Teichmüller spaces and Torelli theorems for hyperkähler manifolds

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ABSTRACT. Kreck and Yang Su recently gave counterexamples to a version of the Torelli theorem for hyperkählerian manifolds as stated by Verbitsky. The initial purpose of this document (which was prepared for a seminar talk) was to extract the correct statement and to give a short proof of it. We also revisit a few of its consequences, some of which are given new (shorter) proofs.

To Shing-Tung Yau, on the occasion of his 70th birth year.

INTRODUCTION

Kreck and Yang Su [13] recently noticed that the Torelli theorem as stated by Verbitsky in [20] cannot hold. This led Verbitsky to post an erratum [21] which purports to resolve the issue. Since many subsequent papers have used his theorem, we thought it worthwhile to offer, what we hope is, a complete account, which starts out from the basics. We decided to set up things a little differently than in the primary sources, as this has the merit of giving shorter proofs and sometimes sharper statements. Among this is our definition of the Teichmüller space $\mathcal{T}$ of *hyperkählerian* complex manifold structures given up isotopy on a fixed compact manifold M and its separated quotient ($\mathcal{T}$ is almost never separated). This should be distinguished from the Teichmüller space $\mathcal{T}_{\mathrm{HK}}$ of *hyperkähler* structures, which is always separated and helps to understand the former. We found it also worthwhile to introduce the Teichmüller space $\mathcal{T}_{\mathbb{H}}$ of Einstein metrics on M, as some properties of interest here are at the end of the day properties of that space. This also leads us to the construction of universal families over the Teichmüller spaces in question, thereby recovering a recent theorem of Markman [15].

To be more concrete, what may distinguish this account from others is perhaps Proposition 2.3 (which is a key to our definition of the Teichmüller spaces), the more prominent role of the twistor families, and the absence of special (customized) topological considerations regarding covering projections (see the proof of Lemma 3.6). Furthermore, we treat a twistor deformation as if its base (a projective line) were a Shimura variety (which it certainly is not), as this yields a simple way to formulate—and leads to a short way to obtain—a recent result of Soldatenkov [19] (qualified by him as 'folklore') and Green-Kim-Laza-Robles [6] on the period map for the full cohomology of a hyperkählerian manifold. Strictly speaking this is independent of the Torelli theorem, but we could not resist to include it, because this merely comes as a bonus after the ground work done here.

Supported by the Chinese National Science Foundation.

We close this introduction with a brief glance backwards along the road traveled so far. Shortly after the Calabi conjecture became Yau's theorem, it was realized by a number of people that this could be a tool for investigating the period map for $K3$-surfaces. The first successful application was independently due to Siu [18] and the author [14], who, by making use of connected chains of twistor conics, proved that the period map for kählerian $K3$-surfaces is surjective. There were no other irreducible hyperkählerian manifolds known at the time, but it was clear that these proofs would extend to that case, once one had some control on the possible Kähler classes. For general hyperkählerian manifolds this was eventually supplied by the work of Huybrechts [10] (which used the Demailly-Paun criterion [3] for the Kähler property as an essential tool). Verbitsky was probably the first to have a clear strategy for using twistor conics to prove injectivity as well. In either case, the earlier use of chains of twistor conics served as a template for establishing properties of the period map. But the proof of Lemma 3.6 now shows that this path is somewhat roundabout in more ways than one, and in the end has prevented us from recognizing the utter simplicity of the situation. Since for K3-surfaces the Demailly-Paun criterion amounts to a classical fact, we can, with this bit of additional hindsight (and ensuing change of the year count), even more concur with Huybrechts, who wrote at the end of his 2011 Bourbaki survey of Verbitsky's work "*To conclude, the Global Torelli theorem for K3 surfaces could have been proved along the lines presented here some thirty years ago*".

It is a pleasure to acknowledge correspondence with Matthias Kreck on some of the issues that arise here. I thank Benson Farb, Dick Hain and Andrei Soldatenkov for feedback on a preliminary draft, in particular Andrei for alerting me to [5].

