H^m(X) \xrightarrow{i^*} H^m(X_\xi) \cong H_m(X_\xi).$$

Since $H_m^{inv}(X_\xi)$ is the part of $H_m(X_\xi)$ that is perpendicular to the kernel of i_* (relative to the intersection pairing on X_ξ), it follows that the image of $i^!$ is contained in $H_m^{inv}(X_\xi)$ and that the two become equal after tensoring with $\mathbb{Q}$. In other words, the cokernel of $i^! : H_{m+2}(X) \to H_m^{inv}(X_\xi)$ is torsion. But the cokernel of $i^!$ can be identified with the cokernel of $i^* : H^m(X) \to H^m(X_\xi)$, which embeds in $H^{m+1}(X, X_\xi)$. According to Remark 3.9, the latter embeds in $\oplus_\rho H^{m+1}(\tilde{e}_\rho, \partial\tilde{e}_\rho)$, which is free abelian. So the cokernel of $i^!$ is torsion free. □

PROPOSITION 5.3. *The following properties are equivalent (where* $i : X_\xi \subset X$ *denotes the inclusion):*

(i) $H_m^{van}(X_\xi; \mathbb{Q}) \cap H_m^{inv}(X_\xi; \mathbb{Q}) = 0$,
(ii) $H_m(X_\xi; \mathbb{Q}) = H_m^{van}(X_\xi; \mathbb{Q}) + H_m^{inv}(X_\xi; \mathbb{Q})$,
(iii) $H_m^{van}(X_\xi; \mathbb{Q})$ *is nondegenerate for the intersection form,*
(iv) $H_m^{inv}(X_\xi; \mathbb{Q})$ *is nondegenerate for the intersection form,*
(v) i_* *restricts to an isomorphism* $H_m^{inv}(X_\xi; \mathbb{Q}) \cong H_m(X; \mathbb{Q})$,
(vi) $i_!(1) \cap : H_{m+2}(X; \mathbb{Q}) \to H_m(X; \mathbb{Q})$ *is an isomorphism,*
(vii) $i_!(1) \cup : H^m(X; \mathbb{Q}) \to H^{m+2}(X; \mathbb{Q})$ *is an isomorphism,*
(viii) $H_m^{van}(X_\xi; \mathbb{C})$ *is reductive as a representation of* $\pi_1(\check{\mathbb{P}} - X^\vee, \xi)$.

PROOF. The equivalence of the first four items follows from the fact that $H_m^{van}(X_\xi; \mathbb{Q})$ and $H_m^{inv}(X_\xi; \mathbb{Q})$ are each others perp with respect to the intersection form (which we know, is nondegenerate). Their equivalence with (v) follows from Corollary 3.11. The equivalence of (v) an (vi) is immediate from Lemma 5.2 and the fact that $i_!(1) \cap = i_* i^!$ and (vii) is simply the dual of (vi). The equivalence of (viii) with (i) follows from Lemma 4.7. □

Examples show that none of the properties (ii), (vi) and (vii) hold with integral coefficients. Note that $i_!(1) \in H^2(X)$ is the restriction of the hyperplane class $\eta \in H^2(\mathbb{P})$; we denote it by η_X. The map $\eta_X \cup : H^\bullet(X; \mathbb{Q}) \to H^\bullet(X; \mathbb{Q})$ is called the *Lefschetz operator* and is usually denoted by L.

THEOREM 5.4 (Hard Lefschetz theorem). *The equivalent properties of Proposition 5.3 hold.*

REMARK 5.5. Lefschetz asserted this in his monograph *l'Analysis Situs et la Géometrie Algébrique* [15], but no one has been able to understand the geometric proof that he gave there. The first accepted proof was due to Hodge and analytic in character, as it was based on the harmonic representation of the complex cohomology. Another proof was given in the 1970s by Deligne. He first proved it for varieties defined over a finite field and then invokes a comparison theorem to pass to the characteristic zero case. It is geometric but in a rather roundabout way. The question of whether a proof in the spirit of Lefschetz exists is still open. In the remainder of this section we derive some consequences of the hard Lefschetz theorem.

Let us note in passing that the validity of the Hard Lefschetz theorem for $X \subset \mathbb{P}$ does not change if we replace η_X by a positive multiple of η_X. So this is really a property of a cohomology class in $H^2(X;\mathbb{Q})$ which is known to be the first Chern class of an ample line bundle.

COROLLARY 5.6. *For $k = 0, \dots, n = \dim X$, capping resp. cupping with η_X^k yields isomorphisms*

$$H_{n+k}(X;\mathbb{Q}) \cong H_{n-k}(X;\mathbb{Q}) \text{ resp. } L^k : H^{n-k}(X;\mathbb{Q}) \cong H^{n+k}(X;\mathbb{Q})$$

are isomorphisms.

PROOF. We only prove the first assertion, as the second then follows via Poincaré duality. We assume $k > 0$ and choose transversal linear section $i' : X' \subset X$ of codimension $k-1$ (so of dimension $n-k+1$) and a transversal hyperplane section $i : X'' \subset X'$, so that $i'' := i'i : X'' \subset X$ is a transversal linear section of codimension k (and hence of dimension $n-k$). Then $\eta_X^k = i''_!(1)$ and so $\eta_X^k \cap = i''_* i''^! = i'_* i_* i^! i'^!$:

$$\eta_X^k \cap : H_{n+k}(X) \xrightarrow{i'^!} H_{n-k+2}(X') \xrightarrow{i_* i^!} H_{n-k}(X') \xrightarrow{i'_*} H_{n-k}(X).$$

The map $i'_* : H_{n-k}(X') \to H_{n-k}(X)$ is an isomorphism by iterated application of the weak Lefschetz theorem. The same is true for $i'^* : H^{n-k}(X) \to H^{n-k}(X')$ and then Poincaré duality gives that $i'^! : H_{n+k}(X) \to H_{n-k+2}(X')$ is an isomorphism as well. The Hard Lefschetz theorem for X' gives that the middle map is an isomorphism after tensoring with $\mathbb{Q}$. □

REMARK 5.7. We may ask under which conditions a morphism $f : X \to \mathbb{P}$ to a projective space (with X a projective manifold, as before), the pull-back $\eta_X := f^*\eta$ satisfies the properties of Corollary 5.6. This is for instance the case when f is finite (for then $f^*\eta$ is then an ample class so that a positive multiple of $f^*\eta$ comes from a projective embedding). But there exist examples for which f contracts irreducible subvarieties of positive dimension. This contracting cannot be too severe, for f will have at least the property that for every irreducible closed subset $i : Z \subset X$, $2\dim Z - \dim f(Z) \le \dim X$. This is perhaps better stated as: the drop in dimension $\dim Z - \dim f(Z)$ is bounded by the codimension $\dim X - \dim Z$ (one then says that f is *semismall*). This is shown as follows. If $d' := \dim f(Z)$, then there is linear subspace of $\mathbb{P}$ of codimension $d'+1$ which misses $f(Z)$. As such a linear subspace represents the cohomology class $\eta^{d'+1}$, it follows that $\eta^{d'+1}\cap f_*([Z]) = 0$ and hence $\eta_X^{d'+1}\cap [Z] = 0$. Since X is projective, the fundamental class $[Z]\in H^{2\dim Z}(X;\mathbb{Q})$ must be nonzero (just intersect it with a linear section of the ambient projective space of codimension $\dim Z$). As $\eta_X^{d'+1}\cap$ is assumed to be injective on the rational homology in degree $> \dim X + d'$, we must have $2\dim Z \le \dim X + d' = \dim X + \dim f(Z)$. If we apply this to $Z = X$, we see, not surprisingly, that $\dim f(X) = \dim X$. So f is generically finite.

It is remarkable that this property is also sufficient: De Cataldo-Migliorini [5] prove that that if f is semismall, then f satisfies both the Weak Lefschetz theorem (relative to the transversal pre-image of a hyperplane in $\mathbb{P}$) and the Hard Lefschetz theorem (relative to the class $f^*\eta$).

Via the preceding corollary the rational homology and cohomology of a projective manifold acquire a very interesting structure. First of all, we obtain a bilinear form on $H^{n-k}(X;\mathbb{Q})$ for $k = 0,\dots,n$ by

$$\Psi : (\alpha,\beta)\in H^{n-k}(X;\mathbb{Q})\times H^{n-k}(X;\mathbb{Q})\mapsto \langle L^k(\alpha\cup\beta)|[X]\rangle\in\mathbb{Q}.$$

This form is $(-1)^{n-k}$-symmetric and Corollary 5.6 implies that it is nondegenerate. It is clear from the definition that for $1\le k\le n$, the map $L : H^{n-k-2}(X;\mathbb{Q})\to H^{n-k}(X;\mathbb{Q})$ preserves this form: $\Psi(\eta_X\cup\alpha,\eta_X\cup\beta) = \Psi(\alpha,\beta)$. In particular, Ψ is nondegenerate on $L(H^{n-k-2}(X;\mathbb{Q}))\subset H^{n-k}(X;\mathbb{Q})$.

DEFINITION 5.8. The *primitive cohomology* of the projective manifold X is

$$P^{n-k}(X) := \mathrm{Ker}\left(L^{k+1} : H^{n-k}(X;\mathbb{Q})\to H^{n+k+2}(X;\mathbb{Q})\right)$$

Note that Poincaré duality identifies $P^{n-k}(X)$ resp. $P^{n-k}(X)^\vee$ also with the kernel of $\eta_X^{k+1}\cap : H_{n+k}(X;\mathbb{Q})\to H_{n-k-2}(X;\mathbb{Q})$ resp. the cokernel of $L^{k+1} : H^{n-k-2}(X;\mathbb{Q})\to H^{n+k}(X;\mathbb{Q})$.

COROLLARY 5.9 (Lefschetz decomposition). *The primitive cohomology is the Ψ-perp of the ideal in $H^\bullet(X;\mathbb{Q})$ generated by η_X and Ψ is nondegenerate on it. In particular, $P^\bullet(X)\to H^\bullet(X;\mathbb{Q})/\eta_X H^\bullet(X;\mathbb{Q})$ is an isomorphism (this makes that $P^\bullet(X)$ acquires the structure of a graded algebra). Moreover, the natural map*

$$\oplus_{k=0}^n \mathbb{Q}[t]/(t^{k+1})\otimes_{\mathbb{Q}} P^{n-k}(X)\to H^\bullet(X)$$

defined by substituting η_X for t is an isomorphism.

PROOF. Let $\alpha\in H^{n-k}(X;\mathbb{Q})$. Then $\alpha\in P^{n-k}(X)$ is equivalent to $\eta_X^{n-k+1}\cup\alpha = 0$. By Poincaré duality, this is equivalent to: $\eta_X^{n-k+1}\cup\alpha\cup\beta = 0$ for all $\beta\in H^{n-k-2}(X;\mathbb{Q})$, in other words, that α is Ψ-perpendicular to $\eta_X\cup H^{n-k-2}(X;\mathbb{Q})$. Since Ψ is nondegenerate on the latter subspace, Ψ is also nondegenerate on its Ψ-perp $P^{n-k}(X)$. All the assertions now follow. $\square$

REMARKS 5.10. The structure that $H^\bullet(X;\mathbb{Q})$ receives from the Lefschetz decomposition can be phrased as defining a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{Q})$ on $H^\bullet(X;\mathbb{Q})$: this Lie algebra has the basis

$$X_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the bracket is then given by $[H, X_\pm] = \pm 2X_\pm$ and $[X_+, X_-] = H$. We may represent this as a Lie algebra of $\mathbb{Q}$-derivations of the algebra $\mathbb{Q}[u,v]$ by

$$X_+ \mapsto u\frac{\partial}{\partial v}, \quad X_- \mapsto v\frac{\partial}{\partial u}, \quad H \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

For any integer $k \geq 0$, the degree k part $\mathbb{Q}[u,v]_k$ of $\mathbb{Q}[u,v]$ is invariant under this action and furnishes an irreducible representation of $\mathfrak{sl}_2(\mathbb{Q})$ of dimension $k+1$. Every finite dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{Q})$ of dimension $k+1$ is isomorphic to $\mathbb{Q}[u,v]_k$ and thus $\mathbb{Q}[u,v]$ may be regarded as the direct sum of all the finite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{Q})$. The monomial $v^k/k! \in \mathbb{Q}[u,v]_k$ is a lowest weight vector of $\mathbb{Q}[u,v]_k$, in the sense that we have an isomorphism of vector spaces

$$\mathbb{Q}[t]/(t^{k+1}) \cong \mathbb{Q}[u,v]_k, \quad t^p \mapsto X_+^p(v^k/k!) = u^p v^{k-p}/(k-p)!.$$

If we use this isomorphism to transfer the action of $\mathfrak{sl}_2(\mathbb{Q})$ to $\mathbb{Q}[t]/(t^{k+1})$, then X_+ acts on $\mathbb{Q}[t]/(t^{k+1})$ as multiplication by t and $H(t^p) = (-k + 2p)t^k$.

Thus the Lefschetz decomposition can be understood as giving $H^\bullet(X;\mathbb{Q})$ the structure of a representation of $\mathfrak{sl}_2(\mathbb{Q})$. This makes X_+ act on $H^r(X;\mathbb{Q})$ as L and H as multiplication by $r-n$ (the action of X_- is then characterized by the property that it is of degree by -2 and $[X_+, X_-]$ acts as H).

Recall that we associated to an endomorphism T of $H^\bullet(X;\mathbb{Q})$ of degree d a class $\gamma(T) \in H^{2n+d}(X \times X;\mathbb{Q})$ and that we called T algebraic if $\gamma(T)$ is an algebraic cohomology class. This is the case for X_+, as

$$\gamma(X_+) = \gamma(L) = (\Delta_X i)_!(1) = \Delta_{X!} i_!(1) \in H^{2n+2}(X \times X;\mathbb{Q}),$$

where $i : X_\xi \subset X$ is a transversal hyperplane section and $\Delta_X : X \hookrightarrow X \times X$ is a diagonal embedding. One of Grothendieck's standard conjectures asserts that $\gamma(X_+)$ and $\gamma(H)$ are also algebraic([3]), but no one seems to have any idea of how to even to approach this question: the only part of the algebraic cohomology of $X \times X$ we can identify

[3]This is in fact a special case of the Hodge conjecture, see [7] for more discussion.

as such in general is the subalgebra generated by the class of the diagonal class and the two pull-backs of the hyperplane class of X and characteristic classes of the tangent bundle and this subalgebra contains in general neither $\gamma(X_-)$ nor $\gamma(H)$.

5.3. An application: the Leray spectral sequence of a projective morphism. This application is the first example of a 'decomposition theorem'. It is about a projective morphism $f : \mathcal{X} \to S$, i.e., for some projective space $\mathbb{P}$, f factors through a closed embedding of $\mathcal{X}$ in $\mathbb{P} \times S$. We assume the morphism to be smooth of relative dimension n: this essentially means that f has smooth fibers of dimension n. The higher direct image $R^q f_*\mathbb{Q})$ has then as its stalk at $s \in S$ the cohomology group $H^q(X_s;\mathbb{Q})$ and the assumption that f is smooth projective, implies that $R^\bullet f_*\mathbb{Q})$ is locally constant: is a local system of graded finite dimensional $\mathbb{Q}$-vector spaces. Notice that we then have a fiberwise Lefschetz decomposition:

$$\oplus_{k=0}^n \mathbb{Q}[t]/(t^{k+1}) \otimes_{\mathbb{Q}} P^{n-k} f_*\mathbb{Q} \cong R^\bullet f_*\mathbb{Q},$$

where $P^{n-k} f_*\mathbb{Q}$ is a local system that appears here as a direct summand of $R^{n-k} f_*\mathbb{Q}$.

THEOREM 5.11 (Deligne [6]). *Let $f : \mathcal{X} \to S$ be a smooth projective morphism of relative dimension n. Then the Leray spectral sequence*

$$E_2^{p,q} := H^p(S, R^q f_*\mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{X};\mathbb{Q})$$

degenerates (so the Leray filtration $\mathcal{L}_\bullet H^r(\mathcal{X};\mathbb{Q})$ on $H^r(\mathcal{X};\mathbb{Q})$ has the property that $\mathcal{L}_q H^r(\mathcal{X};\mathbb{Q})/\mathcal{L}_{q-1} H^r(\mathcal{X};\mathbb{Q})$ is canonically isomorphic to $H^{r-q}(S, R^q f_*\mathbb{Q})$). In particular, $H^r(\mathcal{X};\mathbb{Q}) \to H^0(S, R^r f_*\mathbb{Q})$ is onto and $f^* : H^r(S;\mathbb{Q}) \to H^r(\mathcal{X};\mathbb{Q})$ is into.

PROOF. This is essentially linear algebra. We proceed inductively and assume that we have verified that for some $r \geq 2$, $E_r^{p,q} = E_2^{p,q}$. Since the Lefschetz operator L is given by cupping with the hyperplane class, the Leray spectral sequence is one of $\mathbb{Q}[t]$-modules. Now consider the commutative diagram

$$\begin{array}{ccc} H^p(S, R^{n-k} f_*\mathbb{Q}) & \xrightarrow{d_r} & H^{p+r}(S, R^{n-(k+r-1)} f_*\mathbb{Q}) \\ {\scriptstyle L^{k+1}}\downarrow & & {\scriptstyle L^{k+1}}\downarrow \\ H^p(S, R^{n+k+2} f_*\mathbb{Q}) & \xrightarrow{d_r} & H^{p+r}(S, R^{n+k-r+3} f_*\mathbb{Q}) \end{array}$$

The vertical map on the right is injective by the Hard Lefschetz theorem (use that $k + r - 1 \geq k + 1$). The vertical map on the left is zero when restricted to the direct summand $H^\bullet(S, P^{n-k} f_*\mathbb{Q})$ and hence so is the top horizontal map. In other words, d_r vanishes on the primitive part of the spectral sequence. Since the differentials of the spectral sequence are $\mathbb{Q}[t]$-linear, it then follows that d_r vanishes on all of $H^\bullet(S, R^\bullet f_*\mathbb{Q})$. $\square$

REMARK 5.12. If we regard the relative ample class $\eta_{\mathcal{X}} \in H^2(\mathcal{X};\mathbb{Q})$ as part of our data, then the Leray spectral sequence is one of $\mathfrak{sl}_2(\mathbb{Q})$-representations and the eigenspaces of $H \in \mathfrak{sl}_2(\mathbb{Q})$ split the Leray filtration: the $(q-n)$-eigenspace of H in $H^r(\mathcal{X};\mathbb{Q})$ gets identified with $H^{r-q}(S, R^q f_*\mathbb{Q})$. We may phrase this in terms of the relative Lefschetz operator $L = \eta_{\mathcal{X}} \cup$ only: since it takes $\mathcal{L}_p H^r(\mathcal{X};\mathbb{Q})$ to $\mathcal{L}_{p+2}H^{r+2}(\mathcal{X};\mathbb{Q})$, the kernels of its powers define a filtration which goes in the opposite direction of the Leray filtration: for $0 \le k \le n$, L^{k+1} is zero on $H^{r-n+k}(S, P^{n-k}f_*\mathbb{Q})$, but injective on $H^{r-n+l}(S, Rf_*^{n-l}\mathbb{Q})$ for $l > k$, and so the kernel of the composite map

$$\mathcal{L}_{n-k}H^r(\mathcal{X};\mathbb{Q}) \xrightarrow{L^{k+1}} \mathcal{L}_{n+k+2}H^{r+2k+2}(\mathcal{X};\mathbb{Q}) \twoheadrightarrow H^{r-n-k}(S; R^{n+2k+2}f_*\mathbb{Q})$$

maps isomorphically onto $H^{r+k}(S, P^k f_*\mathbb{Q})$ and we obtain a *relative Lefschetz decomposition*

$$H^\bullet(\mathcal{X};\mathbb{Q}) \cong \oplus_{k=0}^n \mathbb{Q}[t]/(t^{k+1}) \otimes_{\mathbb{Q}} H^\bullet(S, P^{n-k}f_*\mathbb{Q}).$$

We will encounter a generalization, the famous *decomposition theorem*, to arbitrary projective morphisms.

6. The monodromy theorem

6.1. Quasi-unipotence and Zeta function of an endomorphism. Let V be a finite dimensional vector space over a field k of characteristic zero. We say that a transformation $\sigma \in \mathrm{GL}(V)$ is *quasi-unipotent of level* $\le \nu$ (ν is a positive integer) if there exists a positive integer N such that $(\sigma^N - 1)^\nu = 0$. This means that we can find we can find a basis of V on which σ^N can be brought in upper triangular form with 1's on the main diagonal and with Jordan blocks of size at most ν. Equivalently, that there exist a filtration of V of length ν, $V = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^\nu = \{0\}$, by σ-invariant subspaces such that σ acts with finite order on each quotient F^k/F^{k+1}.

If $\sigma \in \mathrm{End}(V)$ is merely an endomorphism, then its *Zeta function* is defined as $Z_\sigma(t) := \det(1-t\sigma)^{-1} \in k(t)$. So its reciprocal is a monic polynomial whose roots are the reciprocals of the nonzero eigenvalues of σ (and whose degree is the rank of σ). Note that this notion is multiplicative for short exact sequences: if W is a σ-invariant subspace, then the Zeta function for the action of σ in V/W is equal to $Z_\sigma/Z_{\sigma|W}$. From this we easily deduce in $k[[t]]$ the identity

$$Z_\sigma(t) = \exp\left(\textstyle\sum_{i=1}^\infty \frac{\mathrm{Tr}(\sigma^i)}{i} t^i\right),$$

for its verification is then reduced to the case when $\dim V = 1$ with σ acting as scalar multiplication by $\lambda \in k$, in which case it amounts to the identity $-\log(1-\lambda t) = \sum_{i=1}^\infty (\lambda t)^i/i$. It is clear that for $\sigma^\vee \in \mathrm{End}(V^\vee)$, $Z_{\sigma^\vee} = Z_\sigma$.

Suppose now V comes with a grading: $V = \oplus_{i\in\mathbb{Z}} V_i$ (we have only finitely many nonzero summands since we still assume $\dim V < \infty$), and that $\sigma_\bullet = (\sigma_i \in \mathrm{End}(V_i))_i$ is a degree preserving endomorphism. The multiplicative property of the Zeta function then suggests that we put

$$Z_{\sigma_\bullet}(t) := \prod_i \det(1 - t\sigma_i)^{(-1)^{i+1}} \in k(t),$$

If we define the *supertrace* of σ as $\mathrm{sTr}(\sigma) := \sum_{i\in\mathbb{Z}}(-1)^i \,\mathrm{Tr}(\sigma_i)$, then the above identity becomes

$$Z(t) = \exp\big(\textstyle\sum_{i=1}^{\infty} \frac{\mathrm{sTr}(\sigma^i)}{i} t^i\big).$$

This graded version of the Zeta function continues to have the multiplicative property for short exact sequences of finite dimensional complexes. If $V_\bullet$ is in fact a complex $\cdots \to V_{k+1} \xrightarrow{\partial_k} V_k \to \cdots$ and $\sigma_\bullet$ is a homomorphism of complexes so that we have induced maps $H_k(\sigma) : H_k(V_\bullet) \to H_k(V_\bullet)$, then this multiplicative property implies that $H_\bullet(\sigma_\bullet)$ and $\sigma_\bullet$ have the same Zeta function. So if $(E^r_{\bullet,\bullet})_r$ is a spectral sequence of finite dimensional bigraded vector spaces converging to some graded space $V_\bullet$, then for an endomorphism $(\sigma^r_{\bullet,\bullet} \in \mathrm{End}(E^r_{\bullet,\bullet}))_r$ converging to $\sigma_\bullet \in \mathrm{End}(V_\bullet)$, the Zeta function of each $\sigma^r_{\bullet,\bullet}$ relative the total grading is equal to the Zeta function of $\sigma_\bullet$.

We apply this to the situation where we are given a topological space Y with finite dimensional rational cohomology and a continuous map $h : Y \to Y$. Then the supertrace of the endomorphism of $H_\bullet(Y;\mathbb{Q})$ induced by h is called the *Lefschetz number* of h and denoted $\Lambda(h)$ so that the preceding discussion yields the Weil identity

$$Z_{h_*}(t) = \exp\big(\textstyle\sum_{i=1}^{\infty} \frac{\Lambda(h^i))}{i} t^i\big).$$

Notice that we get the same Zeta function for the action of h on cohomology, as this is just the dual of h_*. The spectral sequence property shows that if the space Y is filtered by subspaces $\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots Y_n = Y$ with the property that each $H_\bullet(Y_k, Y_{k-1};\mathbb{Q})$ is finite dimensional, and $h(Y_k) \subset Y_k$ for all k, then the Zeta function of h acting in $H_\bullet(Y;\mathbb{Q})$ is the product of the ones of h acting in $H_\bullet(Y_k, Y_{k-1};\mathbb{Q})$, $k = 0, \dots, n-1$.

It is clear that when h is homotopic to the identity, $\Lambda(h)$ is simply the Euler characteristic $e(X)$ of X. We mention without proof that when $h : X \to X$ has no fixed point, then $\Lambda(h) = 0$.

If Y is a compact oriented (topological) manifold without boundary, then $\Lambda(h)$ has the interpretation as the intersection number of the graph of h with the diagonal of Y. This lies at the basis of the Weil's introduction of his Zeta function for a variety defined over a finite field.

6.2. Quasi-unipotence and Zeta function of a monodromy.

THEOREM 6.1 (Clemens, Landman). *Let $\mathcal{Z}$ be a complex manifold of dimension $m+1$ and $f : \mathcal{Z} \to \mathbb{D}$ a proper holomorphic map to an* *open disk* $\mathbb{D} = \{w \in \mathbb{C} : |w| < \varepsilon\}$. *Assume that* f *is submersion over* $\mathbb{D} \smallsetminus \{0\}$ *and let* $\eta \in \mathbb{D} \smallsetminus \{0\}$. *Then the monodromy action of* f *on* $H_k(Z_\eta; \mathbb{Q})$ *is quasi-unipotent of level* $\leq m+1$. *If* f *is projective, then this level is even bounded by* $\min\{k+1, 2m+1-k\}$.

The proof uses in an essential way the resolution theorem which tells us that there exists a projective morphism $\pi : \tilde{\mathcal{Z}} \to \mathcal{Z}$ with $\tilde{\mathcal{Z}}$ nonsingular, π an isomorphism over $\mathcal{Z} - Z_0$ and such that $\tilde{Z}_0 = \pi^{-1} Z_0$ is a simple normal crossing divisor. Since this does not affect the monodromy, we therefore may without loss of generality assume that Z_0 is a normal crossing divisor and is *simple* in the sense that every irreducible component of Z_0 is nonsingular. We want to think of Z_0 as the zero divisor of f, so if $\mathcal{D}$ denotes the collection of irreducible components of the underlying variety, then $Z_0 = \sum_{i \in \mathcal{D}} n_i D_i$, with each n_i a positive integer. We will prove in that case the following more precise assertion.

PROPOSITION 6.2. *Let* $N := \operatorname{lcm}\{n_i : i \in \mathcal{D}\}$ *and* ν *the maximal number of branches of* Z_0 *passing through a point. Then the monodromy of* f, $h_* \in \mathrm{GL}(H_\bullet(Z_\eta; \mathbb{Q}))$, *satisfies* $(h_*^N - 1)^\nu = 0$ *and its Zeta function is equal to the product* $\prod_{i \in \mathcal{D}} (1 - t^{n_i})^{-e(\mathring{D}_i)}$. *Here* $\mathring{D}_i := D_i \smallsetminus \cup_{j \in \mathcal{D} \smallsetminus \{i\}} D_j$ *and* $e(\mathring{D}_i)$ *denotes its Euler number.*

PROOF THAT PROPOSITION 6.2 IMPLIES THEOREM 6.1. In Proposition 6.2 we clearly have $\nu \leq m+1$ and so this implies that h_* is quasi-unipotent of level $\leq m+1$. The second assertion follows by an application of the weak Lefschetz theorem and induction: if $\mathcal{Z} \subset \mathbb{P} \times \mathbb{D}$, then choose a general hyperplane section, so that we get a subfamily $\mathcal{Z}' \subset \mathcal{Z}$. Perhaps some shrinking of $\mathbb{D}$ is needed to make the restriction of $f' : \mathcal{Z} \xrightarrow{f} \mathbb{D}$ also fulfill the hypotheses of Theorem 6.1. Given $k < m$ and $\eta \in \mathbb{D} \smallsetminus \{0\}$, then the natural map $H_k(Z'_\eta; \mathbb{Q}) \to H_k(Z_\eta; \mathbb{Q})$ is surjective. It commutes with the monodromy. By induction the monodromy action on $H_k(Z'_\eta; \mathbb{Q})$ is quasi-unipotent of level $\leq k$ and hence the same is true for the monodromy action on $H_k(Z_\eta; \mathbb{Q})$ and likewise on its dual $H^k(Z_\eta; \mathbb{Q})$. For $k > m$, we use that the monodromy is compatible with Poincaré duality: $H_k(Z_\eta; \mathbb{Q}) \cong H^{2m-k}(Z_\eta; \mathbb{Q})$. □

For the proof of Proposition 6.2 we are going to produce a canonical geometric model for the monodromy, which is due to A' Campo [1].