1. Hyperkählerian manifolds and the twistor construction

The twistor construction. A *holomorphically symplectic manifold* is (in this paper) a simply-connected compact complex manifold X which admits an everywhere nondegenerate holomorphic 2-form. A theorem of Yau asserts that every Kähler form on such a manifold contains in its cohomology class a unique *Kähler-Einstein* metric (which here means that the Ricci form of the metric is zero). This has important consequences for the deformation theory of such X.

Let us first remember that on a finite-dimensional real inner product space V, an endomorphism $E \in \mathrm{End}(V)$ is infinitesimally orthogonal if and only if the form $(v, v') \in V \times V \mapsto \langle v, E(v')\rangle$ is antisymmetric, and that this identifies such endomorphisms with $\wedge^2 V^*$. So the Kähler form, the real and the imaginary part of a symplectic holomorphic 2-form, give three infinitesimal orthogonal transformations of the real tangent bundle. The former reproduces the given complex structure (which is always flat), but the vanishing of the Ricci tensor ensures that the other two are flat as well. The real span of these three transformations is then closed under the Lie bracket, yielding a copy of the Lie algebra of $\mathrm{SO}(3)$ (which is also that of the unit quaternions $\mathbb{H}^1$). If we also add the identity, then their span is even closed under composition and the resulting algebra is a copy $\mathbb{H}_X$ of the quaternions. So $\mathbb{H}_X = \mathbb{R} \oplus \mathbb{H}_X^{\mathrm{pure}}$, with $\mathbb{H}_X^{\mathrm{pure}}$ being the Lie algebra just mentioned. Since the holonomy group of the underlying Riemann manifold will centralize $\mathbb{H}_X$, that group must be contained in a unitary group over the quaternions.

The intersection $\mathbb{H}^1_X \cap \mathbb{H}^{\text{pure}}_X$ (a 2-sphere) is the set of square roots of -1 in $\mathbb{H}_X$. It contains the given complex structure, but we now observe that this is one of many, for every element of this 2-sphere defines a (new) integrable complex structure for which the metric is Kähler. We refer to this family of complex structures as a *twistor deformation* of X. It is also explains why X is called a *hyperkähler manifold* when it is endowed with a Kähler-Einstein metric. If merely a Kähler-Einstein metric exists, then we will say that X is *hyperkählerian*. We say that a hyperkähler manifold X is *irreducible* if it does not decompose nontrivially as the product of two holomorphically symplectic manifolds; this is known to be equivalent to $\dim_{\mathbb{C}} H^{2,0}(X) = 1$ or (by Berger's classification of holonomy groups) that every flat endomorphism of its tangent bundle is contained in the copy of the quaternions defined above.

The twistor construction is best understood by starting out with the underlying Riemann manifold with a metric (that we shall denote by N; the metric is denoted g) of which we assume that the flat endomorphisms of the tangent bundle form a copy $\mathbb{H}_N \subset \operatorname{End}(TN)$ of the quaternions. The last property means that we are in the irreducible case. The multiplicative group $\mathbb{H}^\times_N$ has center $\mathbb{R}^\times$ and its commutator subgroup consists of the unit quaternions $\mathbb{H}^1_N \subset \mathbb{H}^\times_N$ (a copy of $\operatorname{Spin}(3)$). These two meet of course in the center of $\mathbb{H}^1_N$, which is $\mu_2 = \{\pm 1\}$. The Lie algebra of $\mathbb{H}^1_N$ is $\mathbb{H}^{\text{pure}}_N$ and $S_N := \mathbb{H}^1_N \cap \mathbb{H}^{\text{pure}}_N$ is the set of square roots of -1 in $\mathbb{H}_N$ and is a round 2-sphere.

The group $\mathbb{H}^\times_N$ of $\mathbb{H}_N$ acts on the tangent bundle on N. Hence we have a contragradient action of $\mathbb{H}^\times_N$ on the cotangent bundle and therefore on the space of C^∞-forms. The flatness ensures that this action commutes with exterior derivation and its adjoint, so that this action preserves the space of harmonic forms. We identify this space with $H^\bullet(N;\mathbb{R})$, so that $H^\bullet(N;\mathbb{R})$ becomes a $\mathbb{H}^\times_N$-representation. Note that by these conventions, the subgroup $\mathbb{R}^\times \subset \mathbb{H}^\times_N$ defines the opposite grading of $H^\bullet(N;\mathbb{R})$ in the sense that $t \in \mathbb{R}^\times \subset \mathbb{H}^\times_N$ acts on $H^d(N;\mathbb{R})$ as multiplication by t^{-d}. The action of $u \in \mathbb{H}^\times_N$ on $H^{4m}(N;\mathbb{R})$ is scalar multiplication with $(u\overline{u})^{-2m}$ and the linear map $H^\bullet(N;\mathbb{R}) \otimes_{\mathbb{R}} H^\bullet(N;\mathbb{R}) \to H^\bullet(N;\mathbb{R})$ defined by the cup product is one of $\mathbb{H}^\times_N$ representations.

Via the above correspondence, any element of $\mathbb{H}^{\text{pure}}_N$ determines a 2-form on N. This 2-form is harmonic and we thus obtain an embedding of $\mathbb{H}^{\text{pure}}_N$ in $H^2(N;\mathbb{R})$. We shall denote its image by P_N. Since $\mathbb{H}^{\text{pure}}_N$ is naturally oriented, so will be P_N. We shall see that in some sense, this oriented 3-dimensional subspace of $H^2(N;\mathbb{R})$ is almost a complete invariant of the metric g.

It is clear that P_N is invariant under the action of $\mathbb{H}^\times_N$. If we restrict that action to $\mathbb{H}^1_N$, then P_N is essentially the adjoint representation. We transport the norm on $\mathbb{H}^{\text{pure}}_N$ to P_N to obtain a positive quadratic form on P_N. This positive quadratic form defines a conic in the projective plane $\mathbb{P}(\mathbb{C} \otimes_{\mathbb{R}} P_N)$ that we shall denote—for reasons that become clear later—by $\mathcal{D}(P_N)$.

Each $J \in S_N$ defines an integrable complex structure that turns N into a Kähler-Einstein manifold X_J. The elements $\omega \in \mathbb{C} \otimes_{\mathbb{R}} P_N$ that satisfy $\omega(Ja, b) = \omega(a, Jb) = \sqrt{-1}\omega(a, b)$ make up a complex line in $\mathbb{C} \otimes_{\mathbb{R}} P_N$. Indeed, this is just $H^0(X_J, \Omega^2_{X_J})$. Since we have $J^*\omega = -\omega$, and J^* respects the above quadratic form, it follows that the line $H^0(X_J, \Omega^2_{X_J})$ defines a point of $\mathcal{D}(P_N)$. It is an easy exercise to verify that the map $J \in S_N \mapsto [H^0(X_J, \Omega^2_{X_J})] \in \mathcal{D}(P_N)$ is a diffeomorphism which is even conformal.

The associated variation of Hodge structure. We now can state a fundamental theorem of Hitchin-Karlhede-Lindström-Roček (Thm. 3.3 of [9]) in a form that suits our purpose best: it claims that there exists a complex structure on $N\times\mathcal{D}(P_N)$ making it a complex manifold $\mathcal{X}_N$ such that the projection onto $\mathcal{D}(P_N)$ is holomorphic and if $z\in\mathcal{D}(P_N)$ corresponds to $J\in S_N$, then the fiber over z is just X_J. The product metric yields in every fiber a Kähler metric, but, as Hitchin [8] has shown, $\mathcal{X}_N$, does *not* admit a Kähler metric. It is a remarkable fact that the fibers of the other projection onto N define holomorphic sections of $\mathcal{X}_N\to\mathcal{D}(P_N)$ (called by this community *twistor lines*), but with normal bundle isomorphic to a direct sum of $\frac{1}{2}\dim N$ copies of $\mathcal{O}_{\mathcal{D}(P_N)}(1)$. (Its underlying C^∞ vector bundle is indeed trivial: $\frac{1}{2}\dim N$ is even, and $\mathcal{O}_{\mathbb{P}^1}(1)^2$ is C^∞-isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\cong\mathcal{O}_{\mathbb{P}^1}^2$.) So such a section cannot appear as a fiber of a holomorphic map ([1]).