6.3. A 'logmodel' for the geometric monodromy. Let $\hat{\mathbb{D}}$ be the space over $\mathbb{D}$ defined by polar coordinates $(r, u) \in [0, \varepsilon) \times \mathbb{C}_1 \mapsto ru$. This is in fact a construction that is independent of the coordinate on $\mathbb{D}$ defined by the embedding $\mathbb{D} \subset \mathbb{C}$: it is an example of real oriented blowup (namely of 0 in $\mathbb{D}$). The exceptional set (the preimage of the origin) is $\partial\hat{\mathbb{D}}$ and the latter has the intrinsic description as the quotient of $T_0\mathbb{D} \smallsetminus \{0\}$ by the radial action of the multiplicative group $\mathbb{R}_+^\times$. Since $T_0\mathbb{D}\smallsetminus\{0\}$ is a principal homogeneous space for $\mathbb{C}^\times$, there is on $\partial\hat{\mathbb{D}}$ still a residual principal action of the circle group $\mathbb{C}_1 = \mathbb{C}^\times/\mathbb{R}_+^\times$. We shall identify $\mathbb{C}_1$ with the subgroup of $\mathbb{C}^\times$ of complex numbers of norm 1. Our goal is to do something similar for the pair $(\mathcal{Z}, Z_0)$.

We consider $\mathcal{D}$ as indexing a closed covering of Z_0. Its nerve defines an abstract simplicial complex with $\mathcal{D}$ as vertex set: a (finite) subset $\sigma \subset \mathcal{D}$ is a simplex (of dimension $\#(\sigma) - 1$) whenever $D_\sigma := \cap_{i\in\sigma} D_i$ is nonempty. We also put $\mathring{D}_\sigma := D_\sigma \smallsetminus \cup_{j\in\mathcal{D}\smallsetminus\sigma} D_j$, so that $\{\mathring{D}_\sigma\}_\sigma$ is a partition of Z_0 into nonsingular subvarieties.

Consider the normal bundle of D_i in $\mathcal{Z}$ (a holomorphic line bundle). If we take away from its total space the zero section (a holomorphic principal $\mathbb{C}^\times$-bundle) and then divide out by $\mathbb{R}_+^\times$, we get a (real-analytic) principal $\mathbb{C}_1$-bundle T_i over D_i. This bundle appears as the exceptional set (and the boundary) of the real oriented blowup of D_i in $\mathcal{Z}$, denoted $\hat{\mathcal{Z}}_i \to \mathcal{Z}$. We let $\pi : \hat{\mathcal{Z}} \to \mathcal{Z}$ be the fiber product of the maps $\mathcal{Z}_i \to \mathcal{Z}$. So π is proper and surjective and $\hat{\mathcal{Z}}$ is a manifold with boundary $\partial\hat{\mathcal{Z}}$ with corners whose interior $\hat{\mathcal{Z}} \smallsetminus \partial\hat{\mathcal{Z}}$ is the isomorphic preimage under π of $\mathcal{Z} \smallsetminus Z_0$. In particular, the open embedding $\mathcal{Z} \smallsetminus Z_0 \cong \hat{\mathcal{Z}} \smallsetminus \partial\hat{\mathcal{Z}} \subset \hat{\mathcal{Z}}$ is a homotopy equivalence and $\partial\hat{\mathcal{Z}} \subset \hat{\mathcal{Z}}$ is a deformation retract. For a simplex σ of $\mathcal{D}$, put $T_\sigma := \pi^{-1}D_\sigma$ and $\mathring{T}_\sigma := \pi^{-1}\mathring{D}_\sigma$. So T_σ comes with a free ${\mathbb{C}_1}^\sigma$-action. Its orbit space is not D_σ, but a modification $\hat{D}_\sigma$ of D_σ, namely a real oriented blowup of D_σ along its intersection with the D_j, $j \in \mathcal{D} \smallsetminus \sigma$ ($\hat{D}_\sigma$ is manifold with boundary with corners having $\mathring{D}_\sigma$ as its interior). In particular, $\mathring{T}_\sigma \xrightarrow{\pi} \mathring{D}_\sigma$ is a ${\mathbb{C}_1}^\sigma$-principal bundle.

We claim that f lifts uniquely to a map $\hat{f} : \hat{\mathcal{Z}} \to \hat{\mathbb{D}}$ so that we obtain a diagram of continuous maps

$$\begin{array}{ccccc}
\partial\hat{\mathcal{Z}} & \xrightarrow{\subset} & \hat{\mathcal{Z}} & \xrightarrow{\pi} & \mathcal{Z} \\
{\scriptstyle \partial\hat{f}}\downarrow & & {\scriptstyle \hat{f}}\downarrow & & {\scriptstyle f}\downarrow \\
\partial\hat{\mathbb{D}} & \xrightarrow{\subset} & \hat{\mathbb{D}} & \longrightarrow & \mathbb{D}
\end{array}$$

and that this lift enjoys the following properties:

(i) $\hat{f}$ is locally topologically trivial,

(ii) the restriction $\mathring{T}_\sigma \xrightarrow{\partial\hat{f}} \partial\hat{\mathbb{D}}$ is ${\mathbb{C}_1}^\sigma$-equivariant, where ${\mathbb{C}_1}^\sigma$ acts on $\partial\hat{\mathbb{D}}$ via the character $\chi_\sigma : u \in {\mathbb{C}_1}^\sigma \mapsto \prod_{i\in\sigma} u_i^{n_i} \in \mathbb{C}_1$.

So by (ii), $\mathring{T}_\sigma \xrightarrow{(\pi,\partial\hat{f})} \mathring{D}_\sigma \times \partial\hat{\mathbb{D}}$ is a $\mathrm{Ker}(\chi_\sigma)$-principal bundle.

We verify this in terms of local coordinates. At a point p of $\mathring{D}_\sigma$, we choose for every $i \in \sigma$ a local equation z_i for D_i at p such that f is there given as $\prod_{i\in\sigma} z_i^{n_i}$. The z_i's form a part of a coordinate system on a coordinate neighborhood U. More precisely, we can find a U and a holomorphic retraction $\pi_\sigma : U \to \mathring{D}_\sigma \cap U =: \mathring{U}_\sigma$ such that $((z_i)_{i\in\sigma}, \pi_\sigma) : U \to \mathbb{C}^\sigma \times \mathring{U}_\sigma$ is an open embedding. We identify U with its image in $\mathbb{C}^\sigma \times \mathring{U}_\sigma$. Then the passage to polar coordinates for each z_i: $z_i = r_i u_i$ with $r_i \geq 0$, parametrizes the preimage of U in $\hat{\mathcal{Z}}$ and defines the map $\pi^{-1}U \xrightarrow{\pi} U$. The lift $\hat{f}|\pi^{-1}U$ of $f|U$ is given by $((r_i, u_i)_{i\in\sigma}, z') \in \pi^{-1}U \mapsto (\prod_{i\in\sigma} r_i^{n_i}, \prod_{i\in\sigma} u_i^{n_i}) \in \mathbb{D}$. As there is at most one continuous lift of f, this is coordinate independent. It is straightforward to check that it has the stated properties.

We wish to exhibit a geometric monodromy. Observe that for a k-simplex σ, the one parameter subgroup

$$h_{\sigma,\theta} : \theta \in \mathbb{R} \mapsto (\exp(\sqrt{-1}\theta/n_i(k+1)))_{i\in\sigma} \in {\mathbb{C}_1}^\sigma.$$

gives a monodromy flow on the ${\mathbb{C}_1}^\sigma$-principal bundle $\mathring{T}_\sigma \to \mathring{D}_\sigma$, for it follows from property (ii) that then $\partial\hat{f}h_{\sigma,\theta} = \exp(\sqrt{-1}\theta)\partial\hat{f}$. In particular, the restriction of $h_{\sigma,2\pi}$ to the fiber $\mathring{T}_{\sigma,1}$ of $\mathring{T}_\sigma \xrightarrow{\partial\hat{f}} \partial\mathbb{D}$ over 1, serves as a geometric monodromy of $\partial\hat{f}|\mathring{T}_\sigma$. By property (ii) above, $\mathring{T}_{\sigma,1} \to \mathring{D}_\sigma$ is a $\mathrm{Ker}(\chi_\sigma)$-principal bundle.

LEMMA 6.3. *The action of $h_{\sigma,2\pi}$ on the (co)homology of $\mathring{T}_{\sigma,1}$ has order divisible by $\gcd\{n_i\}_{i\in\sigma}$ and its Zeta function is 1 unless σ is a singleton: $\sigma = \{i\}$, in which case it equals $(1-t^{n_i})^{-e(\mathring{D}_i)}$.*

PROOF. We first compute the Zeta function. For this it suffices to determine the Lefschetz numbers of the powers of $h_{\sigma,2\pi}$ acting on $\mathring{T}_{\sigma,1}$. The action of $h^r_{\sigma,2\pi}$ on $\mathring{T}_{\sigma,1}$ is free unless r is a common multiple of the $n_i/\dim(\sigma)$, $i \in \sigma$, in which case it is the identity. So its Lefschetz number is zero unless r is of this form. In that last case, it equals the Euler characteristic of $\mathring{T}_{\sigma,1}$. Now $\mathring{T}_{\sigma,1}$ comes with a free action of $\mathrm{Ker}(\chi_\sigma)$. In particular, the identitity component of $\mathrm{Ker}(\chi_\sigma)$ (which is isomorphic to $\mathbb{C}_1^{\dim(\sigma)}$) acts freely on $\mathring{T}_{\sigma,1}$. Since $\mathbb{C}_1$ has Euler characteristic zero, it follows that the same is true for $\mathring{T}_{\sigma,1}$ unless σ is a singleton, $\{i\}$ say. In that case, $\mathring{T}_i \to \mathring{D}_i$ is a cyclic cover of degree n_i and so has Euler characteristic $n_i e(\mathring{D}_i)$. Summing up, the Lefschetz numbers in question are all zero unless $\sigma = \{i\}$ and r is a multiple of n_i, in which case we get $n_i e(\mathring{D}_i)$. Since

$$\sum_{r=0}^{\infty} \Lambda(h^r_{\{i\},2\pi}|\mathring{T}_{i,1})\frac{t^r}{r} = n_i e(\mathring{D}_i)\sum_{s=1}^{\infty}\frac{t^{n_i s}}{n_i s} = \\ = -e(\mathring{D}_i)\log(1-t^{n_i}) = \log\left((1-t^{n_i})^{-e(\mathring{D}_i)}\right)$$

the Zeta function of $h_{\sigma,2\pi}$ is as asserted.

We have $h_{\sigma,2\pi}$ acting in $\mathring{T}_{\sigma,1}$ as an element of $\mathrm{Ker}(\chi_\sigma)$. The latter acts on the (co)homology via the group of $\pi_0(\mathrm{Ker}(\chi_\sigma))$ (the quotient obtained by dividing out the identity component of $\mathrm{Ker}(\chi_\sigma)$. It is a little exercise to check that this group is cyclic of order $\gcd\{n_i\}_{i\in\sigma}$. $\square$

We may patch these flows together by means of a partition of unity to get a monodromy flow that is globally defined on $\partial\mathcal{Z}$, but it is more canonical to make a further modification of $\partial\mathcal{Z}$ which has no effect on its topological type, but has the advantage that it comes with a geometric monodromy that is intrinsic.

For this we observe that for any simplex σ of $\mathcal{D}$ and any face $\tau \subset \sigma$, T_σ is contained in T_τ. If we denote by $|\sigma|$ the geometric simplex with vertex set σ, then for every simplex σ and face $\tau \subset \sigma$ of that simplex, we regard $|\tau| \times T_\sigma$ as a subspace of both $|\sigma| \times T_\sigma$ and $|\tau| \times T_\tau$ and let $\mathbf{T}$ be the space obtained from the disjoint union of the $|\sigma| \times T_\sigma$ by making the identifications via such inclusions ([4]). Then the natural projection $\mathbf{T} \to \partial\mathcal{Z}_0$ has as fibers geometric simplices and is a homotopy equivalence over $\partial\hat{f}$ (but one can actually show that this map is in fact homotopy equivalent to a homeomorphism). We therefore focus on the 'logmodel' of f,

$$\mathbf{f} : \mathbf{T} \to \partial\mathcal{Z}_0 \xrightarrow{\partial\hat{f}} \partial\mathbb{D}.$$

We show that it comes with canonical monodromy flow $\theta \in \mathbb{R} \mapsto \mathbf{h}_\theta$. We define $\mathbf{h}_\theta$ on $|\sigma| \times T_\sigma$ by using the first component $\rho \in |\sigma|$ to define a one parameter subgroup in ${\mathbb{C}_1}^\sigma$ (and having it act on the second component):

$$\mathbf{h}_{\sigma,\theta} : (\rho, t) = (\rho_i, t_i)_{i\in\sigma} \in |\sigma| \times T_\sigma \mapsto (\rho_i, \exp(\sqrt{-1}\theta\rho_i/n_i)t_i)_{i\in\sigma}$$

It is a monodromy flow, for $\mathbf{f}\mathbf{h}_{\sigma,\theta} = \exp\sqrt{-1}\theta\mathbf{f}\big||\sigma| \times T_\sigma$. Note that when we fix ρ to be the barycenter $\rho_\sigma \in |\sigma|$, then this gives us back the flow $h_{\sigma,\theta}$. For a face $\tau \subset \sigma$, the restrictions of $\mathbf{h}_{\sigma,\theta}$ and $\mathbf{h}_{\tau,\theta}$ to $|\tau| \times T_\sigma$ are the same (for then $\rho_i = 0$ for all $i \in \sigma \smallsetminus \tau$) and so this produces a monodromy flow on $\mathbf{T}$ as asserted. In particular, $\mathbf{h}_{2\pi} : \mathbf{T}_1 \to \mathbf{T}_1$ may now serve as our canonical model for the monodromy. It acts in $\{\rho\} \times T_\sigma$ as the element $(\exp(2\pi\sqrt{-1}\rho_i/n_i))_{i\in\sigma}$ of the torus ${\mathbb{C}_1}^\sigma$.

Proof of Proposition 6.2. Let σ be a simplex of $\mathcal{D}$. We first concentrate on $\mathbf{T}_1|_{|\sigma|\times\mathring{D}_\sigma}$. This is a principal bundle for the group $\mathrm{Ker}(\chi_\sigma : {\mathbb{C}_1}^\sigma \to \mathbb{C}_1)$ and $\mathbf{h}_{\sigma,2\pi}$ acts fiberwise. We embed $\mathring{T}_\sigma$ in $\mathbf{T}_1|_{|\sigma|\times\mathring{D}_\sigma}$ by taking for ρ the barycenter of $|\sigma|$, ρ_σ. Then $\mathbf{h}_{K,2\pi}$ acts on this subspace as $h_{\sigma,2\pi}$. We assert that this inclusion is a homotopy equivalence. Indeed, a deformation retraction is given by

$$(\rho, t) \in \mathbf{T}_1|_{|\sigma|\times\mathring{D}_\sigma} \mapsto$$

$$\Big((1-s)\rho+s\rho_\sigma), \big(\exp(2\pi\sqrt{-1}/n_i.(\rho_i(1-s)+s/|\#(\sigma)|))t_i\big)_{i\in\sigma}\Big) \in \mathbf{T}_1|_{|\sigma|\times\mathring{D}_\sigma}$$

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[4]This is an example of the geometric realization of a simplicial variety.

(where $0 \leq s \leq 1$). Since $\mathbf{h}_{\sigma,2\pi}$ acts on this subspace as $h_{\sigma,2\pi}$, it follows from Lemma 6.3 that it acts on the (co)homology of $\mathbf{T}_1|_{|\sigma|\times \mathring{D}_\sigma}$ with an order dividing N and with Zeta function 1 unless $\sigma = \{i\}$, in which case it is $(1-t^{n_i})^{-e(\mathring{D}_i)}$.

Denote by $Z_0^{(k)}$ the locus where at least $k+1$ members of $\mathcal{D}$ meet. Then $Z_0^{(\nu)} = \emptyset$ by assumption and we have a filtration of Z_0 by closed subsets

$$Z_0 = Z_0^{(0)} \supset Z_0^{(1)} \supset \cdots Z_0^{(\nu-1)} \supset Z_0^{(\nu)} = \emptyset.$$

Let $\partial\hat{\mathcal{Z}}_1^{(k)}$ resp. $\mathbf{T}_1^{(k)}$ denote the preimage of $Z_0^{(k)}$ in $\partial\hat{\mathcal{Z}}_1$ resp. $\mathbf{T}_1$. Notice that then $\partial\hat{\mathcal{Z}}_1^{(k)} \smallsetminus \partial\hat{\mathcal{Z}}_1^{(k+1)}$ is the disjoint union of the $\mathring{T}_{\sigma,1}$ with $\dim(\sigma) = k$. The monodromy $\mathbf{h}_{2\pi}$ preserves the resulting filtration on $\mathbf{T}_1$. Its action on the homology of the fiber pair $(\mathbf{T}_1^{(k)}, \mathbf{T}_1^{(k+1)})$ becomes clear via the isomorphism (duality and excision)

$$\begin{aligned}H_r(\mathbf{T}_1^{(k)}, \mathbf{T}_1^{(k+1)};\mathbb{Q}) &\cong H_r(\partial\hat{\mathcal{Z}}_1^{(k)}, \partial\hat{\mathcal{Z}}_1^{(k+1)};\mathbb{Q}) \cong \\ &H^{2m-r}(\partial\hat{\mathcal{Z}}_1^{(k)} \smallsetminus \partial\hat{\mathcal{Z}}_1^{(k+1)};\mathbb{Q}) \cong \textstyle\bigoplus_{\dim(\sigma)=k} H^{2m-r}(\mathring{T}_{\sigma,1};\mathbb{Q}),\end{aligned}$$

for we just established that the Nth power of the monodromy acts on the latter sum as the identity. We also see that the Zeta function for the monodromy action on $H_\bullet(\mathbf{T}_1^{(k)}, \mathbf{T}_1^{(k+1)};\mathbb{Q})$ is the identity unless $k=0$, in which case we get $\prod_{i\in\mathcal{D}}(1-t^{n_i})^{-e(\mathring{D}_i)}$.

We have a Leray spectral sequence with monodromy action

$$E^1_{p,k} = H_p(\mathbf{T}_1^{(k)}, \mathbf{T}_1^{(k+1)};\mathbb{Q}) \Rightarrow H_{p+k}(\mathbf{T}_1;\mathbb{Q}).$$

This yields a filtration on $H_\bullet(\mathbf{T}_1;\mathbb{Q})$ which is preserved by the monodromy and is such that the Nth power of the monodromy acts as the identity on the successive quotients (these are subquotients of the $H_\bullet(\mathbf{T}_1^{(k)}, \mathbf{T}_1^{(k+1)};\mathbb{Q})$ and there are at most ν such). Both assertions now follow. □

REMARK 6.4. Observe that the intrinsic model for the monodromy $\mathbf{h}_{2\pi}$ has the property that $\mathbf{h}_{2\pi}^r$ acts freely for $1 \leq r < \min_{i\in\mathcal{D}} n_i$. In particular, $\mathbf{h}_{2\pi}$ acts freely if Z_0 is without reduced component.

CHAPTER 2

Perversities

In this chapter spaces are assumed to be locally compact Hausdorff unless otherwise stated.

1. Homological algebra pertaining to sheaves

1.1. Right derived functors. Much of homological algebra is covered by the theory of half exact functors between abelian categories. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be abelian categories such that $\mathfrak{C}$ has enough injective objects. Suppose $T : \mathfrak{C} \to \mathfrak{C}'$ is an additive left exact functor, meaning that it preserves the exactness of exact sequences of the form $0 \to A \to B \to C$. Any object C of $\mathfrak{C}$ admits an injective resolution $C^\bullet \to I^\bullet$. Such an injective resolution is unique up to chain homotopy and so $T(I^\bullet)$ is also defined up to chain homotopy. In particular, $R^kT(C) := H^k(T(I^\bullet))$ is independent of the choice of $C^\bullet \to I^\bullet$; it is the *kth right derived functor* of T. If $R^kT(C) = 0$ for all $k \geq 1$, then C is said to be T-acyclic. For computing the higher derived functors R^kT we may replace resolutions by injective objects by resolutions by T-acyclic objects (it is clear that an injective object is T-acyclic).

This generalizes to chain complexes $C^\bullet$ in $\mathfrak{C}$ that are zero (or just exact) in sufficiently low degree. Let us recall that a map of complexes is called a *quasi-isomorphism* if it induces an isomorphism on all the cohomology groups. We can always find a quasi-isomorphism of $C^\bullet$ with a complex $I^\bullet$ with each I^k injective (see below for a more specific construction). Such a quasi-isomorphism, often referred to as an *injective resolution* of $C^\bullet$, is unique up to chain homotopy so that $T(I^\bullet)$ is also unique up to chain homotopy and $\mathbb{R}^kT(C^\bullet) := H^k(T(I^\bullet))$ is independent of all choices made([1]). It is also clear that this has a functorial character with respect to chain maps: $\phi : C^\bullet \to C'^\bullet$ induces natural morphisms $\mathbb{R}^kT(C^\bullet) \to \mathbb{R}^kT(C'^\bullet)$. It it immediate from the definition that these are isomorphisms when ϕ is a quasi-isomorphism.

For example, if $\mathfrak{C}$ is the category of sheaves on X and Γ is the global section functor, then for every sheaf complex $\mathcal{F}^\bullet$ on X that is exact sufficiently low degree, we have defined $R^k\Gamma(\mathcal{F}^\bullet)$. In that

[1]The notation $\mathbb{R}^kT(C^\bullet)$ (which is a single group) is intended to make the distinction with the graded group $R^kT(C^\bullet) = \oplus_i R^kT(C^i)$, yet many authors write $R^kT(C^\bullet)$ instead.

case we write $\mathbb{H}^k(X, \mathcal{F}^\bullet)$ instead and call it the *hypercohomology* of $\mathcal{F}^\bullet$. A classical example is the holomorphic De Rham complex $\Omega^\bullet_M$ on a complex manifold M. The cohomology of $\Gamma(M, \Omega^\bullet_M)$ is in general not the cohomology of M, but we do have $\mathbb{H}^k(M, \Omega^\bullet_M) = H^\bullet(M; \mathbb{C})$, simply because $\Omega^\bullet_M$ is a resolution of the constant sheaf $\mathbb{C}_M$.

A useful kind of injective resolution of a bounded below complex $C^\bullet$ is obtained via a double complex:

$$\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & I^{k-1,1} & \longrightarrow & I^{k,1} & \longrightarrow & I^{k+1,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & I^{k-1,0} & \longrightarrow & I^{k,0} & \longrightarrow & I^{k+1,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & C^{k-1} & \longrightarrow & C^{k} & \longrightarrow & C^{k+1} & \longrightarrow & \cdots
\end{array}$$

Here the vertical arrows are exact and hence give injective resolutions of the terms of $C^\bullet$ and the squares commute. We will denote this simply by $C^\bullet \to I^{\bullet,\bullet}$. The associated simple complex $s(I^{\bullet,\bullet})$ has terms $s(I^{\bullet,\bullet})^k := \oplus_{i+j=k} I^{i,j}$ with differential $d : I^{i,j} \to I^{i,j+1} \oplus I^{i+1,j}$ being the sum of the vertical differential and $(-1)^i$ times the horizontal differential; this is indeed a complex and the obvious chain map $C^\bullet \to s(I^{\bullet,\bullet})$ is a quasi-isomorphism. The complex $s(I^{\bullet,\bullet})$ comes with two filtrations: the vertical filtration defined by $F^k_{\mathrm{vert}} s(I^{\bullet,\bullet}) := I^{\bullet,\geq k}$ and the horizontal filtration defined by $F^k_{\mathrm{hor}} s(I^{\bullet,\bullet}) := I^{\geq k,\bullet}$. The image of each under T gives rise to a spectral sequence converging to $\mathbb{R}^k T(C^\bullet)$:

$$H^p(R^q T(C^\bullet)) \Rightarrow \mathbb{R}^{p+q} T(C^\bullet) \Leftarrow R^p T(H^q(C^\bullet)).$$

Again we may relax the condition of $I^{p,q}$ being injective to $I^{p,q}$ being T-acyclic. So if $C^\bullet$ is a T-acyclic complex, then $\mathbb{R}^p T(C^\bullet) = H^p(T(C^\bullet))$.

For the De Rham example, the first spectral sequence yields the Hodge De Rham spectral sequence $H^q(M, \Omega^p_M) \Rightarrow H^{p+q}(M; \mathbb{C})$.

This construction is helpful when we have another additive left exact functor $T' : \mathfrak{C}' \to \mathfrak{C}''$ of abelian categories such that T takes injective objects to T'-acyclic ones: then $R(T'T)^k(C)$ is obtained via $\mathfrak{C}'$ by first choosing an injective resolution $C \to J^\bullet$. The cohomology of $T'(T(J^\bullet))$ is by definition $R^\bullet(T'T)(C)$. Since the terms of the complex $T(J^\bullet)$ are T'-acyclic this is also $\mathbb{R}^\bullet T'(T(J^\bullet))$. The other spectral sequence converging to $\mathbb{R}^\bullet T'(T(J^\bullet))$ then gives the *Grothendieck spectral sequence*

$$R^p T'(R^q T(C)) \Rightarrow R^{p+q}(T'T)(C).$$

Note that the hypothesis on T is satisfied if it has a left adjoint $S : \mathfrak{C}' \to \mathfrak{C}$. This means that for any pair $(C, C') \in \mathfrak{C} \times \mathfrak{C}'$, we have a canonical isomorphism $\mathrm{Hom}(C', T(C)) \cong \mathrm{Hom}(S(C'), C)$. Then T

even takes injectives to injectives, for if $I \in \mathfrak{C}$ is injective and $A' \hookrightarrow B'$ is monomorphism in $\mathfrak{C}'$, then $\mathrm{Hom}(B', T(I)) \to \mathrm{Hom}(A', T(I))$ is identified with $\mathrm{Hom}(S(B'), I) \to \mathrm{Hom}(S(A'), I)$ and hence onto. An example of this type is the Leray spectral sequence (see below).

1.2. The derived category. Given an abelian category $\mathfrak{C}$, we denote by $K(\mathfrak{C})$ the category whose objects are complexes in $\mathfrak{C}$ and morphisms are chain homotopy classes of morphisms. This is also an abelian category and the construction is functorial in $\mathfrak{C}$: if $T : \mathfrak{C} \to \mathfrak{C}'$ is an additive functor of abelian categories, then there is clearly an induced functor $K(T) : K(\mathfrak{C}) \to K(\mathfrak{C}')$. If we restrict to complexes that have cohomology in finitely many (resp. negative, resp. positive) degrees, we write $K^b(\mathfrak{C})$ (resp. $K^+(\mathfrak{C})$ resp. $K^-(\mathfrak{C})$).

Assuming $\mathfrak{C}$ has enough injectives, then as noted, any object of $K^+(\mathfrak{C})$ has an injective resolution that is bounded below. Moreover, for every quasi-isomorphism $\phi : I^\bullet \to J^\bullet$ between to such bounded below injective complexes there exists a chain map $\psi : J^\bullet \to I^\bullet$ such that $\psi\phi$ and $\phi\psi$ are chain homotopic to the identity of $I^\bullet$ resp. $J^\bullet$ and so ϕ is then an isomorphism in $K^+(\mathfrak{C})$. This suggests to introduce the *derived category* $D(\mathfrak{C})$ as a 'quotient category' of $K(\mathfrak{C})$ obtained by treating a quasi-isomorphism as an isomorphism, in other words, by declaring that a quasi-isomorphism has an inverse (this is reminiscent of localization) and to do likewise for the variants introduced above: $D^b(\mathfrak{C})$ and $D^\pm(\mathfrak{C})$. In particular, (injective) resolutions define isomorphisms in $D(\mathfrak{C})$. Note that the cohomology functor, $C^\bullet \mapsto H^\bullet(C^\bullet)$, passes via the localization functor $K(\mathfrak{C}) \to D(\mathfrak{C})$. But a derived category is in general no longer an abelian category. In fact, any morphism in $D(\mathfrak{C})$ can be represented by and monomorphism resp. epimorphism of chain complexes([2]).

Since T need not take quasi-isomorphisms to quasi-isomorphisms, the definition of an induced functor $D^+(\mathfrak{C}) \to D^+(\mathfrak{C}')$ needs some care. Assuming $\mathfrak{C}$ has enough injectives, then the above discussion shows that $D^+(\mathfrak{C})$ is equivalent to the full subcategory of $K^+(\mathfrak{C})$ whose objects are injective bounded below complexes. For a left exact functor $T : \mathfrak{C} \to \mathfrak{C}'$, we then define $RT : D^+(\mathfrak{C}) \to D^+(\mathfrak{C})$, by representing an object of $D^+(\mathfrak{C})$ by a bounded below complex in $\mathfrak{C}$ whose objects are injective, or more generally, T-acyclic, and then apply T to it. It is clear that then $\mathbb{R}^k T(C^\bullet) = H^k(RT(C^\bullet))$.