For any $J\in S_N$, the centralizer of J in $\mathbb{H}_N^\times$ is the intersection of $\mathbb{H}_N^\times$ with $\mathbb{R}+\mathbb{R}J$ and so is naturally identified with $\mathbb{C}^\times$. Via this identification, $\zeta\in\mathbb{C}^\times$ acts on $H^{p,q}(X_J)$ as multiplication with $\zeta^{-p}\bar\zeta^{-q}$ and hence we thus recover the Hodge decomposition as an eigenspace decomposition. If we regard $\mathbb{H}_N^\times$ as the group of real points of an algebraic group defined over $\mathbb{R}$, then this copy of $\mathbb{C}^\times$ should also be thus understood, namely as $\mathbb{S}(\mathbb{R})$, where $\mathbb{S}:=\operatorname{Res}_{\mathbb{C}|\mathbb{R}}\mathbb{G}_m$. This is what is called the *Deligne torus*, whose raison d'être is indeed the observation that a finite-dimensional representation of $\mathbb{S}(\mathbb{R})$ on a real vector space endows that vector space with a Hodge structure. Here is then a way to sum this up:

Proposition 1.1. *Let $f:\mathcal{X}_N\to\mathcal{D}(P_N)$ be the projection. Then f is holomorphic and $R^\bullet f_*\mathbb{R}_{\mathcal{X}_N}$ is a constant local system which comes with a natural action of $\mathbb{H}_N^\times$. The action of $\mathbb{H}_N^\times$ on $\mathcal{D}(P_N)$ is transitive and the stabilizer of any $z\in\mathcal{D}(P_N)$ in $\mathbb{H}_N^\times$ is a Deligne torus whose representation on the stalk over z defines the Hodge structure on $H^\bullet(X_z;\mathbb{C})$.*

So $\mathcal{D}(P_N)$ not only plays here the role of a period space, but also parametrizes the elements of a conjugacy class of homomorphisms $\mathbb{S}(\mathbb{R})\to\mathbb{H}^\times$. This is reminiscent of the data that go into the definition of a Shimura variety.

We close this section with:

Lemma 1.2. *The group of automorphisms of a holomorphically symplectic manifold X which fix a given Kähler class, is finite. This is in particular so for the group $\operatorname{Aut}_0(X)$ of automorphisms that are isotopic to the identity. If N is an Einstein manifold as above, then its group of isometries that are isotopic to the identity, $\operatorname{Aut}_0(N)$, coincides with the Aut_0 of every fiber of the associated twistor deformation.*

Proof. For the first assertion, just note that the elements of $\operatorname{Aut}_0(X)$ will fix the Kähler-Einstein metric associated with this Kähler class and since the automorphism group of a Riemann manifold is a compact Lie group, so is $\operatorname{Aut}_0(X)$. But a one-parameter subgroup of $\operatorname{Aut}_0(X)$ determines a nontrivial holomorphic vector field on X, whose contraction with the symplectic form then produces a nontrivial holomorphic 1-form. On a simply-connected complex Kähler manifold, these do not exist. The other assertions are obvious from the preceding discussion. □

[1] This counterpoint between two familiar voices—one holomorphic, the other differential-geometric—has been exploited in mathematical physics to great effect.