1.3. Some basic functors for sheaf cohomology. We recall that for a continuous map $f : Z \to Z'$ there is defined a functor f_*, which assigns to a sheaf $\mathcal{A}$ on Z its direct image of $f_*\mathcal{A}$ and a functor f_*,

[2] But a short exact sequence has a *distinguished triangle* as a substitute, a notion which we will discuss later.

which assigns to a sheaf $\mathcal{A}'$ on Z' the preimage $f^*\mathcal{A}'$([3]). Then (f^*, f_*) is an adjoint pair: we have a natural identification $\mathrm{Hom}(f^*\mathcal{A}', \mathcal{A}) \cong \mathrm{Hom}(\mathcal{A}', f_*\mathcal{A})$. This gives rise to associated adjunction maps: natural sheaf homomorphisms $\mathcal{A}' \to f_*f^*\mathcal{A}'$ and $f^*f_*\mathcal{A} \to \mathcal{A}$ that are associated to the identity of $f^*\mathcal{A}'$ resp. $f_*\mathcal{A}$. But usually this is proved by first establishing the adjunction maps, as these determine the isomorphism and its inverse: given $\phi \in \mathrm{Hom}(f^*\mathcal{A}', \mathcal{A})$ then precomposition of $f_*\phi \in \mathrm{Hom}(f_*f^*\mathcal{A}', f_*\mathcal{A})$ with the adjunction map $\mathcal{A}' \to f_*f^*\mathcal{A}'$ yields an element of $\mathrm{Hom}(\mathcal{A}', f_*\mathcal{A})$, and given $\psi \in \mathrm{Hom}(\mathcal{A}', f_*\mathcal{A})$, postcomposition of $f^*\psi \in \mathrm{Hom}(f^*\mathcal{A}', f^*f_*\mathcal{A})$ with the adjunction map $f^*f_*\mathcal{A} \to \mathcal{A}$ yields an element of $\mathrm{Hom}(f^*\mathcal{A}', \mathcal{A})$. It then amounts to showing that these are each others inverse. Note that pt_{Z*} is the global section functor Γ (and pt_Z^* assigns to a group A the constant sheaf A_Z on Z). The stalk of $f^*\mathcal{A}'$ at z is just $\mathcal{A}'_{f(z)}$ and so f^* is exact (and hence its higher derivatives are zero). Since f_* has a left adjoint, the Grothendieck spectral sequence is valid for the functor f_* and a functor composable with it. For the global section functor Γ on Z', this reproduces the (Leray) spectral sequence $H^p(Z', R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(Z, \mathcal{F})$.

We also have the sheaf $f_!\mathcal{A}$ on Z', the *direct image of $\mathcal{A}$ with proper support*: it is the sheaf associated to the presheaf that assigns to an open $U' \subset X'$ the subgroup of $\mathcal{A}(f^{-1}U')$ of sections that have their support proper over U'. So $\mathrm{pt}_{Z!}$ is the global section functor Γ_c with compact supports.

Sometimes there is defined a functor $\mathcal{A}' \mapsto f^!\mathcal{A}'$ such that $(f_!, f^!)$ is an adjoint pair([4]), meaning that we have a natural identification $\mathrm{Hom}(f_!\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, f^!\mathcal{F}')$. So then the associated adjunction maps go in the opposite direction as for (f^*, f_*): $\mathcal{A} \to f^!f_!\mathcal{A}$ and $f_!f^!\mathcal{A}' \to \mathcal{A}'$. The functor $f^!$ exists for instance when f is the inclusion of a locally closed subspace. As a locally closed subspace is the composite of an open embedding and a closed one, let us first consider these cases separately. We will see in passing how these maps allow us to extend sheaf cohomology to topological pairs for which the second item is a closed or an open subset.

Let $j : O \subset X$ be an open subset with (closed) complement $i : Y := X \smallsetminus O \subset X$ (a closed embedding).

[3]A convenient way to define the preimage is do this first for a closed embedding—then it just the restriction of a sheaf—and for a projection of a product onto a factor—where this is straightforward—and then to define f^* as $i^*\pi^*$, where $i : Z \to Z \times Z'$ is the graph and $\pi : Z \times Z \to Z'$ the projection. This also works well in the setting of ringed spaces, such as schemes, where the product need not have the product topology.

[4]However, such an adjunction is always defined in a derived category. In this setting the relation between the two adjoint pairs also becomes clearer via a duality.

Then $j_!$ is given as extension by zero: for a sheaf $\mathcal{G}$ on O, $j_!\mathcal{G}$ is the sheaf which assigns to an open $U \subset X$, $\mathcal{G}(U)$ if $U \subset O$ and the trivial group otherwise. We have $j^! = j^*$ and indeed, $(j_!, j^*)$ is an adjoint pair. Since j^* has a left adjoint, it takes injectives to injectives. Notice that two adjunction maps for a sheaf $\mathcal{F}$ on X make up a short exact sequence of sheaves on X:

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0. \tag{1}$$

For $\mathcal{F} = \mathbb{Z}_X$, $H^k(X, j_!j^*\mathcal{F})$ is just $H^k(X, Y)$ and so this group may in general be understood as a relative cohomology group with values in $\mathcal{F}$ and the long sequence for cohomology associated to the exact sequence above,

$$\cdots \to H^k(X, j_!j^*\mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(Y, \mathcal{F}) \to H^{k+1}(X, j_!j^*\mathcal{F}) \to \cdots,$$

as the one of the pair (X, Y).

We have $i_! = i_*$, but the definition of $i^!$ is more subtle. Let $\Gamma_Y(\mathcal{F}) \subset \Gamma(\mathcal{F})$ denote the subgroup of sections with support in Y. The functor Γ_Y is left exact. Its kth right derived functor applied to $\mathcal{F}$ is denoted $H^k_Y(X, \mathcal{F})$ and called the *cohomology of $\mathcal{F}$ with support in Y*. We have an exact sequence $0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X \smallsetminus Y, \mathcal{F})$ which can be completed to a short exact sequence ending with 0 if $\mathcal{F}$ is injective. So if we choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ for $\mathcal{F}$, then the long exact sequence associated to $0 \to \Gamma_Y(X, \mathcal{I}^\bullet) \to \Gamma(X, \mathcal{I}^\bullet) \to \Gamma(X \smallsetminus Y, \mathcal{I}^\bullet) \to 0$,

$$\cdots \to H^k_Y(X, \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X \smallsetminus Y, \mathcal{F}) \to H^{k+1}_Y(X, \mathcal{F}) \to \cdots$$

begins with the above sequence. Hence $H^k(X \smallsetminus Y, \mathcal{F})$ can be understood (and is sometimes written) as a relative cohomology group $H^k(X, X \smallsetminus Y; \mathcal{F})$.

We now sheafify the functor Γ_Y. Given an open subset $V \subset Y$, then choose an open $U \subset X$ such that $U \cap Y = V$. Then $\Gamma_V(U, \mathcal{F})$ only depends on V and so we have obtained a presheaf on Y. We denote the associated sheaf by $i^!\mathcal{F}$. It is clearly a subsheaf of $i^*\mathcal{F}$ and you may verify that $(i_*, i^!)$ is an adjoint pair (and hence $i^!$ takes injectives to injectives). Note that the adjunction maps give rise to an exact sequence of sheaves that is the sheaf counterpart of the exact sequence above:

$$0 \to i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}. \tag{2}$$

It too, can be completed to a short exact sequence when $\mathcal{F}$ is injective. The functor $i^!$ is left exact and so it has right derived functors $R^ki^!\mathcal{F}$. Since i^* is exact, the long exact cohomology sequence tells us that $R^1i^!\mathcal{F}$ is the cokernel of $i^!\mathcal{F} \to i^*\mathcal{F}$. For $k > 1$, $R^ki^!\mathcal{F} \cong i^*R^{k-1}j_*j^*\mathcal{F}$ has its support contained in Y and we then sometimes may omit i^*. Note that then the stalk of $R^ki^!\mathcal{F}$ at y is $\varinjlim_{U \ni y} H^{k-1}(U \cap O; \mathcal{F})$. Since

$\Gamma_Y = \Gamma i^!$ as functors, we have a spectral sequence $H^p(Y, R^q i^! \mathcal{F}) \Rightarrow H_Y^{p+q}(X, \mathcal{F})$.

A special case that should be familiar is when X is a manifold, Y is a submanifold of codimension d and $\mathcal{F} = \mathbb{Z}_X$. Then $R^\bullet i^! \mathbb{Z}_X$ is concentrated in degree d and is in this degree locally free of rank one: it is the *orientation sheaf* $o_{Y/X} := R^d i^! \mathbb{Z}_X$ of the normal bundle (which, in case this bundle has been oriented, may be identified with $\mathbb{Z}_Y$). The spectral sequence then identifies $H_Y^k(X, \mathbb{Z}_X)$ with $H^{k-d}(Y, o_{Y/X})$ and the above long exact sequence becomes the Gysin sequence

$$\cdots \to H^{k-d}(Y, o_{Y/X}) \to H^k(X) \to H^k(X \smallsetminus Y) \to H^{k+1-d}(Y) \to \cdots .$$

In fact, if we take for $\mathcal{F}$ a local system (a locally constant sheaf), we still have a long exact sequence of twisted cohomology groups: $R^d i^! \mathbb{Z}_X$ is now the orientation sheaf $o_{Y/X}$ of the normal bundle (a local system of rank one) and we get an exact sequence of local system cohomology

$$\cdots \to H^{k-d}(Y, o_{Y/X} \otimes \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X \smallsetminus Y, \mathcal{F}) \to \\ \to H^{k+1-d}(Y, o_{Y/X} \otimes \mathcal{F}) \to \cdots .$$

The functors $f_*, f^*, f_!$ are functorial in f (e.g., $(fg)_! = f_! g_!$) and hence is this is also true for the adjoint $f^!$ of $f_!$ when it exists (so the above discussion gives its definition when f is a locally closed embedding).

1.4. Sheaf cohomology on a stratified cell complex. Suppose X has the structure of a CW complex with the property that the closure of a cell is a union of cells (we shall refer to this as a *stratified cell complex*). Let $\mathcal{F}$ be a sheaf on X that is locally constant on every (relatively open) cell. Since a relatively open cell e is contractible, $\mathcal{F}$ is in fact constant on it and we put $F_e := H^0(e, \mathcal{F})$. If (e, e') is a pair of cells such that e' is a cell in the closure of e, then we have a homomorphism $\phi_e^{e'} : F_{e'} \to F_e$ and we thus obtain a contravariant functor from the PO-set of cells to the category of abelian groups. Conversely, any such functor defines an sheaf on X that is locally constant on the relatively open cells. The sheaf cohomology $H^\bullet(X, \mathcal{F})$ can be computed via this functor as follows.

For every n-cell e, we have $H_c^k(e, \mathcal{F}) = 0$ unless $k = n$, in which case we get (by the universal coefficient theorem) $H_c^n(e) \otimes F_e$, where $H_c^n(e)$ is infinite cyclic. We shall write $F(e)$ for $H_c^n(e) \otimes F_e$. Since $H_c^n(e)$ is the orientation module of e, we have for every $(n-1)$-cel e' a coboundary map $\delta_e^{e'} : H_c^n(e') \to H_c^n(e)$: if $X_{n-1} \subset X$ denotes the $(k-1)$-skeleton of X, then $\delta_e^{e'}$ is the composite

$$H_c^{n-1}(e') \cong H^{n-1}(X_{n-1}, X_{n-1} \smallsetminus e') \xrightarrow{\delta} H^n(X_{n-1} \cup e, X_{n-1}) \cong H_c^n(e).$$

Denoting by $\mathscr{X}_n$ the collection of n-cells, then

$$\cdots \to \oplus_{e\in\mathscr{X}_{n-1}} F(e) \to \oplus_{e\in\mathscr{X}_n} F(e) \to \oplus_{e\in\mathscr{X}_{n+1}} F(e) \to \cdots$$

where the maps have the 'matrix coefficients' $\delta_e^{e'} \otimes \phi_e^{e'}$ is a complex (the *cellular chain complex*) and its cohomology is $H^\bullet(X, \mathcal{F})$. In case $i : Y \subset X$ is a subcomplex (a closed union of cells) and $j : X \smallsetminus Y \subset X$ its complement, then $\mathcal{F}' := j_! j^* \mathcal{F}$ is of the form above and obtained simply letting F'_e be 0 or F_e according to whether $e \subset Y$ or not. So if X is compact, then this computes $H_c^\bullet(X \smallsetminus Y, j^*\mathcal{F})$. This is even of interest when $j^*\mathcal{F}$ is a constant sheaf like $\mathbb{Z}_{X\smallsetminus Y}$.

EXAMPLE 1.1 (Local system on a surface). The following computation illustrates the technique and will prove useful when we discuss intersection cohomology. Suppose M is a compact connected oriented surface with nonempty boundary and denote by $\mathring{M}$ its interior. We also fix a locally constant sheaf $\mathbb{E}$ on M. Our aim is to compute $H_c^\bullet(\mathring{M}; \mathbb{E})$, $H^\bullet(M; \mathbb{E})$ and the natural map between these two. Note that if $j : \mathring{M} \subset M$, then $H_c^\bullet(\mathring{M}; \mathbb{E}) = H^\bullet(M; j_! j^* \mathbb{E})$ and that $H^\bullet(M; \mathbb{E}) \to H^\bullet(\mathring{M}; \mathbb{E})$ is an isomorphism.

We number the distinct connected components of ∂M by $0, \dots, n$ and choose on the ith component $\partial_i M$ of ∂M a point p_i. We denote by b^i the difference $\partial_i M \smallsetminus \{p_i\}$, regarded as oriented 1-cell (with respect to the orientation it receives from M([5]). We also choose for $i = 1, \dots, n$ an arc b_i from p_0 to p_i (oriented in this manner), and oriented embedded loops a_k, $k = 1, \dots, 2g$ on M emanating from p_0 (where g is the genus of M) such that these arcs are pairwise disjoint away from p_0 and the complement of their union is an open 2-cell d. This clearly gives M the structure of a stratified cell complex for which ∂M is a subcomplex. We choose a universal cover $\tilde{M} \to M$, denote its covering group by Γ (so this can be identified with the fundamental group of M) and put $E := H^0(\tilde{M}; \mathbb{E})$. This defines a representation $\rho : \Gamma \to \mathrm{Aut}(E)$.

Let $D \subset \tilde{M}$ be the closure of some lift of d. Note that D is a 'polygonal' fundamental domain for the action of Γ on $\tilde{M}$. The orientation of $\tilde{M}$ identifies $E(d) = H_c^2(e; \mathbb{E})$ with E; when an element $e \in E$ is viewed as an element of $E(d)$, we shall write it as $e(d)$. We use a similar convention for the other cells. For example, denote by $\tilde{a}_k$ the unique lift of a_k with the property that the orientation coming

[5]Our convention is that if V is a real oriented vector bundle of dimension n and $V_- \subset V$ a closed half space, then the bounding hyperplane ∂V_- is given the orientation characterized by the property that a basis $(e_1, \dots, e_{n-1})$ of ∂V_- is oriented if and only if for every $e \in V \smallsetminus V_-$, $(e, e_1, \dots, e_{n-1})$ is an oriented basis of ∂V_-. This convention is also used for orienting the boundary of an oriented manifold. So the standard orientation of $[0, 1]$ gives $\{1\}$ the positive orientation and $\{0\}$ the negative orientation and the standard orientation of the unit disk in $\mathbb{R}^2$ gives the unit circle the counter clockwise orientation.

from a_k coincides with the orientation induced by ∂D and use this to identify $E(a_k) = H^1_c(a_k, \mathbb{E})$ with E. We denote by α_k the element of Γ that takes the beginning of $\tilde{a}_k$ to its end. The other lift of a_k in D is the transform of $\tilde{a}_k$ by a unique element of Γ that we shall denote by α^k. Similarly we identify $E(b_i) = H^1_c(b_i, \mathbb{E})$ with E and let β^i be the element of Γ that takes the oriented lift $\tilde{b}_i$ of b_i to the other lift of b_i in D. On the other hand, b^i has only one lift $\tilde{b}^i$ in D (yielding the identification $E(b^i) \cong E$) and β^i is also the element of Γ that takes the beginning of $\tilde{b}^i$ to its end. We let $\tilde{p}_i$ be the beginning of $\tilde{b}^i$; this is clearly a lift of p_i and we use this lift to identify $E(p_i)$ with E.

Then the complex yielding $H^\bullet_c(\mathring{M}; \mathbb{E})$ is

$$
\begin{aligned}
(3)\quad 0 \to 0 \to \oplus_{k=1}^{2g} E(a_k) \bigoplus \oplus_{i=1}^{n} E(b_i) \to E(d) \to 0,\\
e(a_k) \mapsto (1 - \rho(\alpha^k))e(d), \quad e(b_i) \mapsto (1 - \rho(\beta^i))e(d)
\end{aligned}
$$

(the first zero is in degree -1). This shows in particular that $H^2_c(\mathring{M}; \mathbb{E})$ can be identified with the group of Γ-coinvariants E_Γ (the biggest quotient of E on which Γ acts trivially).

The complex for $H^\bullet(M; \mathbb{E})$ looks perhaps unnecessarily complicated (we can do with fewer cells), but is helpful if we want to describe $H^1_c(\mathring{M}; \mathbb{E}) \to H^1(M; \mathbb{E})$. Let $\hat{\alpha}_k \in \Gamma$ resp. $\hat{\beta}_i \in \Gamma$ take $\tilde{p}_0$ to the beginning of $\tilde{a}_k$ resp. $\tilde{b}_i$. Then the coboundary on the added cells is

$$
\begin{aligned}
&e(p_i) \mapsto (\rho(\beta^i) - 1)e(b^i) + e(b_i), \quad (i = 1, \dots, n)\\
&e(p_0) \mapsto (\rho(\beta^0) - 1)e(b^0) - \sum_{i=1}^{n} \rho(\hat{\beta}_i^{-1})e(b_i) + \sum_{\pm k=1}^{g} \rho(\hat{\alpha}_k^{-1})(\rho(\alpha_k) - 1)e(a_k),\\
&e(b^i) \mapsto e(d).
\end{aligned}
$$

We can simplify this complex (while not affecting the quasi-isomorphism type) by dividing out by the subcomplex spanned $e(p_i)$ $(i = 1, \dots, n)$ and $e(b^0)$ and their coboundaries. We thus may replace $e(b_i)$ by $-(\rho(\beta^i) - 1)e(b^i)$ for $i = 1, \dots, n$ and we get the quotient (the first zero is in degree -1):

$$
\begin{aligned}
(4)\quad 0 \to E(p_0) \to \oplus_{k=1}^{2g} E(a^k) \bigoplus \oplus_{i=1}^{n} E(b^i) \to 0 \to 0,\\
e(p_0) \mapsto \textstyle\sum_{k=1}^{2g} \rho(\hat{\alpha}_k^{-1})(\rho(\alpha^k) - 1)e(a_k) + \sum_{i=1}^{n} \rho(\hat{\beta}_i^{-1})(\rho(\beta^i) - 1)e(b^i)
\end{aligned}
$$

The cohomology of this complex is $H^\bullet(M; \mathbb{E})$; note this indeed identifies $H^0(M; \mathbb{E})$ with the group of Γ-invariants E^Γ.

We conclude that the map $H^\bullet_c(\mathring{M};\mathbb{E}) \to H^\bullet(M;\mathbb{E})$ is induced by a map of complexes (3)→(4) defined by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{k=1}^{2g} E(a_k) \bigoplus \oplus_{i=1}^{n} E(b_i) & \longrightarrow & E(d) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E(p_0) & \longrightarrow & \oplus_{\pm k=1}^{g} E(a_k) \bigoplus \oplus_{i=1}^{n} E(b^i) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where the central vertical map is the identity on $E(a_k)$ and is on $E(b_i)$ given by $e(b_i) \mapsto (1-\rho(\beta^i))e(b^i)$. This exhibits in particular an explicit description $H^1_c(\mathring{M};\mathbb{E}) \to H^1(\mathring{M};\mathbb{E}) \cong H^1(M;\mathbb{E})$. We shall return to this in Example 3.6, where we will interpret the image of this map as an intersection cohomology group.

1.5. Relative cohomology. It is sometimes convenient to have at our disposal 'relative' and 'support' versions for (hyper)cohomology. Let $i : Y \subset X$ be closed and $j : X \setminus Y \subset X$. Then for any sheaf complex $\mathcal{F}^\bullet$ on X we put

$$\mathbb{H}^\bullet(X, Y; \mathcal{F}^\bullet) := \mathbb{H}^\bullet(X, Rj_!\mathcal{F}^\bullet))$$
$$\mathbb{H}^\bullet_Y(X; \mathcal{F}^\bullet) := \mathbb{H}^\bullet(Y, Ri^!\mathcal{F}^\bullet).$$

With this definition some excision properties become obvious: if we remove from Y a subset that is open in X, then $\mathbb{H}^\bullet(X, Y; \mathcal{F}^\bullet)$ is not affected and if remove from $X \setminus Y$ a subset that is closed in X, then $\mathbb{H}^\bullet_Y(X; \mathcal{F}^\bullet)$ is not affected. These cohomology groups fit in exact sequences

$$\cdots \to \mathbb{H}^k(X, Y; \mathcal{F}^\bullet) \to \mathbb{H}^k(X; \mathcal{F}^\bullet) \to \mathbb{H}^k(Y; \mathcal{F}^\bullet) \to \mathbb{H}^{k+1}(X, Y; \mathcal{F}^\bullet) \to \cdots$$
$$\cdots \to \mathbb{H}^k_Y(X; \mathcal{F}^\bullet) \to \mathbb{H}^k(X; \mathcal{F}^\bullet) \to \mathbb{H}^k(X \setminus Y; \mathcal{F}^\bullet) \to \mathbb{H}^{k+1}_Y(X; \mathcal{F}^\bullet) \to \cdots .$$

1.6. Truncation functors. Let $\mathfrak{C}$ be an abelian category with enough injectives. For a complex $C^\bullet$ in $\mathfrak{C}$ we denote by $\tau_{\leq r}C^\bullet$ the truncated complex which ends as

$$\cdots \to C^{r-2} \to C^{r-1} \to \ker(C^r \to C^{r+1}) \to 0 \to 0 \to \cdots$$

Notice that this is a subcomplex of $C^\bullet$, which induces an isomorphism on cohomology in degree $\leq r$. Its cohomology is of course zero in degree $> r$. If each C^k is injective, then $\ker(C^r \to C^{r+1})$ need not be so, but we can always produce an injective resolution of $\tau_{\leq r}C^\bullet$, for instance by taking

$$\cdots \to C^{r-2} \to C^{r-1} \to C^r \to I^{r+1} \to I^{r+2} \to \cdots ,$$

where $0 \to \ker(C^r \to C^{r+1}) \to C^r \to I^{r+1} \to I^{r+2} \to \cdots$ we get an injective resolution of $\ker(C^r \to C^{r+1})$. Note that $\tau_{\leq r}$ is a functor which takes quasi-isomorphisms to quasi-isomorphisms. Hence it passes to $D(\mathfrak{C})$: $\tau_{\leq r} : D(\mathfrak{C}) \to D(\mathfrak{C})$.

Similarly, we denote by $\tau_{\geq r}C^\bullet$ the truncated complex

$$\cdots \to 0 \to 0 \to \mathrm{Coker}(C^{r-1} \to C^r) \to C^{r+1} \to C^{r+2} \to \cdots$$

Notice that this is a quotient of the quotient complex

$$C^\bullet/\tau_{\leq r-1}C^\bullet : \cdots \to 0 \to 0 \to B^r(C^\bullet) \to C^r \to C^{r+1} \to C^{r+2} \to \cdots ,$$

and that quotient map $C^\bullet/\tau_{\leq r-1}C^\bullet :\to \tau_{\geq r}C^\bullet$ is a quasi-isomorphism, it therefore defines the same object in the derived category. So we may also represent $\tau_{\geq r}C^\bullet$ by $C^\bullet/\tau_{\leq r-1}C^\bullet$. Its cohomology is zero in degree $< r$.

There is an interesting version for sheaf complexes that will play a central role in the construction of the intersection complex. Let $i : Y \subset X$ be an closed subset with complement $j : X \smallsetminus Y \subset X$ and let $\mathcal{F}^\bullet$ be a sheaf complex on $X \smallsetminus Y$. Then we define $\tau^Y_{\leq r-1} j_* \mathcal{F}^\bullet$ as the subcomplex of $j_*\mathcal{F}^\bullet$ obtained by truncation on Y only:
(5)

$$\cdots \to j_*\mathcal{F}^{r-3} \to j_*\mathcal{F}^{r-2} \to (\tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet)^{r-1} \to j_!\mathcal{F}^r \to j_!\mathcal{F}^{r+1} \to \cdots ,$$

where

$$(\tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet)^{r-1} := \mathrm{Ker}(j_*\mathcal{F}^{r-1} \to j_*\mathcal{F}^r \to i_*i^*j_*\mathcal{F}^r)$$

(a less ambiguous notation would be $(\tau^Y_{\leq r-1} j_*)\mathcal{F}^\bullet$). This makes sense as a complex because the kernel of $j_*\mathcal{F}^r \to i_*i^*j_*\mathcal{F}^r$ is $j_!j^*j_*\mathcal{F}^r = j_!\mathcal{F}^r$. We also observe that $j_!\mathcal{F}^{r-1} \subset (\tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet)^{r-1} \subset j_*\mathcal{F}^{r-1}$. The quotient of the first inclusion is easily seen to be $i_*\mathcal{Z}^{r-1}(i^*j_*\mathcal{F}^\bullet)$ and the quotient of the second inclusion maps isomorphically to $i_*\mathcal{B}^r(i^*j_*\mathcal{F}^\bullet)$.

Note that we a filtration

$$j_!\mathcal{F}^\bullet \subset \cdots \subset \tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet \subset \tau^Y_{\leq r} j_*\mathcal{F}^\bullet \subset \cdots \subset j_*\mathcal{F}^\bullet$$

with $\cap_r \tau^Y_{\leq r} j_*\mathcal{F}^\bullet = \tau^Y_{\leq r} j_!\mathcal{F}^\bullet$ and $\cup_r \tau^Y_{\leq r} j_*\mathcal{F}^\bullet = \tau^Y_{\leq r} j_*\mathcal{F}^\bullet$.

LEMMA 1.2. *The above inclusions give rise to exact sequences*

$$0 \to j_!\mathcal{F}^\bullet \to \tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet \to i_*\tau_{\leq r-1}(i^*j_*\mathcal{F}^\bullet) \to 0,$$
$$0 \to \tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet \to j_*\mathcal{F}^\bullet \to i_*\tau_{\geq r}(i^*j_*\mathcal{F}^\bullet) \to 0,$$

that lead to isomorphisms

$$\mathcal{H}^k(\tau^Y_{\leq r} j_*\mathcal{F}^\bullet) \cong \begin{cases} \mathcal{H}^k(j_*\mathcal{F}^\bullet) \text{ for } k < r; \\ \mathrm{Im}\left(\mathcal{H}^r(j_!\mathcal{F}^\bullet) \to \mathcal{H}^r(j_*\mathcal{F}^\bullet)\right) \text{ for } k = r; \\ \mathcal{H}^k(j_!\mathcal{F}^\bullet) \text{ for } k > r. \end{cases}$$

PROOF. The exact sequences follow from observations we already made. So $\mathcal{H}^k(j_!\mathcal{F}^\bullet) \to \mathcal{H}^k(\tau^Y_{\leq r} j_*\mathcal{F}^\bullet)$ is an isomorphism for $k > r$ and onto for $k = r$ and $\mathcal{H}^k(\tau^Y_{\leq r-1} j_*\mathcal{F}^\bullet) \to \mathcal{H}^k(j_*\mathcal{F}^\bullet)$ is an isomorphism for $k < r$ and into for $k = r$. □

Here is a pretty characterization of the functor $\tau^Y_{\leq r-1} j_* j^*$ in cohomological terms.

LEMMA 1.3. *Let $\mathcal{K}^\bullet$ a complex on X which is exact in sufficiently negative degree. Then $\mathcal{K}^\bullet$ is quasi-isomorphic to $\tau^Y_{\leq r-1} j_* j^* \mathcal{K}^\bullet$ if and only if $\mathbb{R}^k i^! \mathcal{K}^\bullet = 0$ for $k \leq r$ and $\mathbb{R}^k i^* \mathcal{K}^\bullet = 0$ for $k \geq r$.*

PROOF. Without loss of generality we may assume that $\mathcal{K}^\bullet$ is injective and zero in sufficiently negative degree. Consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & i_* i^! \mathcal{K}^{r-1} \arrow[r] \arrow[d] & i_* \mathcal{B}^r(i^! \mathcal{K}^\bullet) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] \arrow[d] & \mathcal{K}^{r-1} \arrow[r] \arrow[d] & \mathcal{K}^r \arrow[r] \arrow[d] & \mathcal{K}^{r+1} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & i_*(i^* \mathcal{K}^r / i^* \mathcal{B}^r(i^* \mathcal{K}^\bullet)) \arrow[r] & i_* i^* \mathcal{K}^{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

The top row is a subcomplex and the bottom row a quotient of $\mathcal{K}^\bullet$. As the vertical composites are zero, we think of it as a short complex of complexes

$$0 \to i_* \tau_{\leq r-1}(i^! \mathcal{K}^\bullet) \to \mathcal{K}^\bullet \to i_* \tau_{\geq r+1}(i^* \mathcal{K}^\bullet) \to 0.$$

With the help of exact sequences (1) and (2) we find that corresponding subquotient is

$$(6) \qquad \cdots \to j_* j^* \mathcal{K}^{r-2} \to j_* j^* \mathcal{K}^{r-1} \to \tilde{\mathcal{B}}^r / \mathcal{B}^r(i^! \mathcal{K}^\bullet) \to j_! j^* \mathcal{K}^{r+1} \to \cdots,$$

where $\tilde{\mathcal{B}}^r$ denotes the preimage of $i_* \mathcal{B}^r(i^* \mathcal{K}^\bullet) \subset i_* i^* \mathcal{K}^r$ in $\mathcal{K}^r$. We claim this is quasi-isomorphic to $\tau^Y_{\leq r-1} j_* j^* \mathcal{K}^\bullet$. Indeed, the complex (5) $\tau^Y_{\leq r-1} j_* j^* \mathcal{K}^\bullet$ embeds in the complex (6) with cokernel a complex concentrated in degrees $r-1$ or r and yielding the map of cokernels $\mathcal{B}^r(i^* j_* j^* \mathcal{K}) \to \mathcal{B}(i^* \mathcal{K}) / \mathcal{B}(i^! \mathcal{K})$, which is an isomorphism.

If the vanishing properties are satisfied, then both the top row and the bottom row are exact and so the passage from $\mathcal{K}^\bullet$ to this subquotient is a quasi-isomorphism, which then yields a quasi-isomorphism between $\mathcal{K}^\bullet$ and $\tau^Y_{\leq r-1} j_* j^* \mathcal{K}^\bullet$. The converse is left to you. □

2. Stratified spaces

We continue to assume that spaces are locally compact Hausdorff.

2.1. Stratified spaces and constructible sheaves. We begin with a basic definition.

DEFINITION 2.1. A *stratification* of a space X is a locally finite partition $\mathscr{X}$ of X into connected topological manifolds (referred to as *strata*) that have the property that the closure of every stratum is a union of strata. When X is a complex-analytic (resp. complex-algebraic) variety, then a stratification is called analytic (resp. algebraic) if the closure of each stratum for the Euclidean topology is a subvariety.

So strata are locally closed in X and in the algebraic setting this is even true for the Zariski topology. Note that an open subset of a stratified space is stratified space and so is a closed subset that is a union of strata. The product of two stratified spaces is stratified in an evident manner.

If L is a compact space, then the *open cone over* L, denoted $c(L)$, is the quotient of $[0,1) \times L$ which collapses $\{0\} \times L$ to a point. That point is called the *vertex* (often denoted by o) and L is called the *base* of the cone. If L comes with a stratification $\mathscr{L}$, then $c(L)$ inherits a stratification $c(\mathscr{L})$ having the vertex as a stratum and whose other strata have the form $(0,1)$ times a member of $\mathscr{L}$.

DEFINITION 2.2. We say that a stratification $\mathscr{X}$ of a space X is *conelike* if for every $p \in X$ there exists a compact stratified space $(L_p, \mathscr{L}_p)$ that such that if S_p is the member of $\mathscr{X}$ containing p, then a neighborhood U of p in $(X, \mathscr{X})$ is homeomorphic as stratified space a with the stratified product $c(L_p, \mathscr{L}_p) \times (S_p \cap U)$.

We will call such a homeomorphism an $\mathscr{X}$*-chart* and $(L_p, \mathscr{L}_p)$ a *link* of p relative to $(X, \mathscr{X})$. Such a link need not be unique, but it is easy to see (using the local triviality along the strata that have S_p in their closure) that it is stably so: if $(L'_p, \mathscr{L}'_p)$ is another choice, then $(L'_p, \mathscr{L}'_p) \times \mathbb{R}^N \cong (L_p, \mathscr{L}_p) \times \mathbb{R}^N$ for some N and that moreover either is conelike (we may take $N = 1 + \dim S_p$). Since we assumed S to be connected, the *stable isomorphism type* of $(L_p, \mathscr{L}_p)$ (understood in an obvious manner) only depends on S. (See Siebenmann [19] for more discussion.)

EXAMPLE 2.3. Let Σ be an abstract simplicial complex. Then the relatively open simplices define a stratification of its geometric realization $|\Sigma|$. In this case, we have a natural choice for our charts. To be precise, if σ is a simplex of Σ, then we let U_σ be 'star neighborhood' of $|\mathring{\sigma}|$ in $|\Sigma|$, i.e., the union of relatively open simplices $|\mathring{\tau}|$ with $\tau \supset \sigma$. Recall that if $\sigma \subsetneq \tau$ is a pair of simplices, then $|\tau|$ is naturally the join([6]) $|\sigma| \star |\tau \smallsetminus \sigma|$. This identifies $U_\sigma \cap |\tau|$ with $|\mathring{\sigma}| \times c(|\tau \smallsetminus \sigma|)$. So if $L_\sigma \subset \Sigma$ is the subcomplex whose vertices are the $v \in \Sigma \smallsetminus \sigma$ for which σ and v make up a simplex of Σ and whose simplices are the $\tau \smallsetminus \sigma$ where τ runs over the simplices of A which strictly contain σ, then we have obtained a homeomorphism $U_\sigma \cong |\mathring{\sigma}| \times c(|L_\sigma|)$.

One interest of having such a structure is that it implies that various locally defined (co)homology groups are constant on strata.

DEFINITION 2.4. Given a ring R, then an R_X-module on a stratified space $(X, \mathscr{S})$ is $\mathscr{X}$*-constructible* if this sheaf is locally constant

[6]Recall that the *join* $X \star Y$ of two spaces X, Y is the quotient of $X \times [0,1] \times Y$ by the collapsing maps (projections) $X \times \{0\} \times Y \to X$ and $X \times \{1\} \times Y \to Y$.

and of finite rank when restricted to a stratum (when $R = \mathbb{Z}$, we omit mention of R). When X is a complex-analytic (resp. complex-algebraic) variety, then an R_X-module (or an R_X- homomorphism) on X is said to be *constructible* if it is constructible relative to some analytic (resp. algebraic) stratification.

A stratified cell complex is a stratified space that need not be conelike, but the type of sheaf described there is clearly constructible.

REMARK 2.5. A sheaf homomorphism between two constant sheaves on X is locally constant (constant, when X is connected) and so the same is true for a R_X-homomorphism between two locally constant sheaves. It follows that an R_X-homomorphism between two $\mathscr{X}$-constructible R_X-modules is locally constant on every stratum. So the category of $\mathscr{X}$-constructible R_X-modules makes up a full (abelian) subcategory of the category of R_X-modules on X. Similarly, the analytically (resp. algebraically) constructible R_X-modules and sheaf homomorphisms on a complex-analytic (resp. complex-algebraic) variety make up an abelian category. Given R (usually $\mathbb{Z}$ or a field), then our basic category will be the category of complexes of R_X-modules whose sheaf cohomology groups are constructible relative to some stratification and are nonzero in only finitely many degrees. Such an object is called a *bounded constructible complex of R-modules*.

EXAMPLE 2.6 (Local homology). Assigning to an open subset U of X, the graded group $H_\bullet(X, X - U)$ is a presheaf. The associated sheaf has as its stalk at $p \in X$, the local homology of p in X, $H_\bullet(X, X \smallsetminus \{p\})$. If U_p is a neighborhood of p in X, then this is by excision equal to $H_\bullet(U_p, U_p \smallsetminus \{p\})$, so if we take U_p as above, then this is $H_\bullet((c(L_p), o) \times (S_p, p))$, which by the Künneth theorem is just $o_{S_p,p} \otimes \tilde{H}_{\bullet - \dim S_p - 1}(L_p)$. So this sheaf is locally constant on S_p. This proves that this sheaf is $\mathscr{X}$-constructible.

LEMMA 2.7. *Let $(X, \mathscr{X})$ be a conelike stratified space as above, $i : Y \subset X$ a closed union of strata na $j : X \smallsetminus Y \subset X$ its complement. The functors defined above and their right derivatives take constructible sheaves relative to $\mathscr{X}$ to constructible sheaves relative to $\mathscr{X}|X \smallsetminus Y$ or $\mathscr{X}|Y$ and vice versa.*

PROOF. For the functors, this is straightforward. For their right derivatives, use that the relevant categories of constructible sheaves have enough injectives (or use the explicit descriptions given above). □

Although the significance of the following proposition will become clear later, it seems best to state and prove it now. The function p appearing in this proposition is called a *perversity*.

PROPOSITION 2.8 (The p-intermediate extension). *Let $(X, \mathscr{X})$ be a conelike stratified space, $j : \mathring{X} \subset X$ an open union of strata and $\mathcal{E}^\bullet$ a bounded below complex on $\mathring{X}$ that is $\mathscr{X}|\mathring{X}$-constructible. Put $X' := X \smallsetminus \mathring{X}$. Then for every function $p : Y \in \mathscr{X}|X' \mapsto p_Y \in \mathbb{Z}$, there exists a bounded below $\mathscr{X}$-constructible injective sheaf complex ${}^p j_{!*}\mathcal{E}^\bullet$ on X endowed with a quasi-isomorphism $\alpha : \mathcal{E}^\bullet \to j^* {}^p j_{!*}\mathcal{E}^\bullet$ that is such that for any $Y \in \mathscr{X}'$*

(i) $\mathbb{R}^k i_Y^* {}^p j_{!*}\mathcal{E}^\bullet = 0$ *for* $k \geq p_Y$ *and*
(ii) $\mathbb{R}^k i_Y^! {}^p j_{!*}\mathcal{E}^\bullet = 0$ *for* $k \leq p_Y$.

This extension is natural, i.e., defines a functor $D^+(\mathring{X}) \to D^+(X)$.

When $p \leq q$, then we have a natural transformation ${}^p j_{!*}\mathcal{E}^\bullet \to {}^q j_{!*}\mathcal{E}^\bullet$; we also have natural transformations $Rj_!\mathcal{E}^\bullet \to {}^p j_{!*}\mathcal{E}^\bullet \to Rj^*\mathcal{E}^\bullet$ (which in case X has finitely many strata are realized by taking a perversity function that is sufficiently small resp. sufficiently large).

If $\mathcal{F}^\bullet$ is another bounded below complex on $\mathring{X}$ whose restriction to $\mathring{X}$ is $\mathscr{X}|\mathring{X}$-constructible, then for every $q : Y \in \mathscr{X}|X' \mapsto p_Y \in \mathbb{Z}$ we have a natural map of complexes ${}^p j_{!*}\mathcal{E}^\bullet \otimes {}^p j_{!*}\mathcal{F}^\bullet \to {}^{p+q} j_{!*}(\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet)$.

PROOF. We proceed with induction: we then add a statum one at a time in such a manner that at any stage we are dealing with an open union of strata. To make the induction step, we then may assume that that the extension with the desired properties has already been constructed over the complement of a closed stratum $Y \subset X$. In other words, we may assume that $X' = Y$. Lemma 1.3 tells us that then ${}^p j_{!*}\mathcal{E}^\bullet := R\tau^Y_{\leq p_Y - 1} j_{\mathring{X}*}\mathcal{E}^\bullet$ satisfies the vanishing conditions (i) and (ii) on X.

The second set of assertions is clear and the last assertion follows from the evident fact that $R\tau^Y_{\leq p_Y-1} j_{\mathring{X}*}\mathcal{E}^\bullet \otimes R\tau^Y_{\leq q_Y-1} j_{\mathring{X}*}\mathcal{E}^\bullet$ maps to $R\tau^Y_{\leq p_Y+q_Y-2} j_{\mathring{X}*}(\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet)$ (and hence to $R\tau^Y_{\leq p_Y+q_Y-1} j_{\mathring{X}*}(\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet)$). $\square$

From now on, it will be tacitly assumed that a stratification of a complex-analytic resp. complex-algebraic variety is analytic resp. algebraic. Conelike stratifications exist in this setting: the following result (the proof of which we omit) combines a theorem of Whitney and Thom's first isotopy lemma.

THEOREM 2.9 (Whitney, Thom [23],[20]). *Any complex algebraic resp. analytic variety X admits a conelike stratification $\mathscr{X}$ with the property that if $f : X \to B$ is a differentiable map to a manifold B (meaning that its coordinates are C^∞-dependent of elements of the structure sheaf), then its $\mathscr{X}$-*critical *set (the set of p for which f restricted to the stratum through p has a critical point at p) is closed and f has the* Ehresmann property *relative to $\mathscr{X}$: if f is proper and is without $\mathscr{X}$-critical point, then f is topologically locally trivial relative to $\mathscr{X}$. Moreover, any given partition of X into Zariski locally closed subvarieties can be refined to a conelike stratification.*

REMARK 2.10. The closedness property of the criticial set is just a way of formulating of what is known as the Whitney (a) condition. We will refer to stratification as in Theorem 2.9 as a *Thom-Whitney stratification*.

COROLLARY 2.11. *Let X be a complex algebraic resp. analytic variety, and $f : Y \subset X$ locally closed for the Zariski topology. Then $f_*, f^*, f_!, f^!$ and their right derivatives take constructible sheaves to constructible sheaves.*

PROOF. Choose a conelike stratification $\mathscr{X}$ of X as in Theorem 2.9 which refines the partition $\{Y, \overline{Y} \smallsetminus Y, X \smallsetminus \overline{Y}\}$ of X and is such that the constructible sheaf in question is locally constant on every stratum. Then apply Lemma 2.7. □

3. Intersection cohomology on a conelike stratified space

We here only consider conelike stratified spaces $(X, \mathscr{X})$ with the property that there is a single open stratum $\mathring{X}$ (of dimension d, say). We also fix a field $\mathbf{k}$. The $\mathbf{k}$-dual of a $\mathbf{k}$-vector space V is denoted $V^\vee$.

3.1. Duality on a manifold. Let us first review how Poincaré duality comes about. Assume for this that the stratification is trivial so that $X = \mathring{X}$ is a manifold (without boundary). Denote by o_X the *orientation sheaf* of X, i.e., the sheaf associated to the presheaf $U \subset X \mapsto H_d(X, X \smallsetminus U) = H_d^{cl}(U)$. Its stalk at $p \in U$ is $H_d(X, X \smallsetminus \{p\})$, which is free of rank one and so has exactly two generators (orientations). For an open subset $U \subset X$, $\cap$ defines a map $H^k(U) \times H_d^{cl}(U) \to H_{d-k}^{cl}(U)$. This is an isomorphism if U is an open ball and leads to a homomorphism $H^k(U, o_X) \to H_{d-k}^{cl}(U)$ that is functorial in U (both members define presheaves). With the help of a Čech complex (or equivalently, an iterated Mayer-Vietoris argument), we then see that this is an isomorphism when U can be written as a finite union of open subsets $U_0 \cup U_1 \cup \cdots \cup U_n$ such that each U_i and each intersection $U_{i+1} \cap (U_0 \cup \cdots \cup U_i)$ is homeomorphic to an open ball. This is so when U admits a compactification as a manifold with boundary. If we take our coefficients in the field $\mathbf{k}$, then by the universal coefficient theorem we can as write this as a map $H^k(U, o_X \otimes \mathbf{k}) \to H_c^{d-k}(U; \mathbf{k})^\vee$.

Without much change this generalizes to the situation where $\mathbf{k}$ is replaced by a local system $\mathbb{E}$ of $\mathbf{k}$-vector spaces: we get a map $H^k(U, o_X \otimes \mathbb{E}) \to H_c^{d-k}(U, \mathbb{E}^\vee)^\vee$ with the above properties. We can also express this by saying that if we have two such local systems $\mathbb{E}$ and $\mathbb{F}$ together with a pairing $\psi : \mathbb{E} \otimes_{\mathbf{k}} \mathbb{F} \to o_X \otimes \mathbf{k}$, then

$$H^k(U, \mathbb{E}) \otimes_{\mathbf{k}} H_c^{d-k}(U, \mathbb{F}) \xrightarrow{\cup} H_c^d(U, \mathbb{E} \otimes_{\mathbf{k}} \mathbb{F}) \xrightarrow{\psi} H_c^d(U, o_X \otimes \mathbf{k}) \cong \mathbf{k}$$

is nondegenerate whenever ψ is. We want to generalize this to the case of a nontrivial stratification and then let the role of the open balls be taken by the $\mathscr{X}$-charts U for which U meets only one stratum S in a nonempty closed submanifold $U \cap S$ such that $U \cap S$ an open ball. If we establish a natural duality map for such U, then we will have it for unions of such U of the above type.

We begin with the following observation.

LEMMA 3.1. *Let L be a compact space and let $\mathcal{F}^\bullet$ be a sheaf complex on L. Denote by $\pi : (0,1) \times L \to L$ the projection and by $j : (0,1) \times L \subset c(L)$ the embedding. Then $\mathbb{H}^\bullet(c(L), Rj_!\pi^*\mathcal{F}^\bullet)$ and $\mathbb{H}^\bullet_c(c(L), Rj_*\pi^*\mathcal{F}^\bullet)$ are zero.*

PROOF. Let us first understand these assertions: they mean that if $\mathcal{I}^\bullet$ is an injective resolution of $\pi^*\mathcal{F}^\bullet$, then these identities hold with $Rj_*\pi^*\mathcal{F}^\bullet$ resp. $Rj_!\pi^*\mathcal{F}^\bullet$ replaced by $j_*\mathcal{I}^\bullet$ resp. $j_!\mathcal{I}^\bullet$.

We first check this for the case when L is a point and $\mathcal{F} = \mathbb{Z}$. Then we are considering an injective resolution $\mathcal{I}^\bullet$ of the constant sheaf $\mathbb{Z}_{(0,1)}$ on $(0,1)$. In the first case, we compute cohomology of $(0,1)$ whose support does not have 0 in its closure and in the second case the cohomology of $(0,1)$ whose support is relatively compact in $[0,1)$, i.e., does not have 1 in its closure: this is the cohomology of the pair $([0,1),\{0\})$ resp. $((0,1],\{1\})$ and hence zero.

The general case follows by means of a Künneth argument. For this, we observe that if we replace j by $\tilde{j} : (0,1) \times L \subset [0,1) \times L$, the hypercohomology groups do not change. The Künneth formula shows that the hypercohomology in question is the tensor product of the group computed above (which is zero) with $\mathbb{H}^\bullet(L, \mathcal{F}^\bullet)$. $\square$

COROLLARY 3.2. *In the situation of Lemma 3.1 we have for $p \in \mathbb{Z}$,*

$$\mathbb{H}^k(c(L), R\tau^{\{o\}}_{\le p-1} j_*\pi^*\mathcal{F}^\bullet) = \begin{cases} \mathbb{H}^k(L, \mathcal{F}^\bullet) \text{ for } k \le p-1; \\ 0 \text{ for } k \ge p \end{cases} \quad \text{and}$$

$$\mathbb{H}^k_c(c(L), R\tau^{\{o\}}_{\le p-1} j_*\pi^*\mathcal{F}^\bullet) = \begin{cases} \mathbb{H}^{k-1}(L, \mathcal{F}^\bullet) \text{ for } k \ge p+1; \\ 0 \text{ for } k \le p. \end{cases}$$

PROOF. Let $i : \{o\} \subset c(L)$. Consider the long exact hypercohomology sequence associated to the first exact sequence of Lemma 1.2. If we apply Lemma 3.1, we find that

$$\mathbb{H}^k(c(L), R\tau^{\{o\}}_{\le p-1} j_*\pi^*\mathcal{F}^\bullet) = \mathbb{H}^k(\{o\}, \tau_{\le p-1} i^* Rj_*\pi^*\mathcal{F}^\bullet)$$

and we readily identify the right hand side with $\mathbb{H}^k(L, R\tau_{\le p-1}\mathcal{F}^\bullet)$ and this is $\mathbb{H}^k(L, \mathcal{F}^\bullet)$ for $k \le p-1$ and 0 for $k \ge p$. Similarly the long exact hypercohomology sequence associated to the second exact sequence of Lemma 1.2 gives with the input of Lemma 3.1 that

$$\mathbb{H}^k_c(c(L), R\tau^{\{o\}}_{\le p-1} j_*\pi^*\mathcal{F}^\bullet) = \mathbb{H}^{k-1}(\{o\}, \tau_{\ge p} i^* Rj_*\pi^*\mathcal{F}^\bullet).$$

The right hand side is $\mathbb{H}^{k-1}(L, R\tau_{\ge p}\mathcal{F}^\bullet)$ and for $k \ge p+1$ this is equal to $\mathbb{H}^{k-1}(L, \mathcal{F}^\bullet)$ and zero for $k \le p$. $\square$

3.2. The intersection complex. We continue with our conelike stratified space $(X,\mathscr{X})$. We say that a partitioned space $(Y,\mathscr{Y})$ is 'well-behaved at infinity' if there exists a compact $K\subset Y$ such that the restriction of $\mathscr{Y}$ to the pair $(Y\smallsetminus \mathring{K},\partial K)$ makes the latter is homeomorphic as a partitioned space with the product of partioned spaces $\mathbb{R}_{\ge 0}\times\partial K$, where $\mathbb{R}_{\ge 0}$ is partitioned into $\{0\}$ and $\mathbb{R}_{>0}$ and ∂K has the partition induced by $\mathscr{Y}$.

PROPOSITION 3.3. *Let $\mathbb{E}$ and $\mathbb{F}$ be local systems of finite dimensional $\mathbf{k}$-vector spaces on $\mathring{X}$ and $\psi:\mathbb{E}\otimes_{\mathbf{k}}\mathbb{F}\to o_{\mathring{X}}\otimes\mathbf{k}$ a pairing. Let p and q be maps $\mathcal{X}\to\mathbb{Z}$ such that $p_{\mathring{X}}=p_{\mathring{X}}=0$ and $p_Y+q_Y=\operatorname{codim} Y$. Then we have for every open subset $U\subset X$ a $\mathbf{k}$-linear pairing*

$$\mathbb{H}^k(U,{}^pj_{!*}\mathbb{E})\otimes_{\mathbf{k}}\mathbb{H}_c^{\dim X-k}(U,{}^qj_{!*}\mathbb{F}))\to\mathbf{k}$$

which extends the one on $\mathring{X}$. This pairing is functorial in U and is nondegenerate when ψ is and $(U,\mathscr{X}|U)$ is well-behaved at infinity.

PROOF. We begin with noting that an injective resolutions of $\mathbb{E}$ and $\mathbb{F}$ will do the job on $\mathring{X}$ by ordinary duality discussed above (for $p_{\mathring{X}}=q_{\mathring{X}}=0$). Let $i_Y:Y\subset X$ be a closed stratum and assume we have established the assertions of the proposition on $X\smallsetminus Y$. So if we write $\mathcal{E}^\bullet:={}^pj_{!*}\mathbb{E}|X\smallsetminus Y$ and $\mathcal{F}^\bullet:={}^pj_{!*}\mathbb{F}|X\smallsetminus Y$ and put $j':=X\smallsetminus Y\subset X$, then ${}^pj_{!*}\mathbb{E}=R\tau^Y_{\le p_Y-1}\mathcal{E}^\bullet$ and ${}^pj_{!*}\mathbb{F}=R\tau^Y_{\le q_Y-1}\mathcal{F}^\bullet$.

Let $r=\operatorname{codim} Y$, and let U be an $\mathscr{X}$-chart at a point of Y such that $U\cap Y\cong\mathbb{R}^{d-r}$ and hence $U\cong c(L,\mathcal{L})\times\mathbb{R}^{d-r}$ and $U\smallsetminus Y\cong(0,1)\times L\times\mathbb{R}^{d-r}$. Let $j_L:\mathring{L}\subset L$ and $\mathbb{E}_L$ (a local system on $\mathring{L}$) have the obvious meaning. Now $\mathbb{E}|U\cap\mathring{X}$ is the sheaf preimage of $\mathbb{E}_L$ under the projection $U\cap\mathring{X}\cong(0,1)\times\mathring{L}\times\mathbb{R}^d\to\mathring{L}$ and $\mathcal{E}^\bullet|U\smallsetminus Y$ is an injective resolution of the sheaf preimage of ${}^pj_{L!*}\mathbb{E}_L$ under the projection $U\smallsetminus Y\cong(0,1)\times L\times\mathbb{R}^d\to L$. The Künneth theorem combined with Corollary 3.2 the yields identifications

$$\mathbb{H}^k(U,{}^pj_{!*}\mathbb{E})\cong\begin{cases}\mathbb{H}^k(L,{}^pj_{L!*}\mathbb{E}_L)\text{ for }k\le p_Y-1;\\ 0\text{ for }k\ge p_Y.\end{cases}$$

If we apply the Künneth theorem for hypercohomology with compact supports, the factor $\mathbb{R}^{d-r}$ accounts for a degree shift over $d-r$ and so the corresponding argument gives

$$\mathbb{H}_c^{d-k}(U,\mathbb{F})\cong\begin{cases}\mathbb{H}^{r-k}(L,{}^qj_{L!*}\mathbb{F}_L)\text{ for }r-l\ge q_Y+1;\\ 0\text{ for }r-k\le q_Y.\end{cases}$$

The product decomposition also determines a pairing $\psi_L:\mathbb{E}_L\otimes_{\mathbf{k}}\mathbb{F}_L\to o_{\mathring{L}}\otimes\mathbf{k}$. Since the perversity functions p and q for the induced stratification of L still add up to the codimension function, our inductive set-up implies that we have an induced pairing

$$\mathbb{H}^k(L,{}^pj_{L!*}\mathbb{E}_L)\otimes\mathbb{H}^{r-k}(L,{}^qj_{L!*}\mathbb{F}_L)\to\mathbf{k}$$

which is nondegenerate when ψ_L is. Hence the same is true for $\psi|U\cap \mathring{X}$: we get a pairing

$$\mathbb{H}^k(U, {}^p j_{!*}\mathbb{E}) \otimes_{\mathbf{k}} \mathbb{H}_c^{d-k}(U, {}^q j_{!*}\mathbb{F})) \to \mathbf{k}$$

which is nondegenerate when ψ_L is. This establishes the duality property for an $\mathscr{X}$-chart. A Čech-like (or iterated Mayer-Vietoris) argument then takes care of the general case (we omit the slightly technical issues associated to proving that an open subset on which $\mathscr{X}$ is well-behaved at infinity admits a finite covering with the property that every nonempty intersection is an $\mathscr{X}$-chart). □

We now assume that the dimension of all strata have the same parity (so that every stratum has even codimension). Note that any link of such a stratified space will then have the same property. An important example is a stratified space in the complex analytic setting. We focus on what is called the *middle perversity*, $Y \in \mathscr{X} \mapsto \frac{1}{2}\operatorname{codim}(Y)$ and denote ${}^{\operatorname{codim}_{\mathbb{R}}/2} j_{!*}\mathbb{E}$ as constructed in Proposition 2.8 by $\mathscr{I\!C}^\bullet_X(\mathbb{E})$ (we put in the subscript $_{\mathbb{R}}$ in $\operatorname{codim}_{\mathbb{R}}(Y)$, so that we are free to use codim for complex codimension). Note that this perversity function is self-dual. For $U \subset X$ open, we will abbreviate

$$IH^k(U, \mathbb{E}) := \mathbb{H}^k(U, \mathscr{I\!C}^\bullet_X(\mathbb{E})), \quad IH^k_c(U, \mathbb{E}) := \mathbb{H}^k_c(U, \mathscr{I\!C}^\bullet_X(\mathbb{E}))$$

and call it the *intersection cohomology of U with values in* $\mathbb{E}$. The following theorem simply spells out Propositions 2.8 and 3.3 for the middle perversity.

THEOREM 3.4 (The intersection complex). *The complex $\mathscr{I\!C}^\bullet_X(\mathbb{E})$ is an $\mathscr{X}$-constructible sheaf complex that comes endowed with natural maps $Rj_!\mathbb{E} \to \mathscr{I\!C}^\bullet_X(\mathbb{E}) \to Rj_*\mathbb{E}$ in $D^+(X)$ that extend the identity on $\mathring{X}$. It is functorial in $\mathbb{E}$ and is such that for any stratum Y of $\mathscr{X}$ other than $\mathring{X}$, the following two vanishing conditions hold:*

(i) $\mathbb{R}^k i_Y^* \mathscr{I\!C}^\bullet_X(\mathbb{E}) = 0$ *for* $k \geq \frac{1}{2}\operatorname{codim} Y$ *and*

(ii) $\mathbb{R}^k i_Y^! \mathscr{I\!C}^\bullet_X(\mathbb{E}) = 0$ *for* $k \leq \frac{1}{2}\operatorname{codim} Y$.