2. TEICHMÜLLER SPACES AND PERIOD MAPS

From now on, we fix a compact simply-connected manifold M of dimension $4m$ which admits an irreducible hyperkählerian structure. This structure determines an orientation of M (which we now fix) and an oriented 3-plane P_o in $H^2(M;\mathbb{R})$. Since M is simply-connected, $H := H^2(M;\mathbb{Z})$ is free abelian. According to Bogomolov, Beauville and Fujita there exists a nondegenerate quadratic form $q : H \to \mathbb{Z}$ such that for some positive rational number c, the identity $q(a)^m = c\int_M a^{2m}$ holds for all $a \in H$ and for which $q_\mathbb{R}$ is positive on the oriented 3-plane in P_o (when m is even, the formula determines q up to sign). They prove that form $q_\mathbb{R}$ has signature $(3,n)$, with $n := b_2(M) - 3$.

The Grassmannian of oriented positive 3-planes is contractible (it is the symmetric space of $\mathrm{O}(q_\mathbb{R})$) so that the tautological 3-plane bundle over it is trivial. So the orientation of P_o orients the whole bundle. We refer to this as a *spin structure* on M and make this part of our initial data. We shall only consider hyperkählerian structures that induce the given orientation (but as Soldatenkov [19] has noted, this is in fact automatically the case) and spin structure (for which the same property might hold—by a theorem of Donaldson this is the case for $K3$-surfaces). This spin structure determines for every positive *oriented* 2-plane Π in $H_\mathbb{R}$, a *positive cone*: $\Pi^\perp$ has signature $(1,n)$ and so the set of positive vectors in $\Pi^\perp$ make up an antipodal pair of open cones and the spin structure singles out one of them.

We denote by $h_q : H_\mathbb{C} \times H_\mathbb{C} \to \mathbb{C}$ the hermitian extension of the symmetric bilinear form associated with $q_\mathbb{R}$.

The period manifold. A hyperkählerian structure on M turns M into a Kähler manifold X, so that we have a Hodge decomposition $H_\mathbb{C} = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ with $H^{2,0}(X)$ of dimension 1. Since the cup product preserves the Hodge structure on X, the Hodge type of q will be $(-2,-2)$. The above characterization of q then shows that the Hodge decomposition is orthogonal for h_q, with h_q positive on $H^{2,0}(X) \oplus H^{0,2}(X)$ and of signature $(1,n)$ on $H^{1,1}(X)$. It also follows that $H^{2,0}(X)$ is isotropic for $q_\mathbb{C}$. Since $H^{0,2}(X) = \overline{H^{2,0}(X)}$, the Hodge decomposition is then completely given by the complex line $H^{2,0}(X)$, which, as we just observed, is isotropic for $q_\mathbb{C}$ and positive for h_q. So such Hodge structures are parametrized by an open subset $\mathcal{D}(H_\mathbb{R})$ of the nonsingular quadric $\check{\mathcal{D}}(H_\mathbb{R})$ (of complex dimension $n+1$) in $\mathbb{P}(H_\mathbb{C})$ defined by $q_\mathbb{C}$, namely the locus which parametrizes the lines that are h_q-positive. (The quadric $\check{\mathcal{D}}(H_\mathbb{R})$ is homogenous under its $\mathrm{O}(q_\mathbb{C})$-action and $\mathcal{D}(H_\mathbb{R})$ is an open $\mathrm{O}(q_\mathbb{R})$-orbit in this quadric.) It is clear that such a period manifold $\mathcal{D}(V)$ is defined for any real vector space V equipped with a nondegenerate quadratic form of signature $(p, \dim V - p)$ (but we need $p \geq 2$ to make it nonempty).

Note that a point $z \in \mathcal{D}$ determines an oriented positive 2-plane Π_z in $H_\mathbb{R}$: for the associated Hodge decomposition, the sum $H^{2,0}_z + H^{0,2}_z$ is the complexification of a 2-plane Π_z in $H_\mathbb{R}$, which is indeed canonically oriented (and hence determines a positive cone). Conversely, an oriented positive 2-plane in $H_\mathbb{R}$ determines a point of $\mathcal{D}$.