Moreover, if $\mathbb{E}$ and $\mathbb{F}$ are local systems of finite dimensional $\mathbf{k}$-vector spaces on $\mathring{X}$, then a pairing $\psi : \mathbb{E} \otimes_{\mathbf{k}} \mathbb{F} \to o_{\mathring{X}} \otimes \mathbf{k}$ determines for every open subset $U \subset X$ a pairing

$$IH^k(U, \mathbb{E}) \otimes_{\mathbf{k}} IH^{d-k}_c(U, \mathbb{F}) \to \mathbf{k}$$

which extends the duality map on $\mathring{X}$. This map is functorial in U and is nondegenerate when ψ is and $(U, \mathscr{X}|U)$ is well-behaved at infinity.

REMARKS 3.5. It is clear that any sheaf complex quasi-isomorphic to $\mathscr{I\!C}^\bullet_X(\mathbb{E})$ will have the same properties. (We shall see shortly that the sheaf complex $\mathscr{I\!C}^\bullet_X(\mathbb{E})$ is unique up to quasi-isomorphism.) So by passing to an injective resolution of $\mathscr{I\!C}^\bullet_X(\mathbb{E})$, we may assume that $\mathscr{IC}^\bullet_X(\mathbb{E})$ is an injective complex. Then $\mathscr{IC}^\bullet_X(\mathbb{E})$ is an injective resolution of $\mathbb{E}$ and the properties (i) and (ii) amount to vanishing statements regarding the cohomology sheaves of $i_Y^*\mathscr{IC}^\bullet_X(\mathbb{E})$ and $i_Y^!\mathscr{IC}^\bullet_X(\mathbb{E})$. For $p \in X$, denote by $i_p : \{p\} \subset X$ the inclusion. Property (ii) implies (and is in fact equivalent to) the $\mathbf{k}$-vector spaces $\mathbb{R}^k i_{p*}\mathscr{IC}^\bullet_X(\mathbb{E}) = H^k(\mathscr{IC}^\bullet_X(\mathbb{E})_p)$ and $\mathbb{R}^{d-k} i^!_{p*}\mathscr{IC}^\bullet_X(o_X \otimes \mathbb{E}^\vee)$ being each others dual.

COROLLARY 3.6. *The intersection complex $\mathscr{IC}^\bullet_X(\mathbb{E})$ satisfies the following* support conditions*: for every $k > 0$ the set of $p \in X$ for which $R^k i_p^*\mathscr{IC}^\bullet_X(\mathbb{E}))$ or $\mathbb{R}^{d-k} i_p^!\mathscr{IC}^\bullet_X(\mathbb{E})$ is nonzero is a union of strata of $\mathscr{X}$ of dimension $< d - 2k$. Moreover, for a locally closed submanifold $Z \subset X$ of a stratum of $\mathscr{X}$ that is not open in X, we have*

(i$_Z$) $\mathbb{R}^k i_Z^*\mathscr{IC}^\bullet_X(\mathbb{E}) = 0$ *for* $k \geq \lfloor \frac{1}{2}\operatorname{codim} Z \rfloor$ *and*
(ii$_Z$) $\mathbb{R}^k i_Z^!\mathscr{IC}^\bullet_X(\mathbb{E}) = 0$ *for* $k \leq \lceil \frac{1}{2}\operatorname{codim} Z \rceil$.

PROOF. Clearly, $\mathbb{R}^k i_{p*}\mathscr{IC}^\bullet_X(\mathbb{E})$ is zero for $p \in \mathring{X}$. For $p \notin \mathring{X}$, we have $\mathbb{R}^k i_{p*}\mathscr{IC}^\bullet_X(\mathbb{E}) \cong IH^k(U, \mathbb{E})$ for a chart domain U as above, and this vanishes in degree $\geq \frac{1}{2}\operatorname{codim} S_p$. In other words, $k < \frac{1}{2}\operatorname{codim} S_p$. Hence the locus of p for which $\mathbb{R}^k i_{p*}\mathscr{IC}^\bullet_X(\mathbb{E}) \neq 0$ is contained in the union of the strata of codimension $> 2k$, hence has dimension $< d - 2k$. This implies the support condition. The cosupport condition is verified similarly.

To prove the last statement, let Y be the member of $\mathscr{X}$ which contains Z and denote by $i_{Z/Y} : Z \subset Y$ the inclusion so that $i_Z = i_Y i_{Z/Y}$. Write r resp. s for the codimension of Y resp. Z in X (so that $s \geq r$). If $\mathbb{F}$ is a local system on Y, then $R^k i^*_{Z/Y}\mathbb{F}$ resp. $R^k i^!_{Z/Y}\mathbb{F}$ is zero unless $k = 0$ resp. $k = s - r$. The sheaves $\mathbb{R}^k i_Y^*\mathscr{IC}^\bullet_X(\mathbb{E})$ and $\mathbb{R}^k i_Y^!\mathscr{IC}^\bullet_X(\mathbb{E})$ are locally constant and hence the Grothendieck spectral sequences for $i_Z^* = i^*_{Z/Y} i_Y^*$ and $i_Z^! = i^!_{Z/Y} i_Y^!$ yield an isomorphism $\mathbb{R}^k i_Z^*\mathscr{IC}^\bullet_X(\mathbb{E}) = i^*_{Z/Y}\mathbb{R}^k i_Y^*\mathscr{IC}^\bullet_X(\mathbb{E})$ and $\mathbb{R}^k i_Z^!\mathscr{IC}^\bullet_X(\mathbb{E}) \cong \mathbb{R}^{s-r} i_{Z/Y!}\mathbb{R}^{k-s+r} i_{Y!}\mathscr{IC}^\bullet_X(\mathbb{E})$. The former vanishes for $k \geq r/2$ and *a fortiori* for $k \geq \lfloor s/2 \rfloor$ and the latter vanishes for $k - s + r \leq r/2$, i.e., when $k \leq s - r/2$. Since $s - r/2 \geq \lceil s/2 \rceil$, the asserted vanishing property holds. □

3.3. Uniqueness of the intersection complex. We use the last result to deduce a uniqueness property. It yields a characterization of $\mathscr{IC}^\bullet_X(\mathbb{E})$ as an object of a derived category.

COROLLARY 3.7. *Let $\mathscr{X}'$ be a stratification of X (not necessarily conelike or with only strata of even codimension) which refines $\mathscr{X}$. Let $\mathcal{I}^\bullet$ be an injective $\mathscr{X}'$-constructible bounded below complex of* $\mathbf{k}$*-vector spaces on X endowed with a quasi-isomorphism $\mathbb{E} \to j'^*\mathcal{I}^\bullet$. If $\mathcal{I}^\bullet$ satisfies the support conditions of Corollary 3.6 for every nonopen stratum stratum of $\mathscr{X}'$, then there exists a quasi-isomorphism $\mathscr{IC}^\bullet_X(\mathbb{E}) \to \mathcal{I}^\bullet$*

compatible with the two resolutions of $\mathbb{E}|\mathring{X}'$ *and this quasi-isomorphism is unique up to chain homotopy.*

PROOF. We proceed with induction: we may assume that there is a closed stratum Z of $\mathscr{X}'$ such that $j_Z^*\mathcal{I}^\bullet = j_Z^*\mathscr{IC}^\bullet_X(\mathbb{E})$, where $j_Z : X \smallsetminus Z \subset Z$. Both $\mathcal{I}^\bullet$ and $\mathscr{IC}^\bullet_X(\mathbb{E})$ satisfy the hypotheses of Lemma 1.3 for $r = \lfloor \frac{1}{2}\operatorname{codim} Z \rfloor$; one by assumption and the other by Corollary 3.6. This lemma tells us that both are quasi-isomorphic to $\tau^Z_{\leq r-1} j_{Z*} j_Z^* \mathscr{IC}^\bullet_X(\mathbb{E})$. □

We have at our disposal 'relative' and 'support' versions for intersection cohomology: if $i : Y \subset X$ is closed and $j : X \smallsetminus Y \subset X$, then following our conventions for a general sheaf complex (in Subsection 1.5) we put $IH^\bullet(X, Y; \mathbb{E}) := \mathbb{H}^\bullet(X, j_! \mathscr{IC}^\bullet(\mathbb{E}))$ and $IH^\bullet_Y(X; \mathbb{E}) := \mathbb{H}^\bullet(Y, i^! \mathscr{IC}^\bullet(\mathbb{E}))$. They satisfy the excision properties described there and they fit in exact sequences

$$\cdots \to IH^k(X, Y; \mathbb{E}) \to IH^k(X, \mathbb{E}) \to \mathbb{H}^k(Y, i^*\mathscr{IC}^\bullet_X(\mathbb{E})) \to IH^{k+1}(X, Y; \mathbb{E}) \to \cdots$$

$$\cdots \to IH^k_Y(X; \mathbb{E}) \to IH^k(X, \mathbb{E}) \to IH^k(X \smallsetminus Y, \mathbb{E}) \to IH^{k+1}_Y(X; \mathbb{E}) \to \cdots .$$

If Y is a reasonable subspace of X, then $\mathbb{H}^k(Y, i^*\mathscr{IC}^\bullet_X(\mathbb{E}))$ should be interpreted as the intersection cohomology of a regular neighborhood U_Y of Y with values in $\mathbb{E}$. But even if $\mathbb{E} = \mathbf{k}$ and Y is a union of strata (so that $IH^k(Y, \mathbf{k})$ is defined), this need not be identifyable with $\mathbb{H}^k(Y, i^*\mathscr{IC}^\bullet_X(\mathbf{k}))$. For this to be so, the singularities of X in Y should not be substantially worse than the singularities of Y and $\mathbb{E}$ must already be defined on an open-dense subset of Y (beware though that this only applies to $IH^\bullet$ and not to $IH^\bullet_c$). In this situation we have a Thom isomorphism and a Gysin map, as explained below.

3.4. The Gysin map. We begin with the following observation. Let $\mathcal{F}^\bullet$ be a sheaf complex on a space Y of finitely generated $\mathbf{k}_Y$-modules and let $\mathbb{Z}_Y \to \mathcal{I}^\bullet$ be an injective resolution of the constant sheaf. Then $\mathcal{F}^\bullet \to \mathcal{I}^\bullet \otimes \mathcal{F}^\bullet$ is an injective resolution by $\mathbf{k}_Y$-modules and we have a chain map

$$\Gamma(Y, \mathcal{I}^\bullet) \otimes \Gamma(Y, \mathcal{F}^\bullet) \to \Gamma(Y, \mathcal{I}^\bullet \otimes \mathcal{F}^\bullet).$$

If $\mathcal{F}^\bullet$ is injective, then this defines a graded map

$$H^\bullet(Y) \otimes \mathbb{H}^\bullet(Y, \mathcal{F}^\bullet) \to \mathbb{H}^\bullet(Y, \mathcal{F}^\bullet)$$

that gives $\mathbb{H}^\bullet(Y, \mathcal{F}^\bullet)$ the structure of a $H^\bullet(Y)$-module. If $\mathcal{F}^\bullet$ is not injective, then we replace it by an injective resolution of $\mathbf{k}$-modules and the resulting module structure is independent of this choice. This module structure is also natural in the sense that a homomorphism of sheaf complexes $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of $\mathbf{k}$-modules induces a map $\mathbb{H}^\bullet(Y, \mathcal{F}^\bullet) \to \mathbb{H}^\bullet(Y, \mathcal{G}^\bullet)$ of $H^\bullet(Y)$-modules. This structure enters in a version of the Thom isomorphism.

PROPOSITION 3.8 (Gysin map). *Let $(X, \mathscr{X})$ be a conelike stratified space and $i : Y \subset X$ a closed subspace of codimension r transversal to $\mathscr{X}$ in the sense that we can cover Y by open subsets U such that $(U, \mathscr{X}|U) \cong \mathbb{R}^r \times (U \cap Y, \mathscr{X}|U \cap Y)$. Then $o_{Y/X} := R^\bullet i^! \mathbb{Z}_X$ is a local system of rank one concentrated in degree r.*

If $\mathscr{X}$ has only strata of even codimension and $\mathbb{E}$ is a local system on its union $\mathring{X}$ of open-dense strata, then we have a natural (Thom) *isomorphism $IH^{\bullet-r}(Y; o_{Y/X} \otimes i^*\mathbb{E}) \cong IH^\bullet_Y(X; \mathbb{E})$. The associated* Gysin map

$$IH^{\bullet-r}(Y; o_{Y/X} \otimes i^*\mathbb{E}) \cong IH^\bullet_Y(X; \mathbb{E}) \to IH^\bullet(X; \mathbb{E})$$

has the property that if $\mathbb{E}$ is a local system of finite dimensional **k***-vector spaces and $(X, \mathscr{X})$ is of pure dimension and well-behaved at infinity (e.g., compact), then it is the dual of $i^* : IH^\bullet_c(X; o_X \otimes \mathbb{E}^\vee) \to IH^\bullet_c(Y; i^*(o_X \otimes \mathbb{E}^\vee))$.*

PROOF. For U as in the lemma, the Künneth theorem gives us an isomorphism $IH^\bullet_{U\cap Y}(U; \mathbb{E}) \cong H^\bullet_{\{0\}}(\mathbb{R}^r) \otimes IH^\bullet(U \cap Y, i^*\mathbb{E})$. Since $H^\bullet_{\{0\}}(\mathbb{R}^r)$ is infinite cyclic and concentrated in degree r we get an isomorphism $IH^\bullet_{U\cap Y}(U; \mathbb{E}) \cong IH^\bullet(U \cap Y; H^\bullet_{\{0\}}(\mathbb{R}^r) \otimes i^*\mathbb{E})$. It follows that $R^\bullet i^! \mathbb{Z}_X$ is a local system of rank one and that $\mathscr{IC}^{\bullet-r}_Y(o_{Y/X} \otimes i^*\mathbb{E})$ may serve as a representative of $Ri^! \mathscr{IC}^\bullet_X(\mathbb{E})$.

We thus obtain an isomorphism $IH^\bullet_Y(X; \mathbb{E}) \cong IH^{\bullet-r}(Y; o_{Y/X} \otimes i^*\mathbb{E})$ as asserted.

In the situation of the second paragraph of the lemma, we observe that the dual of the Gysin map $IH^{k-r}(Y; o_{Y/X} \otimes i^*\mathbb{E}) \to IH^k(X; \mathbb{E})$ is the form

$$IH^{\dim X-k}_c(X; o_X \otimes \mathbb{E}^\vee) \to IH^{\dim Y+r-k}(Y; o_Y \otimes o^\vee_{Y/X} \otimes i^*\mathbb{E}^\vee)$$

Since we have a natural identification $o_Y \otimes o^\vee_{Y/X} \cong o_X|\mathring{Y}$ and $\dim X = \dim Y + r$ this can indeed be regarded a map $IH^{\dim X-k}_c(X; o_X \otimes \mathbb{E}^\vee) \to IH^{\dim X-k}(Y; i^*(o_X \otimes \mathbb{E})^\vee)$. For $Y \subset X$ replaced by $Y \cap U \subset U$ as above, this is easily checked to be given by restriction. The general case then follows with a Mayer-Vietoris argument. $\square$

3.5. The IC complex in the algebraic and analytic setting. The preceding shows at least that the quasi-isomorphism type of our intersection complex can be phrased entirely in vanishing conditions, provided that the the stratification refines a conelike one. But in the complex-analytic or complex-algebraic setting this assumption is moot, as any two Thom-Whitney stratifications admit a common refinement (Theorem 2.9).

> **Nota Bene:** *Since we are now returning to the analytic/algebraic situation,* $\dim$ *and* codim *means* complex *(co)dimension.*

In this setting a natural class of local systems is encountered as follows. Let X be a projective variety, $\mathring{X} \subset X$ a Zariski open-dense subset and $f : Z \to \mathring{X}$ a projective and smooth morphism of relative dimension d. Then for $k \leq d$, the direct cohomology sheaf $R^k f_* \mathbb{Q}_Z$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces. We say that $\mathbb{E}$ *is of geometric origin* if it appears as a direct summand of such a system (a theorem of Deligne asserts that such an $\mathbb{E}$ is semisimple: any local subsystem is a direct summand), or is obtained from such summands by means of tensor products and direct sums. We will find that these comprise essentially all the local systems we run into.

For an analytic variety X we denote by $D_{\mathrm{an}}(X)$, $D^b_{\mathrm{an}}(X)$, $D^{\pm}_{\mathrm{an}}(X)$ the derived category of analytically constructible of sheaves with the appropriate bounded condition. For its complex-algebraic analogue we replace the subscript $_{\mathrm{an}}$ by a simple $_c$: $D_c(X, \mathbf{k})$ etc.; if consider constructible sheaves of $\mathbf{k}$-vector spaces instead, then we will write $D^b_{\mathrm{an}}(X, \mathbf{k})$ etc.

COROLLARY 3.9. *Let X be an irreducible complex-analytic variety and $\mathbb{E}$ a locally constant sheaf of finite rank over a Zariski open-dense subset $\mathring{X} \subset X$. Then the image of $\mathbb{E}$ in $D^b_{\mathrm{an}}(\mathring{X})$ extends to a unique object $\mathscr{IC}^\bullet_X(\mathbb{E})$ of $D^b_{\mathrm{an}}(\mathring{X})$ with the property that for every irreducible closed subvariety Z, $i_Z : Z \subset X$:*

Support condition: $\mathbb{R}^k i_Z^* \mathscr{IC}^\bullet_X(\mathbb{E})$ *vanishes on a Zariski open-dense subset of Z for $k \geq \operatorname{codim} Z$ (unless $Z = X$ and $k = 0$),*

Cosupport condition: $\mathbb{R}^k i^!_Z \mathscr{IC}^\bullet_X(\mathbb{E})$ *vanishes on a Zariski open-dense subset of Z for $k \leq \operatorname{codim} Z$ (unless $Z = X$ and $k = 0$).*

The same is true relative to the field $\mathbf{k}$: $\mathbb{E}$ is then given as a local system of finite dimensional $\mathbf{k}$ vector spaces and determines an object of $D^b_{\mathrm{an}}(\mathring{X}; \mathbf{k})$.

PROOF. Given a Thom-Whitney stratification $\mathscr{X}$, then by Corollary 3.7 a complex $\mathscr{IC}^\bullet_X(\mathbb{E})$ has been constructed with the stated vanishing properties. Since any two such stratifications have a common refinement, the uniqueness also follows from 3.7. $\square$

The importance of the following consequence will become apparent when we discuss the abelian category of perverse sheaves. If $C^\bullet$ is a complex in an abelian category, then for any integer $n \in \mathbb{Z}$, $C[n]^\bullet$ denotes the same complex shifted n units to the left: $C[n]^k := C^{n+k}$.

We will make use of the following fact: if X and Y are closed subsets of a space W and $\mathcal{F}$ is a sheaf on X, then it is immediate from the definition that $i^!_Y i_{X*} \mathcal{F}$ is the direct image of $i^!_{X \cap Y/X} \mathcal{F}$ on Y. Since the direct image under a closed embedding is exact, the same is true for a bounded below complex $\mathcal{F}^\bullet$ on X: $\mathbb{R}^k i^!_Y i_{X*} \mathcal{F}^\bullet$ is the direct image of $\mathbb{R}^k i^!_{X \cap Y/X} \mathcal{F}^\bullet$ on Y.

COROLLARY 3.10. *Assume that in the situation of Corollary 3.9, X is a closed analytic subset of an analytic variety W via $i_X : X \subset W$.*

Then $\mathcal{P}^\bullet := i_{X*}\mathscr{IC}^\bullet_X\mathbb{E})[\dim X]$ satisfies the following (so-called) perversity *conditions: for every irreducible closed subset $i_Y : Y \subset W$, $\mathbb{R}^k i_Y^*\mathcal{P}^\bullet$ resp. $\mathbb{R}^k i_Y^!\mathcal{P}^\bullet$ vanishes on a Zariski open-dense subset of Y for $k \geq -\dim Y$, resp. $k \leq -\dim Y$, unless $Y = X$, in which these inequalities hold only strictly.*

PROOF. By the above remarks, $\mathbb{R}^k i_Y^*\mathcal{P}^\bullet$ resp. $\mathbb{R}^k i_Y^!\mathcal{P}^\bullet$ is the direct image of $\mathbb{R}^k i_{X\cap Y/X}^*\mathcal{P}^\bullet$ resp. $\mathbb{R}^k i_{X\cap Y/X}^!\mathcal{P}^\bullet$ on Y. Now apply Corollary 3.9. □

So a finite direct sum $\oplus_\alpha i_{X_\alpha/W*}\mathscr{IC}^\bullet_{X_\alpha}\mathbb{E}[\dim X_\alpha]$ of such sheaf complexes also satisfies these perversity conditions. Its algebraic counterpart can be stated somewhat more concisely using the Zariski topology (but comes with the same proof):

THEOREM 3.11. *Let X be an irreducible complex algebraic variety with generic point ξ and let $\mathbb{E} \in D_c^b(\xi)$. Then there exists a unique extension $\mathscr{IC}^\bullet_X(\mathbb{E}) \in D^b(X)$ of $\mathbb{E}$ with the property that for every (possibly nonclosed) point η of X, $i_\eta : \{\eta\} \subset X$, $R^k i_\eta^*\mathscr{IC}^\bullet_X(\mathbb{E})$ for $k \geq \operatorname{codim}\eta$ (unless η is the generic point of X and $k = 0$) and $R^k i_\eta^!\mathscr{IC}^\bullet_X(\mathbb{E})$ for $k \leq \operatorname{codim}\eta$ (unless $k = \dim X$ and η is a closed point). The formation of $\mathscr{IC}^\bullet_X(\mathbb{E})$ is functorial in $\mathbb{E}$ and its image in $D_c^b(X, \mathbf{k})$ satisfies the duality property for every open subset of X of finite type.*

If X is a closed subset of a variety W via $i_X : X \subset W$, then $\mathcal{P}^\bullet := i_{X*}\mathscr{IC}^\bullet_X\mathbb{E})[\dim X]$ satisfies the perversity conditions: for every $\eta \in W$, we have $\mathbb{R}^k i_\eta^*\mathcal{P}^\bullet = 0$ for $k \geq -\dim\eta$ and $\mathbb{R}^k i_\eta^!\mathcal{P}^\bullet = 0$ for $k \leq -\dim\eta$ unless $Y = X$, in which these inequalities hold only strictly.

REMARK 3.12. This formulation makes sense in any cohomology theory for algebraic varieties over an algebraically closed field for which we have a notion of a local system (and hence that of a constructible sheaf), and functors i^* and $i^!$. Étale cohomology is such a theory and indeed, the analogue of this theorem holds there [2]. In this setting the orientation sheaf is responsible for a 'Tate twist' (whose characteristic zero analogue will be discussed later).

EXAMPLE 3.13. Let $Y \subset \mathbb{P}^N(\mathbb{C})$ be a nonsingular projective variety of dimension $n-1$ and let $X \subset \mathbb{C}^{N+1}$ be the associated affine cone of dimension n. We compute $IH^\bullet(X;\mathbb{Q})$. Then $\mathring{X} := X \smallsetminus \{0\}$ is a $\mathbb{C}^\times$-bundle over Y. If $\eta_Y \in H^2(Y;\mathbb{Q})$ denotes the hyperplane class, then the Gysin exact sequence is

$$\cdots \to H^{k-2}(Y;\mathbb{Q}) \xrightarrow{\eta_Y\cup} H^k(Y;\mathbb{Q}) \to H^k(\mathring{X};\mathbb{Q}) \to \\ \to H^{k-1}(Y;\mathbb{Q}) \xrightarrow{\eta_Y\cup} H^{k+1}(Y;\mathbb{Q}) \to \cdots$$

combined with the hard Lefschetz theorem shows that for $k \leq n-1$, $H^k(\mathring{X};\mathbb{Q})$ is just the primitive cohomology of Y in that degree: $P^k(Y)$. Now note that X has the structure of an open cone (with basis the intersection of X with the unit sphere of $\mathbb{C}^{N+1}$). Since $IH^k(X;\mathbb{Q})$ is zero in degree $\geq n$, it follows that $IH^\bullet(X;\mathbb{Q}) = \oplus_{k=0}^{n-1} P^k(Y)$. Similarly, $IH_c^k(X;\mathbb{Q})$ is zero for $k \leq n$. For $k > n$ it is equal to $H_c^k(\mathring{X};\mathbb{Q})$ and using duality on $\mathring{X}$ we can identify this with $H_{2n-k}(\mathring{X};\mathbb{Q})$, which is just the dual of $P^k(Y)$.

The following proposition furnishes another class of examples.

PROPOSITION 3.14. *Let X be a compact irreducible analytic variety of dimension n whose singular set is finite. Then in the sequence*

$$H_c^k(X_{reg};\mathbb{Q}) \to IH^k(X,\mathbb{Q}) \to H^k(X_{reg};\mathbb{Q})$$

the second map is an isomorphism for $k < n$ and injective for $k = n$, whereas the first map is an isomorphism for $k > n$ and surjective for $k = n$ (so that $IH^n(X;\mathbb{Q})$ can be identified with the image of $H_c^n(X_{reg};\mathbb{Q}) \to H^n(X_{reg};\mathbb{Q})$). In terms of these isomorphisms, the duality property is inherited by Poincaré duality on X_{reg} (which identifies $H^k(X_{reg};\mathbb{Q})$ with $H_c^{2n-k}(X_{reg};\mathbb{Q})^\vee$).

PROOF. We may assume that $n > 0$. If $j : X_{reg} \subset X$ is the inclusion and $i : X_{sing} \subset X$ its complement, then we have corresponding maps

$$Rj_!\mathbb{Q}_{X_{reg}} \subset R\tau_{\leq n-1}^{X_{sing}} j_*\mathbb{Q}_{X_{reg}} \subset Rj_*\mathbb{Q}_{X_{reg}}.$$

The cokernel of the first inclusion is $i_*\tau_{\leq n-1}i^*Rj_*\mathbb{Q}_{X_{reg}}$ and of the second inclusion $i_*\tau_{\geq n}i^*Rj_*\mathbb{Q}_{X_{reg}}$. Since X_{sing} is finite, it has no cohomology in nonzero degree, and hence $\mathbb{H}^k(X_{sing}, \tau_{\leq n-1}i^*Rj_*\mathbb{Q}_{X_{reg}}) = H^k(\tau_{\leq n-1}\Gamma(X_{sing}, i^*Rj_*\mathbb{Q}_{X_{reg}}))$ and this is zero for $k > 0$ and hence also for $k \geq n$. It follows that $H_c^k(X_{reg};\mathbb{Q}) \to IH^k(X,\mathbb{Q})$ is an isomorphism for $k < n$ and injective for $k = n$. The other assertion is obtained in a similar (but easier) fashion, for $\mathbb{H}^k(X_{sing}, \tau_{\geq n}i^*Rj_*\mathbb{Q}_{X_{reg}}) = 0$ for $k \leq n-1$ always. ☐

3.6. The case of a curve. In a sense this is a continuation of Example 1.1. Let X be a nonsingular (connected) projective curve, and $j : U \subset X$ the complement of a finite subset $S \subset X$. Suppose we are also given a universal cover $\pi : \tilde{U} \to U$ with covering group Γ and a finite dimensional $\mathbf{k}$-representation of Γ: $\rho : \Gamma \to \mathrm{GL}(E)$. This defines a local system $\mathbb{E}$ on U (as the quotient of the trivial local system $E \times \tilde{U} \to \tilde{U}$ by the diagonal action of Γ) and any local system of finite dimensional $\mathbf{k}$-vector spaces on U so arises.

LEMMA 3.15. *We then have $\mathscr{IC}_X^\bullet(\mathbb{E}) = \tau_{\leq 0}Rj_*\mathbb{E} = j_*\mathbb{E}$ and*

$$IH^k(X,\mathbb{E}) = H^k(X, j_*\mathbb{E}) \cong \begin{cases} \text{the space of } \Gamma\text{-invariants } E^\Gamma, \text{ for } k = 0, \\ \mathrm{Im}(H_c^1(U,\mathbb{E}) \to H^1(U,\mathbb{E})) \text{ for } k = 1, \\ \text{the space } \Gamma\text{-covariants } E_\Gamma \text{ for } k = 2, \end{cases}$$

where in the last case, we used the complex orientation of X (for other values of k we get zero).

PROOF. We know that $H^0(X, j_*\mathbb{E}) = H^0(U, \mathbb{E}) \cong E^\Gamma$. The natural map $j_!\mathbb{E} \to j_*\mathbb{E}$ is injective with kernel a sheaf with finite support. The associated long exact sequence then shows that $H^1(X, j_!\mathbb{E}) \to H^1(X, j_*\mathbb{E})$ is onto and $H^2(X, j_!\mathbb{E}) \to H^2(X, j_*\mathbb{E})$ is an isomorphism. But $H^i(X, j_!\mathbb{E}) = \mathbb{H}^i(X, Rj_!\mathbb{E}) = H^i_c(U, \mathbb{E})$. We know that $H^2_c(U, \mathbb{E})$ can be identified with E_Γ. The natural map $j_*\mathbb{E} \to Rj_*\mathbb{E}$ is injective with cokernel a sheaf complex that is quasi-isomorphic to the skyscraper sheaf $R^1 j_*\mathbb{E}$ placed in degree 1. This implies that $H^1(X, j_*\mathbb{E}) \to \mathbb{H}^1(X, Rj_*\mathbb{E})$ is injective. Since we can identify $\mathbb{H}^1(X, Rj_*\mathbb{E})$ with $H^1(U, \mathbb{E})$, it follows that $H^1(X, j_*\mathbb{E})$ can be identified with the image of $H^1_c(U, \mathbb{E}) \to H^1(U, \mathbb{E})$, as asserted. □

The image of $H^1_c(U; \mathbb{E}) \to H^1(U; \mathbb{E})$ can be computed by means of Example 1.1. It is perhaps no surprise that the image of $H^1_c(U; \mathbb{E}) \to H^1(U; \mathbb{E})$ appeared in the mathematical literature before the advent of intersection cohomology. An instance is that in the Eichler-Shimura isomorphism in the theory of automorphic forms [9].

We now drop the assumption that the complex-analytic variety X be irreducible, but still assume that we are given Zariski-open dense subset $\mathring{X}$. If $\{X_\alpha\}_\alpha$ is the collection of irreducible components of X, then $\{\mathring{X}_\alpha := \mathring{X} \cap X_\alpha\}_\alpha$ is the collection of connected components of $\mathring{X}$ (which can be of varying dimension), and we can then also allow for a local system $\mathbb{E}$ on $\mathring{X}$ of finite dimensional $\mathbf{k}$-vector spaces, $\mathbb{E}_\alpha := \mathbb{E}|\mathring{X}_\alpha$ to have different rank for different α. We then pass to the normalization $\pi : \hat{X} \to X$. This is a finite morphism with connected components the normalizations $\hat{X}_\alpha$ of X_α (so irreducible). Then $\mathscr{IC}^\bullet_{\hat{X}}(\pi^*\mathbb{E})$ is well-defined and we put $\mathscr{IC}^\bullet_X(\mathbb{E}) := \pi_*\mathscr{IC}^\bullet_{\hat{X}}(\pi^*\mathbb{E})$ (π_* is exact and so replacing π_* by $R\pi_*$ makes no difference). It is then clear that

$$\begin{aligned} IH^k(X; \mathbb{E}) := \mathbb{H}^k(X, \pi_*\mathscr{IC}^\bullet_{\hat{X}}(\pi^*\mathbb{E})) \cong \mathbb{H}^k(\hat{X}, \mathscr{IC}^\bullet_{\hat{X}}(\pi^*\mathbb{E})) = \\ = \textstyle\prod_\alpha \mathbb{H}^k(\hat{X}_\alpha, \mathscr{IC}^\bullet_{\hat{X}_\alpha}(\mathbb{E}_\alpha)) = \prod_\alpha IH^k(\hat{X}_\alpha; \mathbb{E}_\alpha). \end{aligned}$$

This is of course a valid definition only if we prove that for an irreducible X, normalization has no effect on its intersection cohomology. This is straightforward to check: choose compatible Thom-Whitney stratifications for both X and $\hat{X}$ and observe that a truncated extension on X is the direct image of a truncated extension on $\hat{X}$.

4. Weak Lefschetz theorems for singular varieties

4.1. Isolated singular points of analytic functions. We here assume given a (closed) analytic set X endowed with an analytic Thom-Whitney stratification $\mathscr{X}$, a complex-analytic function $f : X \to \mathbb{C}$ and an isolated critical point p of f relative to $\mathscr{X}$. We want to understand the topology of f near p; in fact we will show that f has there a model that looks a bit like the model we constructed for a quadratic singularity in Section 3.

For this we may always replace X by a neighborhood if p and assume that $f(p) = 0$. So we may as well assume that X is a closed analytic subset of an open subset U of some $\mathbb{C}^N$. We can also assume that $\{p\}$ is a stratum, for if S_p strictly contains $\{p\}$, then we can refine the stratification breaking S_p up into $\{p\}$ and $S \smallsetminus \{p\}$ and still have a Thom-Whitney stratification. By shrinking U we may then assume that any stratum Y distinct from $\{p\}$ has p in it closure and that $f|Y$ is a submersion. Then the restriction of $\mathscr{X}$ to $f^{-1}(0)$ is also a Thom-Whitney stratification. The function $r : X \to \mathbb{R}$, $z \in X \mapsto \|z - p\|^2$, will have p as isolated singular point and the same is true for its restriction to $f^{-1}(0)$. We now assume $\varepsilon > 0$ such that $r|f^{-1}(0)$ is proper over $[0, \varepsilon]$ and has no positive critical value relative to $\mathscr{X}|f^{-1}(0)$. We then choose $\eta > 0$ such that (f, r) has no critical value in $D_\eta \times \{\varepsilon\}$ (here D_η denotes the closed disk $|w| \leq \eta$ as usual) and put $D := D_\eta$, $B := B_\varepsilon(p) \cap f^{-1}D_\eta$ (where $B_\varepsilon(p)$ is the closed ε-ball centered at p), $\dot{B} := \partial B_\varepsilon(p) \cap f^{-1}D$ (the boundary of B relative to f) and we continue to denote by f the map $B \to D$ induced by f. The complete boundary ∂B of B is the union of $\dot{B}$ and $B_{\partial D}$ with the intersection of these two pieces is $\dot{B}_{\partial D}$. The stratification $\mathscr{X}$ induces a conelike stratification $\mathscr{B}$ of B for which $\dot{B}$, $B_{\partial D}$ and $\dot{B}_{\partial D}$ are unions of strata. We have taken care that now $(\dot{B}, \mathscr{B}|\dot{B})$ is locally trivial over D, $(B, \mathscr{B})$ is locally trivial over $D \smallsetminus \{0\}$ and $(B, \mathscr{B})$ is homeomorphic to the closed cone over $(\partial B, \mathscr{B}|\partial B)$, where $\partial B := \dot{B} \cup B_{\partial D}$. Notice that ∂B serves as a link of p in X. We shall refer to B as a *singularity box* for f at p

The stratified local triviality of $B_{\partial D}$ enables us to construct a relative homeomorphism $H : ([0, 2\pi], \{0, 2\pi\}) \times B_\eta \to (B_{\partial D}, B_\eta \cup \dot{B})$ that covers $\eta \in [0, 2\pi] \mapsto \eta e^{\sqrt{-1}\theta} \in \partial D$ with $H(0, z) = z$ and is compatible with a stratified trivialization of $\dot{B}_{\partial D} \to \partial D$ that extends to $\dot{B} \to D$. This implies that the geometric monodromy is one of stratified spaces: this is a map $h := H_{2\pi} : (B_\eta, \mathscr{B}|B_\eta) \to (B_\eta, \mathscr{B}|B_\eta)$ that is the identity on $\dot{B}_\eta$. Its stratified isotopy class determines all the topological information present here in the sense that we can reconstruct $f : (B, \mathscr{B}) \to D$ from this: first use h to reconstruct $B_{\partial D} \to \partial D$ so that $\dot{B}_{\partial D} \to \partial D$ has been trivialized (so identified with $\partial D \times \dot{B}_\eta$), then glue on $D \times \dot{B}_\eta$ to reconstruct $\partial f : \partial B \to D$. And finally, extend ∂f to the closed cone over ∂B by $(t, z) \mapsto t\partial f(z)$; this map is topologically equivalent to f.

Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathscr{X}$-constructible sheaves on X. Then h is covered by an isomorphism of complexes $h^*(\mathcal{F}^\bullet|B_\eta) \cong \mathcal{F}^\bullet|B_\eta$ that is the identity on ∂B_η. So it defines a variation map

$$\mathrm{var}(h) : \mathbb{H}^\bullet(\mathring{B}_\eta; \mathcal{F}^\bullet) \cong \mathbb{H}^\bullet(B_\eta; \mathcal{F}^\bullet) \to \mathbb{H}^\bullet(B_\eta, \partial B_\eta; \mathcal{F}^\bullet) \cong \mathbb{H}^\bullet_c(\mathring{B}_\eta; \mathcal{F}^\bullet),$$

where $\mathring{B}_\eta := B_\eta \setminus \dot{B}_\eta$.

Now consider the long exact sequence of hypercohomology for the pair $(\partial B, B_\eta \cup \dot{B})$:

$$\cdots \to \mathbb{H}^k(\partial B, \mathcal{F}^\bullet) \to \mathbb{H}^k(B_\eta \cup \dot{B}, \mathcal{F}^\bullet)) \to \mathbb{H}_c^{k+1}(\partial B \setminus (B_\eta \cup \dot{B}), \mathcal{F}^\bullet) \to \\ \to \mathbb{H}^{k+1}(\partial B, \mathcal{F}^\bullet) \to \cdots$$

PROPOSITION 4.1 (Variation sequence). *The above long exact sequence for the pair $(\partial B, B_\eta \cup \dot{B})$ can be identified with the sequence*

$$\cdots \to \mathbb{H}^k(\partial B, \mathcal{F}^\bullet) \to \mathbb{H}^k(\mathring{B}_\eta, \mathcal{F}^\bullet) \xrightarrow{\mathrm{var}(h)} \mathbb{H}_c^k(\mathring{B}_\eta, \mathcal{F}^\bullet) \to \mathbb{H}^{k+1}(\partial B, \mathcal{F}^\bullet) \to \cdots$$

PROOF. First observe that $B_\eta \cup \dot{B}$ contains B_η as a deformation retract, even in a stratified sense. This implies that the restriction maps $\mathbb{H}^k(B_\eta \cup \dot{B}, \mathcal{F}^\bullet) \to \mathbb{H}^k(B_\eta, \mathcal{F}^\bullet) \to \mathbb{H}^k(\mathring{B}_\eta, \mathcal{F}^\bullet)$ are isomorphisms for all k. The stratified relative homeomorphism H introduced above gives via the Künneth theorem $\mathbb{H}_c^{k+1}(\partial B \setminus (B_\eta \cup \dot{B}), \mathcal{F}^\bullet) \cong H_c^1((0, 2\pi)) \otimes \mathbb{H}_c^k(\mathring{B}_{-\eta}, \mathcal{F})$. We identify $H_c^1((0, 2\pi)) \cong H^1([0, 2\pi], \{0, 2\pi\}) \cong \tilde{H}^0(\{0, 2\pi\})$ with $\mathbb{Z}$ with the generator corresponding to the function that is 1 on 2π and -1 on 0. So this identifies $\mathbb{H}_c^{k+1}(\partial B \setminus (B_\eta \cup \dot{B}), \mathcal{F}^\bullet)$ with $\mathbb{H}_c^k(\mathring{B}_\eta, \mathcal{F}^\bullet)$.It then remains to identify the connecting map with the variation map. This is left to you. □

So the above proposition yields for ordinary cohomology the sequence

$$\cdots \to H^k(\partial B, \mathbb{E}) \to H^k(\mathring{B}_\eta, \mathbb{E}) \xrightarrow{\mathrm{var}(h)} H_c^k(\mathring{B}_\eta, \mathbb{E}) \to H^{k+1}(\partial B, \mathbb{E}) \to \cdots$$

If we are given a local system $\mathbb{E}$ of finite rank on the union of open strata of X, this yields for intersection cohomology

$$\cdots \to IH^k(\partial B, \mathbb{E}) \to IH^k(\mathring{B}_\eta, \mathbb{E}) \xrightarrow{\mathrm{var}(h)} IH_c^k(\mathring{B}_\eta, \mathbb{E}) \to IH^{k+1}(\partial B, \mathbb{E}) \to \cdots$$

In case X is of pure of dimension n, we may here replace $IH^\bullet(\partial B, \mathbb{E})$ by its more local description

$$IH^k(\partial B, \mathbb{E}) \cong \begin{cases} IH^k(B, \mathbb{E}) = \mathcal{H}^k(\mathscr{IC}_X^\bullet(\mathbb{E})_p) = R^k i_{\{p\}}^* \mathscr{IC}_X^\bullet(\mathbb{E}) \ (k < n), \\ IH_c^{k+1}(B, \mathbb{E}) \cong R^{k+1} i_{\{p\}}^! \mathscr{IC}_X^\bullet(\mathbb{E}) \quad (k \geq n). \end{cases}$$

By virtue of the conical structure, B resp. $\mathring{B}$ contains as a closed subset the closed cone $\mathrm{Cone}(B_\eta)$ resp. the open cone $c(B_\eta)$.

LEMMA-DEFINITION 4.2. *We have a natural isomorphism*

$$\mathbb{H}_c^{k+1}(c(B_\eta), \mathcal{F}^\bullet) \cong \mathbb{H}^{k+1}(B, B_\eta; \mathcal{F}^\bullet).$$

We call this the vanishing cohomology of $(f, \mathcal{F}^\bullet)$ *at* p *in degree* k *and denote it by* $R^k\phi_{f,p}\mathcal{F}^\bullet$.

PROOF. The $\mathscr{X}$-constructibility of $\mathcal{F}^\bullet$ implies that $\mathbb{H}_c^{k+1}(c(B_\eta), \mathcal{F}^\bullet)$ may be identified with $\mathbb{H}_c^{k+1}(\mathrm{Cone}(B_\eta), B_\eta); \mathcal{F}^\bullet)$, the claimed isomorphism follows readily from the observation that $\mathrm{Cone}(B_\eta) \subset B$ induces an isomorphisms on $\mathscr{X}$-constructible hypercohomology. □

Let us see what we get in case $\mathcal{F}^\bullet$ is a single $\mathscr{X}$-constructible sheaf $\mathcal{F}$ put in degree zero. Then $H^k(B, \mathcal{F}) \cong H^k(\mathrm{Cone}(B_\eta), \mathcal{F}) \cong H^k(c(B_\eta), \mathcal{F})$ is trivial unless $k = 0$, in which case we get the stalk $\mathcal{F}_p = i_p^*\mathcal{F}$. Since the stratification is conelike at p, the exact sequence for the pair $(c(B_\eta), c(B_\eta) \smallsetminus \{o\})$ yields an exact sequence

$$0 \quad \to \quad i_p^!\mathcal{F} \quad \to \quad \mathcal{F}_p \quad \to \quad H^0(B_\eta, \mathcal{F}) \quad \to \quad H_c^1(c(B_\eta), \mathcal{F}) \quad \to \quad 0$$

and isomorphisms $H_c^k(c(B_\eta), \mathcal{F}) \cong H^{k-1}(B_\eta, \mathcal{F})$ for $k \geq 2$. In other words,

$$R^k\phi_{f,p}\mathcal{F} \cong \begin{cases} i_p^!\mathcal{F} \quad (k = -1), \\ \mathrm{Coker}(\mathcal{F}_p \cong H^0(B, \mathcal{F}) \to H^0(B_\eta, \mathcal{F}))\, (k = 0), \\ H^k(B_\eta, \mathcal{F}) \quad (k > 1). \end{cases}$$

Our first goal is to prove:

THEOREM 4.3 (Weak Lefschetz theorem in the singular setting). *Let $\Omega \subset \mathbb{C}^N$ be a convex domain in $\mathbb{C}^N$, $X \subset \Omega$ a closed analytic subset([7]) of dimension n and $\mathscr{X}$ a finite analytic Thom-Whitney stratification such that X is well-behaved at infinity with respect to $\mathscr{X}$. Then X has the homotopy type a finite CW complex of dimension $\leq n$ and for any $\mathscr{X}$-constructible sheaf $\mathcal{F}$, $H^\bullet(X, \mathcal{F})$ is zero in degree $> n$.*

REMARK 4.4. The proof also yields some information in degree n: we will find that the map $H_c^n(X, \mathcal{F}) \to H^n(X, \mathcal{F})$ factors through an epimorphism from the vanishing cohomology in degree n as

$$H_c^n(X, \mathcal{F}) \to \oplus_\rho R^{n-1}\phi_{f,p_\rho}\mathcal{F} \twoheadrightarrow H^n(X, \mathcal{F}),$$

where the middle term is defined as below.

The property that $(X, \mathscr{X})$ is well-behaved at infinity implies that the following (somewhat technical) lemma holds, which we state without proof.

LEMMA 4.5. *For almost every complex-linear form f on $\mathbb{C}^N$ there exist convex domains $C' \subset C \subset \mathbb{C}$ and a stratified homeomorphism $h : (X, \mathscr{X}) \to (X_C, \mathscr{X}|X_C)$ such that h is the identity on $X_{C'}$, the critical points of $f|(X, \mathscr{X})$ are finite in number and have distinct values, all contained in C', and f is over C* well-behaved at infinity, *meaning that there exists an open subset $X_C^\infty \subset X_C$ such that $X_C \smallsetminus X_C^\infty$ is proper over C and the pair consisting of the closure of X_C^∞ in X_C and its boundary in X_C is locally trivial over C relative to the partition $\mathscr{X}$ induces on it.*

[7] This implies that X is a Stein space.

4.2. The vanishing intersection cohomology module. The preceding lemma enables us to study the topology of X using Lefschetz techniques in a stratified setting: inasmuch as our concern is the topology of X, X can be replaced by X_C and then X_C is locally topologically trivial over C except for finitely many fibers where the change in topology only manifests itself in a single point. So we essentially proceed as in the proof of the weak Lefschetz theorem 3.1: for each critical point p_ρ of f on $(X, \mathscr{X})$, we choose a singularity box $B_\rho \xrightarrow{f} D_\rho$ for f at p_ρ such that the disks D_ρ are pairwise disjoint and lie in C'. Choose $\eta_\rho \in \partial D_\rho$ and write B_{η_ρ} for B_{ρ,η_ρ}. Then choose $w \in \partial C'$ and an arc γ_ρ in $\overline{C}'$ connecting w with η_ρ whose relative interiors do not meet and do not meet any D_σ. A trivialization of f over γ_ρ yields an embedding $k_\rho : B_{\eta_\rho} \hookrightarrow X_w$. The argument of *loc. cit.* shows that X has the homotopy type of an 'iterated mapping cone' K of the resulting map $\sqcup_{\rho=1}^r B_{\eta_\rho} \to X_w$: the quotient of $\sqcup_{\rho=1}^r [0,1] \times B_{\eta_\rho} \sqcup X_w$ with $(0,z) \in [0,1] \times B_{\eta_\rho}$ is identified with $k_\rho(z) \in X_w$ and where we collapse $\{1\} \times B_{\eta_\rho}$ to a singleton $\{v_\rho\} \subset K$. So K contains each closed cone $\mathrm{Cone}(B_{\eta_\rho})$ and $\mathrm{Cone}(B_{\eta_\rho}) \smallsetminus X_w$ is the open cone $c(B_{\eta_\rho})$. We regard X_K as subspace of X. It is then a deformation retract in a stratified sense. So this construction also to passes the sheaf level as long as we deal with $\mathscr{X}$-constructible bounded below sheaf complexes: then the natural map $\mathbb{H}^k(X, \mathcal{F}^\bullet) \to \mathbb{H}^k(X_K, \mathcal{F}^\bullet)$ is an isomorphism. So we have an exact sequence

$$(7) \quad \cdots \to \oplus_\rho R^{k-1}\phi_{f,p_\rho}\mathcal{F}^\bullet \to \mathbb{H}^k(X, \mathcal{F}^\bullet) \to \mathbb{H}^k(X_w, \mathcal{F}^\bullet) \to \\ \to \oplus_\rho R^k\phi_{f,p_\rho}\mathcal{F}^\bullet \to \cdots$$

in which we see appear the vanishing cohomology. This applies in particular to intersection cohomology: if $\mathbb{E}$ is a local system on the union of open strata, then we have an exact sequence

$$(7_{IH}) \quad \cdots \to \oplus_\rho R^{k-1}\phi_{f,p_\rho}\mathscr{IC}^\bullet_X(\mathbb{E}) \to IH^k(X, \mathbb{E}) \to IH^k(X_w, \mathbb{E}) \to \\ \to \oplus_\rho R^k\phi_{f,p_\rho}\mathscr{IC}^\bullet_X(\mathbb{E}) \to \cdots .$$

This has a counterpart for compactly supported cohomology. For this the most intuitive approach is perhaps give X a partial boundary which makes f proper. Since we will not do a homotopy discussion, this is not necessary and we proceed as follows. First observe that our local triviality assumption implies that $\mathbb{H}^k_c(X_{C'}, \mathcal{F}^\bullet) \to \mathbb{H}^k_c(X, \mathcal{F}^\bullet)$ is an isomorphism. Let E denote the union of $C \smallsetminus C'$, the arcs γ_ρ and the disks D_ρ. This is a closed subset of C. Our assumptions regarding local triviality and a Künneth formula imply that $\mathbb{H}^k_c(X \smallsetminus X_E, \mathcal{F}^\bullet) \cong \mathbb{H}^{k-2}_c(X_w, \mathcal{F}^\bullet)$. On the other hand, excision yields an identification of $\mathbb{H}^k_c(X_E, \mathcal{F}^\bullet)$ with the vanishing cohomology $\oplus_\rho \mathbb{H}^k_c(c(B_{\rho,\eta_\rho}), \mathcal{F}^\bullet)$. Thus the exact sequence for the pair (X, X_E) becomes

$$(7_c)\quad \cdots \to \mathbb{H}_c^{k-2}(X_w, \mathcal{F}^\bullet) \to \mathbb{H}_c^k(X, \mathcal{F}^\bullet) \to \oplus_\rho R^{k-1}\phi_{f,p_\rho}\mathcal{F}^\bullet \to \to \mathbb{H}_c^{k-1}(X_w, \mathcal{F}^\bullet) \to \cdots .$$

which in a sense is indeed dual to (7). In the case of intersection cohomology it yields

$$(7_{IH_c})\quad \cdots \to IH_c^{k-2}(X_w, \mathbb{E}) \to IH_c^k(X, \mathbb{E}) \to \oplus_\rho R^{k-1}\phi_{f,p_\rho}\mathscr{IC}^\bullet_X(\mathbb{E}) \to \to IH_c^{k-1}(X_w, \mathbb{E}) \to \cdots .$$

PROOF OF THEOREM 4.3. We proceed with induction on $n = \dim X$. Observe that X_w and each $\mathring{B}_{\rho,\eta_\rho}$ is given as a closed analytic subset in an open convex subset of $\mathbb{C}^N$ of finite type and is of dimension $\leq n-1$. By our induction hypothesis they have the homotopy type of a finite cell complex of dimension $\leq n-1$. It follows that K has the homotopy type of a finite cell complex of dimension $\leq n$.

The proof of the second assertion is similar. In the exact sequence (7) we can replace $R^{k-1}\phi_{f,p_\rho}\mathcal{F}^\bullet$ by $H^{k-1}(B_{\eta_\rho}, \mathcal{F})$ (for $k \geq 2$) or (for $k = 1$) by a quotient thereof. Our induction induction hypothesis tells us that $H^k(B_{\eta_\rho}, \mathcal{F})$ is zero for $k > \dim \mathring{B}_{\eta_\rho}$. This implies that $R^{k-1}\phi_{f,p_\rho}\mathcal{F}^\bullet = 0$ for $k > 1 + \dim_{\mathbb{C}} \mathring{B}_{\eta_\rho}$ and hence for $k > n$. Similarly, our induction hypothesis applied to X_w says that $H^k(X_w, \mathcal{F}) = 0$ for $k > \dim X_w$ and $H_c^k(B_{\eta_\rho}, \mathcal{F}) = 0$ for $k < \dim_{\mathbb{C}} \mathring{B}_{\rho,\eta}$. So $H^k(X, \mathcal{F}) = 0$ for $k > \dim X$. This also shows that the natural map $\oplus_\rho R^{n-1}\phi_{f,p_\rho}\mathcal{F}^\bullet \to H^n(X, \mathcal{F})$ is onto. □

COROLLARY 4.6. *Let $X \subset \Omega$ be as in Theorem 4.3 and $\mathcal{F}^\bullet$ a bounded below sheaf complex on X that is constructible relative to a finite Thom-Whitney stratification of X. If for an integer d, $\dim \operatorname{Supp} \mathcal{H}^k(\mathcal{F}^\bullet) \leq d-k$ for all k, then $\mathbb{H}^k(X, \mathcal{F}^\bullet) = 0$ for $k > d$. In particular, if $\mathbb{E}$ a locally constant on an open-dense subset $\mathring{X}$ of X, then $IH^k(X, \mathbb{E}) = 0$ for $k > \dim X$.*

PROOF. We have a spectral sequence $E_2^{pq} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet)$. We have $H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) = H^p(\operatorname{Supp} \mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{H}^q(\mathcal{F}^\bullet))$ and by assumption, the latter is zero for $p+q > d$. Hence $\mathbb{H}^k(X, \mathcal{F}^\bullet) = 0$ for $k > d$. □

The following Corollary is a straightforward consequence of duality when $\mathbb{E}$ is a local system of finite dimensional **k**-vector spaces. The proof for the more general case when $\mathbb{E}$ is just a locally constant sheaf involves slightly more.

COROLLARY 4.7. *Suppose X is of pure dimension n.*

(i_n) *If X is as in the weak Lefschetz theorem 4.3, then $IH_c^k(X, \mathbb{E}) = 0$ for $k < n$.*

(ii_n) *In the situation of 4.1, the vanishing intersection cohomology at p, $R^\bullet\phi_{f,p}\mathscr{IC}^\bullet_X(\mathbb{E})$, is concentrated in degree $n-1$.*

PROOF. We first prove that $(i_{n-1}) \Rightarrow (ii_n)$. Then $IH^k(\mathring{B}_\eta) = 0$ for $k \geq n$. Consider the exact sequence

$$\cdots \to IH^{k-1}(\partial B, B_\eta; \mathbb{E}) \to IH^k(B, \partial B; \mathbb{E}) \to IH^k(B, B_\eta; \mathbb{E}) \to IH^k(\partial B, B_\eta; \mathbb{E}) \to \cdots$$

With identifications made earlier, we can rewrite this as an exact sequence

$$\cdots \to IH_c^{k-2}(\mathring{B}_\eta; \mathbb{E}) \to R^k i_p^! \mathscr{IC}_X(\mathbb{E}) \to R^{k-1}\phi_{f,p}\mathscr{IC}^\bullet_X(\mathbb{E}) \to IH_c^{k-1}(\mathring{B}_\eta; \mathbb{E}) \to \cdots$$

For $k \leq n-1$, $R^k i_p^! \mathscr{IC}_X(\mathbb{E}) = 0$ by construction and $IH_c^{k-1}(\mathring{B}_\eta; \mathbb{E}) = 0$ by assumption (i_{n-1}). It follows that $R^k\phi_{f,p}\mathscr{IC}^\bullet_X(\mathbb{E}) = 0$ for $k < n-1$ and so $R^\bullet\phi_{f,p}\mathscr{IC}^\bullet_X(\mathbb{E})$ is indeed concentrated in degree $n-1$.

The implication $(i_{n-1}) \wedge (ii_n) \Rightarrow (i_n)$ is an immediate consequence of the exact sequence (7_{IH_c}). $\square$

REMARK 4.8. We may now form a diagram with exact row and column and with the lower south-west arrow being the obvious one (it is in a sense self-dual):

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & IH^{n-1}(\mathring{B}_\eta; \mathbb{E}) \arrow[d] \arrow[dr, "var(h)"] & & \\
0 \arrow[r] & R^n i_p^! \mathscr{IC}^\bullet_X(\mathbb{E}) \arrow[r] \arrow[dr] & R^{n-1}\phi_{f,p}\mathscr{IC}^\bullet_X(\mathbb{E}) \arrow[r] \arrow[d] & IH_c^{n-1}(\mathring{B}_\eta; \mathbb{E}) \arrow[r] & 0 \\
 & & R^n i_p^* \mathscr{IC}^\bullet_X(\mathbb{E}) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

COROLLARY 4.9. *Let X be an irreducible projective variety of pure dimension n and $\mathbb{E}$ a local system on its generic point. Let $i : X_\xi \subset X$ be a hyperplane section. Then $IH^k(X \smallsetminus X_\xi; \mathbb{E}) = 0$ for $k > n$ and hence $IH_c^k(X \smallsetminus X_\xi; \mathbb{E}) = 0$ for $k < n$. If X_ξ is generic, or more generally, when $i^*\mathscr{IC}^\bullet_X(\mathbb{E})$ can serve as $\mathscr{IC}^\bullet(X_\xi, i^*\mathbb{E})$, so that the natural map $IH^k(X; \mathbb{E}) \to \mathbb{H}^k(X_\xi, i^*\mathbb{E})$ is defined, then the latter is an isomorphism in degree $< n-1$ and injective in degree $n-1$.*

PROOF. The first assertion is immediate from Corollaries 4.6 and 4.7 and the second follows from the exact sequence

$$\cdots \to IH_c^{k-1}(X\smallsetminus X_\xi;\mathbb{E}) \to IH^k(X;\mathbb{E}) \to IH^k(X_\xi; i^*\mathbb{E}) \to \\ \to IH_c^k(X\smallsetminus X_\xi;\mathbb{E}) \to \cdots \qquad \square$$

Here is another application.

COROLLARY 4.10. *Suppose that in the situation of Proposition 4.1, B is of pure (complex) dimension n. Then for $k<n-1$ the restriction map $IH^k(B,\mathbb{E})\to IH^k(\mathring{B}_\eta,\mathbb{E})$ is an isomorphism and (hence) the monodromy acts trivially on $IH^k(\mathring{B}_\eta;\mathbb{E})$, whereas for $k=n-1$ the situation is described by the exact sequence*

$$0\to R^{n-1}i_p^*\mathscr{IC}^\bullet_X(\mathbb{E})\to IH^{n-1}(\mathring{B}_\eta,\mathbb{E})\xrightarrow{\mathrm{var}(h)} IH_c^{n-1}(\mathring{B}_\eta,\mathbb{E})\to \\ \to R^{n+1}i_p^!\mathscr{IC}^\bullet_X(\mathbb{E})\to 0$$

PROOF. We know that $IH^k(\mathring{B}_\eta,\mathbb{E})=0$ for $k\ge n$. Applying this to $\mathbb{E}^\vee$ and dualizing gives that $IH_c^k(\mathring{B}_\eta,\mathbb{E})=0$ for $k\le n-2$. Substitute this in the variation sequence of Proposition 4.1 and use that $R^ki_p^*\mathscr{IC}^\bullet_X(\mathbb{E})\cong IH^k(B,\mathbb{E})\to IH^k(\partial B,\mathbb{E})$ is an isomorphism for $k<n$ and $IH^k(\partial B,\mathbb{E})\to IH_c^{k+1}(B,\mathbb{E})\cong R^{k+1}i_p^!\mathscr{IC}^\bullet_X(\mathbb{E})$ is an isomorphism for $k\ge n$. $\square$

Note that the dual of the exact sequence above yields the one for $o_X\otimes\mathbb{E}^\vee$.

5. The hard Lefschetz theorem for intersection cohomology

5.1. Tate twists. Let C be a nonsingular complex-algebraic curve and $p\in C$. Then $H^\bullet_{\{p\}}(C)$ is concentrated in degree 2. In fact, a complex-analytic chart $z:U_p\to\mathbb{C}$ at p with $z(p)=0$ yields an isomorphism $H^k_{\{p\}}(C)\cong H^k_{\{0\}}(\mathbb{C})=H^2(\mathbb{C},\mathbb{C}\smallsetminus\{0\})$ and this is zero unless $k=2$, whereas for $k=2$ the boundary map gives an isomorphism $H^2_{\{0\}}(\mathbb{C})\cong H^1(\mathbb{C}\smallsetminus\{0\})$, which using the standard orientation of $\mathbb{C}$, can be identified with $\mathbb{Z}$. But in algebraic geometry there is no prefered orientation, as the two are interchanged under complex conjugation. Perhaps the best way to understand this is to replace our field of definition $\mathbb{C}$ by a topological field K that is known to be isomorphic to $\mathbb{C}$, without having specified such an isomorphism (note that there are two of these, interchanged by complex conjugation). Assuming that C is defined over K, we may represent $H^2_{\{p\}}(C)$ in an algebraic De Rham setting by choosing a rational differential form α on C with at p a pole of order one with residue 1: so $\alpha=z^{-1}dz+\alpha'$ with α' regular at p. The way this relates to $H^2_{\{p\}}(C)$ is that when $U_p\subset C$ is a small open disk centered at p, then $H^2_{\{p\}}(C;K)\xleftarrow{\cong}H^1(U_p\smallsetminus\{p\};K)=$

$\mathrm{Hom}(H_1(U_p \smallsetminus \{p\}), K)$. Now α determines a class $[\alpha] \in H^2_{\{p\}}(C, K)$ and the Cauchy residue formula shows that when this class is viewed as an element of $\mathrm{Hom}(H_1(U_p \smallsetminus \{p\}), K)$, it defines an isomorphism of $H_1(U_p \smallsetminus \{p\})$ onto $2\pi\sqrt{-1}\mathbb{Z} \subset K^+$, where we note that the latter is a well-defined subgroup of the additive group K^+ of K (it is also the kernel of the exponential homomorphism $z \in K^+ \mapsto e^z \in K^\times$). This isomorphism does not depend on the choice of α.

We will write $\mathbb{Z}(1)$ for the subgroup $2\pi\sqrt{-1}\mathbb{Z} \subset K^+$, keeping in mind that it has no distinguished generator, and we shall refer to it as the *Tate group*. When $m \in \mathbb{Z}$, we write $\mathbb{Z}(m)$ for the $|m|$-fold tensor power of $\mathbb{Z}(1)$ (when $m > 0$) or the dual of this (when $m < 0$), agreeing that $\mathbb{Z}(0) = \mathbb{Z}$. Since $\mathbb{Z}(1)$ contains no root of unity, $\mathbb{Z}(m)$ can also be regarded as the subgroup $(2\pi\sqrt{-1})^m\mathbb{Z}$ of K^+. For any abelian group A, $A(n)$ will stand for $A \otimes \mathbb{Z}(n)$. Returning to (C, p) above, we found an isomorphism $H_1(U_p \smallsetminus \{p\})) \cong \mathbb{Z}(1)$. This dualizes to a *canonical* isomorphism $H^2_{\{p\}}(C) \cong \mathbb{Z}(-1)$, equivalently, it provides a *canonical generator* (namely our $[\alpha]$) of $H^2_{\{p\}}(C)(1) \cong \mathrm{Hom}(H_1(U_p \smallsetminus \{p\}), \mathbb{Z}(1))$.

Tate twists will serve us well as a bookkeeping device for orientation choices; at the same time they help us remember that there is involved a homology class in degree 2 ([8]). The following illustrates this. If M is a complex (or rather, K-)manifold of dimension m and $p \in M$, then $H^{2m}_{\{p\}}(M) \cong H^{2m}_{\{0\}}(K^m) \cong H^2_{\{0\}}(K)^{\otimes m}$ (by the Künneth formula) and so $H^{2m}_{\{p\}}(M) \cong \mathbb{Z}(-m)$ canonically. It follows that $H^{2m}_c(M) \cong \mathbb{Z}(-m)$ when M is connected. Hence $H^{2m}_c(M; \mathbb{Z}(m)) = H^{2m}_c(M)(m)$ (rather than $H^{2m}_c(M)$) has a canonical generator. Our insistence on algebro-geometric naturality also affects duality pairings: if $\mathbb{E}$ is a local system of finite dimensional $\mathbf{k}$-vector spaces (with $\mathbf{k} \supset \mathbb{Q}$) on M and M is connected and of finite type, then the pairing

$$H^k(M; \mathbb{E}) \otimes H^{2m-k}_c(M, \mathbb{E}^\vee(m)) \to H^{2m}_c(M, \mathbf{k}(m)) \cong \mathbf{k}$$

is perfect. This generalizes to the setting of intersection cohomology.

We have a similar discussion for a hyperplane class: given a projective space $\mathbb{P}$ over K of positive dimension, then for any projective line $\ell \subset \mathbb{P}$, the restriction map $H^2(\mathbb{P}) \to H^2(\ell) \cong \mathbb{Z}(-1)$ is an isomorphism and hence $H^2(\mathbb{P})(1)$ (rather than $H^2(\mathbb{P})$) has a natural generator. We often denote this generator by η. This means that if $i : X \subset \mathbb{P}$ is a projective variety, then $\eta_X := i^*\eta \in H^2(X)(1)$ and cupping with η^k defines a map $H^i(X) \to H^{i+2k}(X)(k)$.

[8] But their usefulness, especially in Hodge theory, goes well beyond that. The same applies to their counterparts on other cohomology theories, such as étale cohomology.

In the setting of the Hard Lefschetz theorem in Corollary 5.6 (so where X is nonsingular of dimension n) we obtained a nondegenerate pairing on the primitive cohomology. We write this here as

$$\Psi^k : (\alpha, \beta) \in P^k(X;\mathbb{Q}) \times P^k(X;\mathbb{Q}) \mapsto \alpha \cup \beta \cup \eta_X^{n-k} \in H^{2n}(X;\mathbb{Q})(n-k).$$

This pairing is clearly $(-1)^k$-symmetric. Since $H^{2n}(X;\mathbb{Q})(n-k) \cong \mathbb{Q}(-n) \otimes_{\mathbb{Z}} \mathbb{Z}(n-k) = \mathbb{Q}(-k)$, this makes Ψ^k ressemble the intersection pairing on the middle dimensional cohomology of a k-dimensional projective manifold. For good reason: the Lefschetz theorems show that if $i : Y \subset X$ is transversal linear section of dimension k, then i^* identifies $(P^k(X;\mathbb{Q}), \Psi^k)$ with a subspace of $H^k(Y;\mathbb{Q})$ endowed with its intersection pairing.

In the remainder of this section, we assume that $X \subset \mathbb{P}$ is a projective variety of pure dimension $n = m + 1$, and $\mathbb{E}$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces on a Zariski open-dense subset $\mathring{X} \subset X$ that is the union of open strata of a Thom-Whitney stratification $\mathscr{X}$ of X. So $IH^\bullet(X;\mathbb{E})$ has the structure of a $H^\bullet(X;\mathbb{Q})$-module. In particular, we have defined a Lefschetz operator

$$L := \eta_X \cup : IH^k(X;\mathbb{E}) \to IH^{k+2}(X;\mathbb{E}(1)),$$

where $\eta_X \in H^2(X, \mathbb{Z}(1))$ is the hyperplane class. We will write this as a composite of two maps involving a hyperplane section. Let $H \subset \mathbb{P}$ be a hyperplane that is transversal to $\mathscr{X}$ and put $i : X_H := X \cap H \subset X$. So $\dim X_H = m$.

COROLLARY 5.1 (to Proposition 3.8). *The Lefschetz operator is the composite*

$$L : IH^k(X;\mathbb{E}) \xrightarrow{i^*} IH^k(X_H;\mathbb{E}) \xrightarrow{i^{*\vee}} IH^{k+2}(X;\mathbb{E}(1)),$$

where $i^{*\vee}$ is the dual of $i^* : IH^{2m-k}(X;\mathbb{E}^\vee(m)) \to IH^{2m-k}(X_H;\mathbb{E}^\vee(m))$. The map $i^{*\vee}$ can also be identified with the Gysin map

$$IH^k(X_H;\mathbb{E}) \cong IH^{k+2}_{X_H}(X;\mathbb{E}(1)) \to IH^{k+2}(X;\mathbb{E}(1)),$$

where the first map is a Thom isomorphism.

PROOF. The assumption that H is transversal to $(X, \mathscr{X})$ implies that $\mathscr{X}$ induces on X_H a Thom-Whitney stratification so that Proposition 3.8 applies. The second assertion also follows from that Proposition because the middle isomorphism $IH^k(X_H;\mathbb{E}) \cong IH^{k+2}_{X_H}(X;\mathbb{E}(1))$ is defined by cupping with the Euler class of the normal bundle of H restricted to X_H, i.e., with η_{X_H}. $\square$

We have seen that $i^* : IH^k(X;\mathbb{E}) \to IH^k(X_H;\mathbb{E})$ is an isomorphism for $k < m$ and is injective for $k = m = n - 1$. We define as in the nonsingular setting the *invariant* and the *vanishing* intersection cohomology:

$$IH^m_{inv}(X;\mathbb{E}) := \mathrm{Im}\left(i^* : IH^m(X;\mathbb{E}) \hookrightarrow IH^m(X_H;\mathbb{E}))\right)$$
$$IH^m_{van}(X;\mathbb{E}) := \mathrm{Ker}\left(i^{*\vee} : IH^m(X_H;\mathbb{E}) \twoheadrightarrow IH^{m+2}(X;\mathbb{E}(1))\right)$$

In order that $i^{*\vee}$ can be regarded an adjoint we will assume that $\mathbb{E}$ comes with a nondenegerate pairing of the following type.

DEFINITION 5.2. We will call a pairing $\Psi : \mathbb{E} \otimes_{\mathbf{k}_{\mathring{X}}} \mathbb{E} \to \mathbf{k}_{\mathring{X}}(-w)$ which is nondegenerate and $(-1)^w$-symmetric a *quasi-polarization of weight* w on $\mathbb{E}$.

This definition includes the case of a single vector space (take $X = \mathrm{pt}$).

A quasi-polarization identifies $\mathbb{E}$ with its dual in a particular way. Let us also note that the quasi-polarized local systems of $\mathbf{k}$-vector spaces are closed under tensor products and homogenenous direct sum (i.e., when the summands have the same weight). It is of course possible that a subsystem of a quasi-polarized local system $(\mathbb{E}, \Psi)$ to be degenerate for relative to Ψ.

EXAMPLE 5.3. Let $f : Z \to \mathring{X}$ be a projective and smooth morphism of relative dimension d. Then for $k \leq d$, the primitive cohomology sheaf $P^k f_*\mathbb{Q}_Z$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces and comes with a polarization of weight k as defined in Section 5.

LEMMA 5.4. *Suppose* $\mathbb{E}$ *endowed with a* quasi-polarization of weight w *on* $\mathbb{E}$*). Then the duality pairing defines nondegenerate pairings*

$$\Psi^k_X : IH^k(X;\mathbb{E}) \times IH^{2n-k}(X;\mathbb{E}) \to \mathbf{k}(-n-w)$$

and we have $\Psi^k_X(\alpha,\beta) = (-1)^{k+w}\Psi^{2n-k}_X(\beta,\alpha)$ *(so that thes direct sum* $IH^k(X;\mathbb{E}) \oplus IH^{2n-k}(X;\mathbb{E})$ *is quasi-polarized of weight* $n+w$*).*

Moreover, $i^{*\vee} : IH^m(X_H;\mathbb{E}) \twoheadrightarrow IH^{m+2}(X;\mathbb{E}(1))$ *is the adjoint of* $\pm i^* : IH^m(X;\mathbb{E}^\vee(m)) \to IH^m(X_H;\mathbb{E}^\vee(m))$ *relative to the nondegenerate pairing* $\Psi^m_X : IH^m(X;\mathbb{E}) \times IH^{m+2}(X;\mathbb{E}) \to \mathbf{k}(-1-m-w)$ *and the quasi-polarization* $\Psi := \Psi^m_{X_H} : IH^m(X_H;i^*\mathbb{E}) \times IH^m(X_H;i^*\mathbb{E}) \to \mathbf{k}(-m-w)$*. In particular,* $IH^m_{inv}(X;\mathbb{E})$ *and* $IH^m_{van}(X;\mathbb{E})$ *are each others perp with respect to* Ψ*.*

PROOF. The pairing $IH^k(X;\mathbb{E}) \times IH^{2n-k}(X;\mathbb{E}^\vee(n)) \to \mathbf{k}$ plus the isomorphism $\mathbb{E}^\vee \cong \mathbb{E}(w)$ defined by the polarization yields the nondegenerate pairing $IH^k(X;\mathbb{E}) \times IH^{2n-k}(X;\mathbb{E}(n+w)) \to \mathbf{k}$. After 'twisting' this with $\mathbf{k}(-n-w)$, we get Ψ^k_X. The relation between Ψ^k_X and Ψ^{2n-k}_X can be traced down to the Koszul rule for the cup product (we omit the details). The last assertion is immediate from Corollary 5.1 and the definition. $\square$

We have the following generalization of Proposition 5.3.

PROPOSITION 5.5. *The following properties are equivalent:*

(i) $IH^m_{van}(X_H; i^*\mathbb{E}) \cap IH^m_{inv}(X_H; i^*\mathbb{E}) = 0$,
(ii) $IH^m(X_H; i^*\mathbb{E}) = IH^m_{van}(X_H; i^*\mathbb{E}) + IH^m_{inv}(X_H; i^*\mathbb{E})$,
(iii) $i^{*\vee}$ *restricts to an isomorphism* $IH^m_{inv}(X_H; \mathbb{E}) \xrightarrow{\cong} IH^{m+2}(X; \mathbb{E}(1))$,
(iv) $L : H^m(X; \mathbb{E}) \to H^{m+2}(X; \mathbb{E}(1))$ *is an isomorphism.*

When $\mathbb{E}$ *comes with a quasi-polarization of weight* w*, these properties are also equivalent to:*

(v) $IH^m_{van}(X_H; \mathbb{E})$ *is nondegenerate relative to* Ψ,
(vi) $IH^m_{inv}(X_H; \mathbb{E})$ *is nondegenerate relative to* Ψ,

PROOF. This is straightforward linear algebra (use Lemma 5.4 and the fact that $L = i^{*\vee} i^*$). □

The following theorem is due to Beilinson, Bernstein, and Deligne-Gabber [2]. It has later been generalized by Morihiko Saito, Mochizuki and Sabbah.

THEOREM 5.6 (Hard Lefschetz theorem for intersection cohomology). *For* $X \subset \mathbb{P}$ *of pure dimension* n *and* $\mathbb{E}$ *semisimple, the equivalent properties of Proposition 5.5 all hold.*

With this in hand, the 'hard Lefschetz program' for the nonsingular case Section 5 can be carried out without any essential change. For $k = 0, \dots, n$, we define the *primitive part* $P^k_{IH}(X; \mathbb{E})$ of $IH^k(X, \mathbb{E})$ as the kernel of the restriction $L^{n-k+1}|IH^k(X, \mathbb{E})$.

COROLLARY 5.7. *In the situation of Theorem 5.6, the Lefschetz operator* $L^k : IH^{n-k}(X, \mathbb{E}) \to IH^{n+k}(X, \mathbb{E})(k)$ *is an isomorphism and if* $\mathbb{E}$ *comes with a quasi-polarization of weight* w*, then*

$$\Psi : (\alpha, \beta) \in P^k_{IH}(X; \mathbb{E}) \times P^k_{IH}(X; \mathbb{E}) \mapsto \alpha \cup \beta \cup \eta_X^{n-k} \in \mathbb{Q}(-k - w)$$

defines a polarization of weight $k + w$ *on* $P^k_{IH}(X; \mathbb{E})$.

So the action of L on the Tate twisted intersection cohomology gives rise to a Lefschetz decomposition of $\oplus_{i=0}^{2n} IH^i(X, \mathbb{E})$ up to Tate twists (namely $\oplus_{k=0}^{n} \mathbb{Q}[t]/(t^{k+1}) \otimes_{\mathbb{Q}} P_{IH}^{n-k}(X)$) and this is part of a natural representation of $\mathfrak{sl}_2(\mathbb{Q})$ on the left hand side. The counterpart of Deligne's theorem is also rather formal. It says that if $f : \mathcal{X} \to S$ is a flat morphism of relative dimension n that is topologically local trivial, then the Leray spectral sequence

$$E_2^{p,q} := H^p(S, R^q f_* \mathscr{IC}^\bullet_{\mathcal{X}}(\mathbb{E}) \Rightarrow IH^{p+q}(\mathcal{X}; \mathbb{E})$$

degenerates, with the proof in fact showing that $\mathbb{R} f_* \mathscr{IC}^\bullet_{\mathcal{X}}(\mathbb{E})$ is quasi-isomorphic to the direct sum $\oplus_q R^q f_* \mathscr{IC}^\bullet_{\mathcal{X}}(\mathbb{E})$ (see Remark 5.12). There is however a far reaching generalization of this property to the case of an arbitrary projective morphism. In its basic form it states:

THEOREM 5.8. *Let $f : X \to Y$ be a projective morphism of complex varieties. Then $\mathbb{R}f_*\mathscr{IC}^\bullet_X(\mathbb{Q})$ decomposes in the derived category into a direct sum of shifted intersection cohomology sheaves, to be precise, there exists a finite collection of triples $(Z_\alpha, \mathbb{E}_\alpha, n_\alpha)$, where $i_\alpha : Z_\alpha \subset Y$ is an irreducible subvariety of Y, $\mathbb{E}_\alpha$ an irreducible quasipolarized local system of finite dimensional $\mathbb{Q}$-vector spaces on the general point of Z_α and $n_\alpha \in \mathbb{Z}$ such that we have in $D^b_c(Y)$*

$$Rf_*\mathscr{IC}^\bullet_X(\mathbb{Q}) \sim \oplus_\alpha i_{\alpha *}\mathscr{IC}^{n_\alpha+\bullet}_{Z_\alpha}(\mathbb{E}_\alpha).$$

6. Perversity

6.1. Triangulated categories. Let $\mathfrak{C}$ be an abelian category. The *shift operator* T assigns to a complex $C^\bullet$ in $\mathfrak{C}$ that same complex shifted by a unit to the left: $T(C^\bullet)^k = C^{k+1}$ and with the differential multiplied by -1. This shift operator is clearly invertible and preserves quasi-isomorphisms. We often write $C[n]^\bullet$ for $T^n(C^\bullet)$. So for a single term complex with an object C of $\mathfrak{C}$ in degree zero, $C[n]$ places that object in degree $-n$.

This shift operator appears naturally in the mapping cone construction: if $f : A^\bullet \to B^\bullet$ is a morphism in $K(\mathfrak{C})$, then we can form a double complex by putting $A^\bullet$ in row -1 and $B^\bullet$ in row 0:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & B^k & \xrightarrow{d_B} & B^{k+1} & \xrightarrow{d_B} & \cdots \\
\uparrow & & f^k\uparrow & & f^{k+1}\uparrow & & \uparrow \\
\cdots & \longrightarrow & A^k & \xrightarrow{d_A} & A^{k+1} & \xrightarrow{d_A} & \cdots
\end{array}$$

The associated single complex is $B^\bullet \oplus A[1]^\bullet$ (so $(B^\bullet \oplus A^\bullet[1])^k = B^k \oplus A^{k+1}$) whose differential takes $y \in B^k$ to $d_B(y) \in B^{k+1}$ and $x \in A[1]^k = A^{k+1}$ to $(f(x), -d_A(x)) \in B^{k+1} \oplus A[1]^{k+1}$. This complex is called the *mapping cone* of f and denoted $C^\bullet_f$. Observe that it contains $B^\bullet$ as a subcomplex: $g : B^\bullet \hookrightarrow C^\bullet_f$, so that $A^\bullet[1]$ becomes the quotient complex: $h : C^\bullet_f \twoheadrightarrow A[1]$. This makes f appear as the differential of a short exact sequence of complexes (after renumbering $A^\bullet$): we have a long exact sequence:

$$\cdots \to H^k(A^\bullet) \xrightarrow{H^k(f)} H^k(B^\bullet) \xrightarrow{H^k(g)} H^k(C^\bullet_f) \xrightarrow{H^k(h)} H^{k+1}(A^\bullet) \to \cdots$$

This also shows that f is a quasi-isomorphism if and only if $C^\bullet_f$ is exact, or equivalently, represents the zero element of the derived category $D(\mathfrak{C})$. We may combine our chain maps into a sequence $A^\bullet \to B^\bullet \to C^\bullet_f \to A[1]^\bullet$ and observe that if we take cohomology in degree zero of the 'complex of complexes' $\cdots \to A^\bullet[k] \to B^\bullet[k] \to C^\bullet_f[k] \to A[k+1]^\bullet \to \cdots$, we recover the associated long exact sequence above. You may check that this construction is functorial if f:

any commutative square in $K(\mathfrak{C})$ exends to commutative ladder

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C_f^\bullet & \longrightarrow & A[1]^\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C_{f'}{}^\bullet & \longrightarrow & A'[1]^\bullet \end{array}$$

Since we want to treat the three complexes (apart from their indexing) on an equal footing, we prefer to write the above sequence as a triangle:

$$\begin{tikzcd} A^\bullet \arrow[rr, "f"] & & B^\bullet \arrow[dl, "g"] \\ & C_f^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

There is a symmetry here that becomes only apparent if we pass to the derived category $D(\mathfrak{C})$. We first observe that this notion can be passed on to $D(\mathfrak{C})$ by agreeing that a morphism from a diagram in $K(\mathfrak{C})$ to another diagram in $K(\mathfrak{C})$ of the same type is a quasi-isomorphism, if each of the defining arrows of that morphism is one. If we now apply the preceding construction to $g : B^\bullet \to C_f^\bullet$:

$$\begin{tikzcd} B^\bullet \arrow[rr, "g"] & & C_f^\bullet \arrow[dl, "h"] \\ & C_g^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

then you may check that the obvious projection of $C_g^k = C_f^k \oplus B^{k+1} = (B^k \oplus A^{k+1}) \oplus B^{k+1} \to A^{k+1}$ defines a quasi-isomorphism $C_g^\bullet \to A[1]^\bullet$ which fits in a quasi-isomorphism of the triangle $B^\bullet \to C_f^\bullet \to C_g^\bullet \to B[1]^\bullet$ associated to g to the rotated triangle associated to f: $B^\bullet \to C_f^\bullet \to A^\bullet[1] \to B[1]^\bullet$, where the last map takes $x \in A[1]^\bullet$ to $-f(x) \in B[1]^\bullet$. Doing this twice (or doing this backward) shows that the triangle associated to $v : C_f^\bullet \to A[1]^\bullet$ is quasi-isomorphic to $C_f^\bullet \to A[1]^\bullet \to B[1]^\bullet \to C_f[1]^\bullet$.

Let us then call a diagram in $K(\mathfrak{C})$ of the form

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

a *distinguished triangle* in $K(\mathfrak{C})$ if it is quasi-isomorphic to one of the type above. Such a triangle gives rise to a long exact sequence

$$\cdots \to H^k(A^\bullet) \to H^k(B^\bullet) \to H^k(C^\bullet) \to H^{k+1}(A) \to \cdots$$

and one third of a full turn takes this triangle into another such (yielding the same long exact sequence with a shift). The distinguished triangles serve here as a substitute for short exact sequences in $K(\mathfrak{C})$. By endowing the category $D(\mathfrak{C})$ with the shift operator T and its collection of distinguished triangles (images of distinguished triangles in $K(\mathfrak{C})$), it is able to remember a bit about its origin as a derived category. The properties of this notion have been collected and formalized by Verdier [21] into what is called a 'triangulated category':

DEFINITION 6.1. A *triangulated category* is an additive category $\mathfrak{D}$, endowed with an invertible additive automorphism T and a collection of diagrams $A \to B \to C \to A[1](:= T(A))$ called *distinguished triangles* satisfying four properties, three of which are:

TR1: For any object A of $\mathfrak{D}$, the triangle $A \xrightarrow{=} A \to 0 \to A[1]$ is distinguished, any $\mathfrak{D}$-morphism $A \to B$ fits in distinguished triangle $A \to B \to C \to A[1]$ and any triangle isomorphic to a distinguished triangle is distinguished.

TR2: A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished if and only if its rotation $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-T(f)} B[1]$ is.

TR3: Given two distinguished triangles (say, $A \to B \to C \to A[1]$ and $A' \to B' \to C' \to A'[1]$), then a morphism between the subdiagrams defined by their first sides (so going from $A \to B$ to $A' \to B'$) extends to a morphism between these triangles.

The morphism whose existence is postulated in (TR3) need not be unique.

Let $\mathfrak{D}$ satisfy the three properties above. For objects X, Y of $\mathfrak{C}$ and $n \in \mathbb{Z}$, we put $\mathrm{Hom}^n_{\mathfrak{C}}(X, Y) := \mathrm{Hom}_{\mathfrak{C}}(X, Y[n])$.

LEMMA 6.2. *If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is a distinguished triangle of $\mathfrak{D}$, then for every object X of $\mathfrak{D}$, the following sequences are exact:*

$$\cdots \to \mathrm{Hom}^n_{\mathfrak{D}}(X, A) \to \mathrm{Hom}^n_{\mathfrak{D}}(X, B) \to \mathrm{Hom}^n_{\mathfrak{D}}(X, C) \to \mathrm{Hom}^{n+1}_{\mathfrak{D}}(X, A) \to \cdots$$

$$\cdots \to \mathrm{Hom}^n_{\mathfrak{D}}(C, X) \to \mathrm{Hom}^n_{\mathfrak{D}}(B, X) \to \mathrm{Hom}^n_{\mathfrak{D}}(A, X) \to \mathrm{Hom}^{n-1}_{\mathfrak{D}}(C, X) \to \cdots$$

PROOF. We only do this for the first sequence; the proof of the second is similar. We verify that for every morphism $u : X \to A$, the composite $X \to A \to B \to C$ is zero. For this we note that u determines an obvious morphism from $X \xrightarrow{=} X$ to $A \to B$. By (TR3) this extends to a morphism from the distinguished triangle $X \xrightarrow{=} X \to 0 \to X[1]$ to the distinguished triangle $A \to B \to C \to A[1]$ and so $X \to A \to B \to C$ is zero as asserted.

We next check that if the composite of u with $A \to B$ is zero, then u lifts to $C[-1]$: then u determines an obvious morphism from $X \to 0$ to $A \to B$ and by (TR3) this extends to a morphism from the distinguished triangle $X \xrightarrow{=} X \to 0 \to X[1]$ to the distinguished triangle $C[-1] \to A \to B \to C$. This proves that u lifts to $C[-1]$. $\square$

Property (TR1) postulates that every morphism $f : A \to B$ fits in a triangle $A \to B \to C \to A[1]$. If $A \to B \to C' \to A[1]$ is another such triangle, then (TR3) asserts the existence of a morphism $\gamma : C \to C'$ defining a morphism of triangles. This defines for every X a morphism between the two long exact sequences associated to $\mathrm{Hom}_{\mathfrak{D}}(X, -)$. This is an isomorphism by the 5-lemma and so $\gamma_* : \mathrm{Hom}_{\mathfrak{D}}(X, C) \cong \mathrm{Hom}_{\mathfrak{D}}(X, C')$ is an isomorphism functorial in X. The Yoneda lemma then implies that γ is an isomorphism (but we repeat that γ need not be unique).

We now formulate Property (TR4), known as the *axiom of octahedron*, and refer to section $\S 1.1$ of [2] for a more thorough discussion. It formalizes the properties in $K(\mathfrak{C})$ of the relations between the distinguished triangles can be formed from nested complexes $W_0^\bullet \subset W_1^\bullet \subset W_2^\bullet$: the quotient complex $W_2^\bullet/W_1^\bullet$ is also a quotient of two quotient complexes (namely $W_1^\bullet/W_0^\bullet$ and $W_2^\bullet/W_0$) and this observation can be formulated in terms of distinguished triangles. Concretely:

TR4: if we have a diagram $W_0 \to W_1 \to W_2$ in $\mathfrak{D}$ and distinguished triangles $W_0 \to W_1 \to W_{10} \to W_0[1]$ and $W_1 \to W_2 \to W_{21} \to W_1[1]$, so that we can form the diagram

$$\begin{tikzcd}
W_{21} \arrow[dd, dashed, "{[1]}"'] \arrow[dr, "{[1]}"] & & W_2 \arrow[ll] \\
& W_1 \arrow[ur] \arrow[dl] & \\
W_{10} \arrow[rr, "{[1]}"] & & W_0 \arrow[ul] \arrow[uu, dashed]
\end{tikzcd},$$

where the dashed arrows are defined as composites making the two lateral triangles commute, then we can obtain the outer square also by means of two other distinguished triangles with a common vertex, namely

$$\begin{tikzcd}
W_{21} \arrow[dd, "{[1]}"'] & & W_2 \arrow[ll, dashed] \arrow[dl] \\
& W_{20} \arrow[ul] \arrow[dr, "{[1]}"] & \\
W_{10} \arrow[ur] \arrow[rr, dashed, "{[1]}"] & & W_0 \arrow[uu]
\end{tikzcd}$$

We will encounter triangulated categories $\mathfrak{D}$ that are the derived category of inequivalent abelian categories. It is therefore useful to have a structure on $\mathfrak{D}$ which helps us to find such an abelian category. The following notion is the answer to that: it mimics the properties of the full subcategory $D^{\geq 0}(\mathfrak{C})$ resp. $D^{\leq 0}(\mathfrak{C})$ of objects in $D(\mathfrak{C})$ that are exact in degree < 0 resp. > 0. The intersection of these two subcategories is equivalent to $\mathfrak{C}$.

DEFINITION 6.3. A *t-structure* on a triangulated category $\mathfrak{D}$ consists of giving two full subcategories $\mathfrak{D}^{\geq 0}$ and $\mathfrak{D}^{\leq 0}$ so that if we put $\mathfrak{D}^{\geq n} := T^{-n}(\mathfrak{D}^{\geq 0})$ and $\mathfrak{D}^{\leq n} := T^{-n}(\mathfrak{D}^{\leq 0})$, then

- **(t1):** $\mathfrak{D}^{\leq 0} \subset \mathfrak{D}^{\leq 1}$ and $\mathfrak{D}^{\geq 0} \supset \mathfrak{D}^{\geq 1}$,
- **(t2):** $\mathrm{Hom}_{\mathfrak{D}}(A, B) = 0$ when $A \in \mathfrak{D}^{\leq 0}$ and $B \in \mathfrak{D}^{\geq 1}$,
- **(t3):** Every $A \in \mathfrak{D}$ is an extension of an object of $\mathfrak{D}^{\geq 1}$ by one of $\mathfrak{D}^{\leq 0}$, that is, fits in a distinguished triangle $\tau_{\leq 0}A \to A \to \tau_{\geq 1}A \to \tau_{\leq 0}A[1]$ with $\tau_{\leq 0}A \in \mathfrak{D}^{\leq 0}$ and $\tau_{\geq 1}A \in \mathfrak{D}^{\geq 1}$.

When endowed with a t-structure, $\mathfrak{D}$ is called a *t-category*; we then refer to the full subcategory $\mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$ as its *heart*.

REMARKS 6.4. These properties indeed hold when $\mathfrak{D} = D(\mathfrak{C})$. For (t1) and (t3) this is obvious. For (t2) observe that for every $f \in \mathrm{Hom}_{D(\mathfrak{C})}(A^\bullet, B^\bullet)$, the composite $\tau_{\leq 0}A^\bullet \to A^\bullet \xrightarrow{f} B^\bullet \to \tau_{\geq 1}B^\bullet$ is zero and that the first and the last map are quasi-isomorphisms in case $A^\bullet \in D(\mathfrak{C})^{\leq 0}$ and $B^\bullet \in D(\mathfrak{C})^{\geq 1}$.

Note that properties (t1) and (t2) imply that when $A \in \mathfrak{D}^{\leq k}$ and $Y \in \mathfrak{D}^{\geq l}$ with $k < l$, then $\mathrm{Hom}_{\mathfrak{D}}(A, B) = 0$.

Assigning to A the morphism $\tau_{\leq 0}A \to A$ defines in fact a right adjoint for the inclusion $\mathfrak{D}^{\leq 0} \subset \mathfrak{D}$ and hence is unique. To see this, consider for every $X \in \mathfrak{D}^{\leq 0}$ the exact sequence

$$\mathrm{Hom}(X, (\tau_{\geq 1}A)[-1]) \to \mathrm{Hom}(X, \tau_{\leq 0}A) \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, \tau_{\geq 1}A).$$

Since $(\tau_{\geq 1}A)[-1] \in \mathfrak{D}^{\geq 2}$, the extremal terms vanish by the previous remark so that $\mathrm{Hom}(X, \tau_{\leq 0}A) \xrightarrow{\cong} \mathrm{Hom}(X, A)$. It also follows that the morphism $\tau_{\leq n}A := T^{-n}\tau_{\leq 0}T^nA \to A$ is a right adjoint for the inclusion $\mathfrak{D}^{\leq n} \subset \mathfrak{D}$. A similar argument shows that the morphism $A \to \tau_{\geq n}A := T^{1-n}\tau_{\geq 1}T^{n-1}A$ is a *left* adjoint for the inclusion $\mathfrak{D}^{\geq n} \subset \mathfrak{D}$: for every $X \in \mathfrak{D}^{\geq n}$, the natural map $\mathrm{Hom}(\tau_{\geq n}A, X) \xrightarrow{\cong} \mathrm{Hom}(A, X)$ is an isomorphism.

We also note that if $A \to B \to C \to A[1]$ is distinguished with $A, C \in \mathfrak{D}^{\leq k}$, then $B \in \mathfrak{D}^{\leq k}$. For this it suffices to show that we have $\mathrm{Hom}(B, X) = 0$ when $X \in \mathcal{D}^{\geq k+1}$ and this follows from the exactness of the Hom-sequence $0 = \mathrm{Hom}_{\mathfrak{D}}(C, X) \to \mathrm{Hom}_{\mathfrak{D}}(B, X) \to \mathrm{Hom}_{\mathfrak{D}}(A, X) = 0$. Similarly, $A, C \in \mathfrak{D}^{\geq k}$ implies $B \in \mathfrak{D}^{\geq k}$. In particular, $A, C \in \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$ implies $B \in \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$.

We want to prove that $\mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$ is an abelian category. The construction of the mapping cone in an abelian category $\mathfrak{C}$ suggests how to proceed, for if $f : A \to B$ is a $\mathfrak{C}$-morphism, then its kernel and cokernel can be expressed in terms of the mapping cone and the truncation operators: the mapping cone C_f is the complex $0 \to A \xrightarrow{f} B \to 0$, where B is put in degree 0 and hence $\tau_{\geq 0} C_f = \mathrm{Coker}(f)$ and $\tau_{\leq -1} C_f = \mathrm{Ker}(f)[1]$.

Returning to the situation of Definition 6.3, put $\mathfrak{C} := \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$ and let $f : A \to B$ be a morphism in $\mathfrak{C}$. We then form a distinguished triangle $A \to B \to C \to A[1]$. Since $B \to C \to A[1] \to B[1]$ is also distinguished, the fact that B and $A[1]$ lie in $\mathfrak{D}^{\geq -1} \cap \mathfrak{D}^{\leq 0}$ implies that the same holds for C: $C \in \mathfrak{D}^{\geq -1} \cap \mathfrak{D}^{\leq 0}$. It follows that both $\tau_{\leq -1} C[-1]$ and $\tau_{\geq 0} C$ are in $\mathfrak{C}$. We then invoke (t3) to write C as an extension of an element of $\mathfrak{C}$ by an element of $\mathfrak{C}[1]$ and this yields a square (of which the dashed arrows are obtained as composites):

$$
\begin{tikzcd}
\tau_{\leq -1}C \arrow[dr] \arrow[dd, dashed, "\alpha"', "{[1]}"] & & \tau_{\geq 0}C \arrow[ll, "{[1]}"'] \\
& C \arrow[ur] \arrow[dl, "{[1]}"'] & \\
A \arrow[rr, "f"] & & B \arrow[ul] \arrow[uu, dashed, "\beta"]
\end{tikzcd}
,
$$

PROPOSITION 6.5. *Then $\alpha[-1] : \tau_{\leq -1} C[-1] \to A$ and $\beta : B \to \tau_{\geq 0} C$ is a kernel resp. cokernel for f in $\mathfrak{C}$ (so that $\mathfrak{C}$ has kernels and cokernels). The cokernel of $\alpha[-1]$ may be identified with the kernel of β and thus $\mathfrak{C}$ becomes an abelian category.*

If $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathfrak{D}$ with A and C in $\mathfrak{C}$, then $B \in \mathfrak{C}$ and this produces an exact sequence in $\mathfrak{C}$. Moreover, every exact sequence in $\mathfrak{C}$ is of this form.

The functors $\tau_{\leq k}$ and $\tau_{\geq l}$ commute (their composite is zero when $k < l$) and $H^0 : \mathfrak{D} \to \mathfrak{C}$, $A \mapsto \tau_{\geq 0}\tau_{\leq 0} A \cong \tau_{\leq 0}\tau_{\geq 0} A$ defines a cohomological functor: if we put $H^k := T^{-k} H^0 T^k$, then a distinguished triangle in $\mathfrak{D}$ gives rise to a long exact sequence in $\mathfrak{C}$.

In case, $\cap_{n\in\mathbb{Z}} \mathfrak{D}^{\leq n} = 0 = \cap_{n\in\mathbb{Z}} \mathfrak{D}^{\geq n}$ (we then say that the t-structure is nondegenerate*), $A \in \mathfrak{D}^{\leq 0}$ resp. $A \in \mathfrak{D}^{\geq 1}$ is equivalent to $H^k(A) = 0$ for all $k > 0$ resp. for all $k \leq 0$.*

The proofs are not difficult and can be found in §1.3 of [2].

6.2. Our main example: perverse sheaves. Let W be a complex-algebraic variety of finite type. We fix a ring R and take as our central category the derived category $D^b_c(W; R)$. We will define a t-structure as follows.

THEOREM-DEFINITION 6.6 (Abelian category of perverse sheaves). *Let $p : \mathbb{Z}_{\geq p} \to \mathbb{Z}$ be a function and denote by ${}^p D^{\leq 0}(W; R) \subset D^+_c(W; R)$*

resp. ${}^pD^{\geq 0}(W;R) \subset D^b_c(W;R)$ *be the full subcategory of complexes* $\mathcal{P}^\bullet$ *of* R_X*-modules with the property that for every closed irreducible subvariety* $i_Y : Y \subset W$*,* $\mathbb{R}^k i_Y^* \mathcal{P}^\bullet$ *resp.* $\mathbb{R}^k i_Y^! \mathcal{P}^\bullet$ *vanishes on a Zariski open-dense subset of* Y *for* $k > p(\dim Y)$ *resp. for* $k < p(\dim Y)$*. This pair defines a nondegenerate* t*-structure on* $D^b_c(W;R)$*. The associated abelian category defined by their intersection is called the category of* p-perverse sheaves *on* W*, denoted here by* ${}^p\mathfrak{P}(W;R)$*.*

The most interesting case is when $p(n) = -n$; we then simply refer to the corresponding abelian category as the one of perverse sheaves and denote it simply by $\mathfrak{P}(W;R)$). So according to Corollary 3.10, the direct image of a shifted intersection cohomology sheaf, $\mathscr{IC}^\bullet_X(\mathbb{E})[\dim X]$, with $X \subset W$ irreducible and $\mathbb{E}$ a local system of $\mathbb{Q}_X$-modules of finite rank on a Zariski open-dense subset of X is a member of $\mathfrak{P}(W)$. We will that see for when we consider $\mathbf{k}_X$-modules, then simple $\mathbb{E}$ (i.e., $\mathbb{E}$ contains no proper nonzero subsheaf) these yields the simple objects of $\mathfrak{P}(W,\mathbf{k})$.

PROOF OF THEOREM 6.6. We verify properties (t1) through (t3) of Definition 6.3. Property (t1) holds trivially. Let us write $\mathfrak{D}_W$ for $D^b_c(W;R)$.

For checking (t2), let $\mathcal{F}^\bullet \in {}^p\mathfrak{D}^{\leq 0}_W$ and $\mathcal{G}^\bullet \in {}^p\mathfrak{D}^{\geq 1}_W$ be injective and let $\phi \in \mathrm{Hom}_{\mathfrak{D}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$. To prove that ϕ is zero in $\mathfrak{D}_W$, we proceed with induction on the depth of a conelike stratification adapted to the situation: choose a stratification of W with respect to which $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are constructible and let $Y \subset W$ be a closed stratum. Put $j : W \smallsetminus Y \subset W$. With induction (on the local depth of the stratification) we may assume that $j^*\phi : j^*\mathcal{F}^\bullet \to j^*\mathcal{G}^\bullet$ is zero in $\mathfrak{D}_{W\smallsetminus Y}$, or equivalently, that the composite of ϕ with $\mathcal{G}^\bullet \to j_*j^*\mathcal{G}^\bullet$ is zero. This means that ϕ factors through a morphism $\mathcal{F}^\bullet \to i_!i^*\mathcal{G}^\bullet$. Via adjointness, the latter is simply the morphism $i^*\phi : i^*\mathcal{F}^\bullet \to i^*\mathcal{G}^\bullet$. Now $\mathcal{H}^k(i^*\mathcal{F}^\bullet) = i^*\mathcal{H}^k(\mathcal{F}^\bullet)$ is locally constant on Y and zero for $k > p(0)$, so that $\tau_{\leq p(0)} i^*\mathcal{F}^\bullet \to i^*\mathcal{F}^\bullet$ is a $\mathfrak{D}_Y$-isomorphism. Similarly, we find that $i^*\mathcal{G}^\bullet \to \tau_{\geq p(0)+1} i^*\mathcal{G}^\bullet$ is a $\mathfrak{D}_Y$-isomorphism. Since the composite $\tau_{\leq p(0)} i^*\mathcal{F}^\bullet \to i^*\mathcal{F}^\bullet \to i^*\mathcal{G}^\bullet \to \tau_{\geq p(0)+1} i^*\mathcal{G}^\bullet$ is zero in $\mathfrak{D}_Y$, the same is true for $i^*\phi$ and we conclude that ϕ is zero in $\mathfrak{D}_W$.

For (t3), let $\mathcal{F}^\bullet \in \mathfrak{D}_W$ be injective and proceed as under (t2): we choose a stratification of W with respect to which $\mathcal{F}^\bullet$ is constructible, let $Y \subset W$ be a closed stratum and assume that we have done our job on $W \smallsetminus Y$, i.e., we have constructed a distinguished triangle

$$ {}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet \to j^*\mathcal{F}^\bullet \to {}^p\tau_{\geq 1} j^*\mathcal{F}^\bullet \to ({}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet)[1] $$

with ${}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet \in {}^p\mathfrak{D}^{\leq 0}_W$ and ${}^p\tau_{\geq 1} j^*\mathcal{F}^\bullet \in {}^p\mathfrak{D}^{\geq 1}_W$. We then may be tempted to take for ${}^p\tau_{\leq 0}\mathcal{F}^\bullet$ the Y-truncation of $j_*{}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet$, $\tau^Y_{\leq 0} j_*({}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet)$, but this does not come naturally with a morphism to $\mathcal{F}^\bullet$. We therefore replace ${}^p\tau_{\leq 0} j^*\mathcal{F}^\bullet$ by the mapping cone of the composite $u : \mathcal{F} \to j_*\mathcal{F}^\bullet \to$

$j_*{}^p\tau_{\geq 1}j^*\mathcal{F}^\bullet$ shifted by one: this yields a morphism $\mathcal{C}_u^\bullet[-1] \to \mathcal{F}^\bullet$ and in view of the exact triangle displayed above, $j^*\mathcal{C}_u^\bullet[-1]$ is canonically $\mathfrak{D}_{W\smallsetminus Y}$-isomorphic to ${}^p\tau_{\leq 0}j^*\mathcal{F}^\bullet$. So $j^*\mathcal{C}_u^\bullet[-1]$ is an extension of ${}^p\tau_{\leq 0}j^*\mathcal{F}^\bullet$ across Y which still maps to $\mathcal{F}^\bullet$. Then we take ${}^p\tau_{\leq 0}\mathcal{F}^\bullet := \tau^Y_{\leq p(0)}\mathcal{C}_u^\bullet[1]$ together with its natural morphism to $\mathcal{F}^\bullet$ and we let $\mathcal{F}^\bullet \to {}^p\tau_{\geq 1}\mathcal{F}^\bullet$ be the mapping cone of the morphism $\mathcal{F}^\bullet \to {}^p\tau_{\leq 0}\mathcal{F}^\bullet$. You may check that the resulting triangle is as required.

The nondegeneracy of this t-structure is clear. $\square$

THEOREM 6.7. *A simple object of ${}^p\mathfrak{P}(W, \mathbf{k})$, (i.e., an object admitting no proper subobject) is of the form $i_{X*}{}^p j_{\mathring{X}!*}\mathbb{E})$, where $i_X : X \subset W$ is irreducible and closed, $j_{\mathring{X}} : \mathring{X} \subset X$ Zariski open-dense and $\mathbb{E}$ a simple local system of finite dimensional $\mathbf{k}$-vector spaces on $\mathring{X}$. Every object of ${}^p\mathfrak{P}(W, \mathbf{k})$ admits a finite (Jordan-Hölder) filtration by perverse subsheaves such that the successive graded pieces are simple.*

OUTLINE OF THE PROOF. We first show that an object of ${}^p\mathfrak{P}(W, \mathbf{k})$ admits a Jordan-Hölder) filtration by perverse subsheaves such that the successive graded pieces are of the stated form. Let $\mathcal{P}$ be an injective complex on W representing such an object. Choose a stratification of W with respect to which $\mathcal{P}$ is constructible, let $i : Y \subset W$ be a closed stratum with complement $j : W \smallsetminus Y \subset W$, write r for $p(\dim Y)$ and put $\mathcal{Q} := j^*\mathcal{P}$. Assume with induction that we have constructed a finite filtration $0 = \mathcal{Q}_a \subset \cdots \subset \mathcal{Q}_b = \mathcal{Q}$ such that $\mathcal{Q}_k/\mathcal{Q}_{k-1}$ is isomorphic to the direct image of an element the form ${}^p j_{\mathring{X}'_k!*}\mathbb{E}_k$ with X'_k an irreducible closed subset of $W \smallsetminus Y$ and $\mathbb{E}_k$ a simple local system on a Zariski open dense subset of X'_k. Then $\mathcal{P}' := {}^p j_{!*}\mathcal{Q} = \tau^Y_{\leq r-1}j_*\mathcal{Q}$ is an object of $\mathfrak{P}(W, \mathbf{k})$ filtered by the $\mathcal{P}'_k := {}^p j_{!*}\mathcal{Q}_k = \tau^Y_{\leq r-1}j_*\mathcal{Q}_k$. Now $\mathcal{P}'_k/\mathcal{P}'_{k-1}$ is isomorphic to the direct image on X of $\tau^Y_{\leq r-1}{}^p j_{\mathring{X}'_k!*}\mathbb{E}_k$ and the latter is just ${}^p j_{\mathring{X}_k!*}\mathbb{E}_k$, where X_k is the closure of X'_k in X_k.

The cohomology sheaf $R^k i_Y^*\mathcal{P}$ is zero in degree $> r$ and $R^k i_Y^!\mathcal{P}$ is zero in degree $< r$. Consider the local systems $\mathbb{E} := R^r i_Y^*\mathcal{P}$ and $\mathbb{F} := R^r i_Y^!\mathcal{P}$. Then the natural maps $\mathcal{P} \to i_*i^*\mathcal{P}$ and $i_*i^!\mathcal{P} \to \mathcal{P}$ induce the maps in $\mathfrak{P}(X, \mathbf{k})$:

$$\alpha : i_{Y*}\mathbb{F}[-r] = i_{Y*}\tau_{\leq r}i_Y^!\mathcal{P}^\bullet \to \mathcal{P}^\bullet$$
$$\beta : \mathcal{P}^\bullet \to i_{Y*}\tau_{\geq r}i_Y^*\mathcal{P}^\bullet = i_{Y*}\mathbb{E}[-r].$$

Here α is a monomorphism and β an epimorphism and $\beta\alpha = 0$. The subquotient $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ is an object of $\mathfrak{P}(X, \mathbf{k})$ which we may identify with $\mathcal{P}'$. The choice of a Jordan-Hölder decomposition of $\mathbb{E}$ and $\mathbb{F}$ (as local systems) then yields one for $\mathcal{P}$.

It now remains to show that any nonzero morphism

$$i_{X*}{}^p j_{\mathring{X}!*}\mathbb{E} \to i_{X'*}{}^p j_{\mathring{X}!*}\mathbb{E}'$$

is an isomorphism. This is left to you: first show that we must have $X' = X$ and then observe that such a morphism is given by map of local systems $\mathbb{E} \to \mathbb{E}'$ over on an open-dense subset of X. Since both $\mathbb{E}$ and $\mathbb{E}'$ are simple such a map will be an isomorphism. $\square$

So for the middle perversity a simple object (of $\mathfrak{P}(W, \mathbf{k})$) is of the form $i_{X*}\mathscr{IC}^\bullet_X(\mathbb{E})[\dim X]$. Let us say that an object of $D^b_c(W; \mathbf{k})$ is *semisimple* if it is the direct sum of shifted simple objects of $\mathfrak{P}(W, \mathbf{k})$. The decomposition theorem can be strengthened as follows.

THEOREM 6.8. *For a projective morphism $f : W \to W'$ of complex varieties of finite type, Rf_* takes semisimple objects to semisimple objects.*

6.3. Link with $\mathscr{D}$-modules in a nutshell. The point of departure is the following observation: let $\mathbb{E}$ be a local system on a nonsingular variety W of finite dimensional complex vector spaces. Then $\mathcal{O}_W \otimes_{\mathbb{C}} \mathbb{E}$ comes with a flat connection $\nabla : \mathcal{O}_W \otimes_{\mathbb{C}} \mathbb{E} \to \Omega^1_W \otimes_{\mathbb{C}} \mathbb{E}$, $f \otimes e \mapsto df \otimes e$, whose flat sections give us back $\mathbb{E}$. Conversely, a locally free $\mathcal{O}_W$-module $\mathcal{E}$ of finite rank endowed with a flat connection $\nabla : \mathcal{E} \to \Omega^1_W \otimes_{\mathcal{O}_W} \mathcal{E}$ has as its subsheaf of flat sections a local system $\mathbb{E}$ contained in $\mathcal{E}$ with the property that the natural map $\mathcal{O}_W \otimes_{\mathbb{C}} \mathbb{E} \to \mathcal{E}$ is an isomorphism. The presence of a flat connection ∇ on $\mathcal{E}$ can also be expressed as a module structure. Covariant derivation defines an action of the Lie sheaf of vector fields $\theta_W = \mathcal{H}om_{\mathcal{O}^1_W}(\Omega_W, \mathcal{O}_W)$ on W (by $\xi \mapsto \nabla_\xi$: we have $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$). So if $\mathscr{D}_W$ denotes the sheaf of universal enveloping algebras of θ_W, then by its universal property, $\mathscr{D}_W$ will act in $\mathcal{E}$. This is quasi-coherent and locally free as a $\mathcal{O}_W$-module; in fact, if $\partial_1, \dots, \partial_m$ is a basis of θ_U for some open $U \subset W$, then the noncommutative monomials in $\partial_1^{k_1} \cdots \partial_m^{k_m}$, where $(k_1, \dots, k_m)$ runs over $\mathbb{Z}^m_{\geq 0}$ (we not assume that the ∂d_i mutually commute), make up a basis $\mathcal{O}_W$-modules. It is clear that the $\mathscr{D}_W$-module structure on $\mathcal{E}$ allows us to recover the flat connection. The simplest $\mathscr{D}_W$-module is perhaps $\mathcal{O}_W$. But Ω^k_W is also one. The notion of a solution of a system of partial differential equations on M can be now expressed algebraically: Given a finite set of partial differential operators $P_1, \dots, P_r \in H^0(W, \mathscr{D}_W)$, then form the $\mathscr{D}_W$-module $M := \mathscr{D}_W / (\sum_i \mathscr{D}_W P_i)$ and note that $f \in \mathcal{O}_W$ is a local solution to the system of PDE's $P_i(f) = 0$, $i = 1, \dots, r$ if and only we have a homomorphism of $\mathscr{D}_W$-modules $M \to \mathcal{O}_W$ which sends the image of 1 in M to f. So $\mathcal{H}_{\mathscr{D}_W}(M, \mathcal{O}_M)$ has the interpretion as the sheaf of local solutions to this system.

As any universal enveloping algebra, $\mathscr{D}_W$ comes with an increasing filtration $\mathscr{D}_{W,\leq\bullet}$, where $\mathscr{D}_{W,\leq k}$ is the image of the k-fold tensor power of θ_W, with the property that $[\mathscr{D}_{W,\leq k}, \mathscr{D}_{W,\leq l}] \subset \mathscr{D}_{W,\leq k+l-1}$. So $\operatorname{gr} \mathscr{D}_{W,\leq\bullet}$ is a commutative algebra, in this case, the symmetric algebra $\operatorname{Sym}^\bullet_{\mathcal{O}_W} \theta_W$ whose relative spec defines the total space $T^\vee W$ of the cotangent bundle of W. A finitely generated $\mathscr{D}_W$-module $\mathcal{E}$ is quasi-coherent $\mathcal{O}_W$-module and admits an increasing filtration $F_\bullet\mathcal{E}$

such that $\mathscr{D}_{W,\leq k}F_l\mathcal{E} \subset F_{k+l}\mathcal{E}$. This implies that $\mathrm{gr}^F_\bullet \mathcal{E}$ is a coherent $\mathrm{Sym}^\bullet_{\mathcal{O}_W} \theta_W$-module. As such is has a support in $T^\vee W$. This support has dimension $\geq \dim W$ and is independent of the chosen filtretion. We say that W is *holonomic* if we have equality. So then this support is pure dimension $\dim W$. We say that W is *regular*, if a certain finiteness condition is fulfilled which makes that solutions along curves have moderate growth. One form of the Riemann-Hilbert correspondence asserts that the category or perverse sheaves with complex coefficients, the abelian category $\mathfrak{P}(W, \mathbb{C})$, is equivalent to the category of finitely generated regular holonomic $\mathscr{D}_M$-modules (this is independently due to Kashiwara and Mebkhout).

Let us indicate what this correspondence is like in the 'simple' case of a shifted intersection cohomology sheaf. When $\mathring{X} \subset W$ is an irreducible nonsingular subvariety and $\mathbb{E}$ is a local system of finite dimensional complex vector spaces on $\mathring{X}$, then we may form the $\mathscr{D}_{\mathring{X}}$-module $\mathcal{O}_{\mathring{W}} \otimes_{\mathbb{C}} \mathbb{E}$. There is direct image construction (which is the ordinary direct image when $\mathring{X}$ is dense in W) that turns this into a $\mathscr{D}_W$-module that is both regular and holonomic. This is indeed the $\mathscr{D}_W$-module associated to $i_{X*}\mathscr{IC}^\bullet_X(\mathbb{E})$.

6.4. Link with Hodge theory in a nutshell. The generalized Riemann-Hilbert correspondence as described above plays a major role in putting the theory (following Morihiko Saito) in the context of Hodge theory. This leads to putting a mixed Hodge structure on the intersection cohomology group $IH^k(X;\mathbb{Q})$ (X a complex variety of finite type), which is pure when X is irreducible and proper. T o be precise, if X is irreducible and $\mathbb{E}$ is variation of polarized Hodge structure of weight w, then $IH^k(X;\mathbb{E})$ has a natural mixed Hodge structure which is pure of weight $w + k$ when X is proper. the structure is however already present on the sheaf level.

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