Remark 2.1. If $P \subset H_\mathbb{R}$ is a positive 3-plane, then $\mathcal{D}(P) = \mathcal{D}(H_\mathbb{R}) \cap \mathbb{P}(P_\mathbb{C}) = \check{\mathcal{D}}(H_\mathbb{R}) \cap \mathbb{P}(P_\mathbb{C})$ is a conic. We prefer to call this a *twistor conic* rather than twistor line, since that name had already been taken (in the early literature of the subject a twistor line is a section of a twistor deformation). A twistor conic is a maximal irreducible compact subspace of $\mathcal{D}$. Its Douady space is identified with the Grassmannian $\mathrm{Gr}_3^+(H_\mathbb{C})$ of h_q-positive complex 3-planes in $H_\mathbb{C}$, where one should note that the projective plane defined by such a 3-plane meets $\mathcal{D}$ in a nonsingular conic. This is a bounded symmetric domain for $\mathrm{U}(h_q)$ whose real part is the symmetric space $\mathrm{Gr}_3^+(H_\mathbb{R})$ of $\mathrm{O}(q)$ which parametrizes the twistor conics. We wonder whether this space parametrizes geometric structures on M (in a manner that for real 3-planes gives us the structure of an Einstein metric).

Teichmüller spaces. The *Teichmüller space* $\mathcal{T}(M)$ of M is for the moment just a set, namely the set of hyperkählerian structures on M given up to C^∞-isotopy. By assigning to a hyperkählerian complex structure on M the associated Hodge decomposition on H, we obtain the *period map*

$$\mathcal{P} : \mathcal{T}(M) \to \mathcal{D}(M).$$

The Kodaira-Spencer theory suggests that $\mathcal{T}(M)$ has the structure of a (perhaps non-separated) complex manifold and the local Torelli theorem would then tell us that $\mathcal{P}$ is a local isomorphism. We will establish this when we have at our disposal Proposition 2.3 below. Let us first observe that for a twistor family this gives us the period map discussed earlier. More precisely, if M is endowed with an Einstein metric and the resulting Riemann manifold is denoted N, so that we then have defined a twistor family $\mathcal{X}_N \to \mathcal{D}(P_N)$, then:

Lemma 2.2. *The action of the group $\mathbb{H}_N^1$ of unit quaternions on $H_\mathbb{R}$ leaves $q_\mathbb{R}$ invariant. We have $\mathcal{D}(P_N) = \mathcal{D}(H_\mathbb{R}) \cap \mathbb{P}(P_\mathbb{C})$ and the tautological map $\mathcal{D}(P_N) \to \mathcal{T}(M)$ composed with $\mathcal{P}$ is the inclusion $\mathcal{D}(P_N) \subset \mathcal{D}(H_\mathbb{R})$.* $\square$

The proof is left as an exercise. Since we have fixed M, we shall from now on write $\mathcal{D}$ for $\mathcal{D}(H_\mathbb{R})$ and $\mathcal{T}$ for $\mathcal{T}(M)$.

Parts of the following proposition appear in somewhat different incarnations (at least implicitly) in various places in the literature (and then with somewhat different proofs), which makes it hard to give it a proper attribution. The archetypical version is certainly the Main Lemma of Burns-Rapoport [2], which it amplifies and generalizes. We here replace their use of Bishop's analyticity theorem by a properness theorem of Fujiki (which was not available at the time). Part (iv) is due to Hassett-Tschinkel ([7], Thm. 2.1).

Proposition 2.3. *Let $\pi : \mathcal{X} \to U$ and $\pi' : \mathcal{X}' \to U$ be proper holomorphic families of hyperkählerian manifolds over the same simply-connected complex manifold U. Suppose we are given an isomorphism between the associated variations of Hodge structure in degree two: $\phi : R^2\pi_*\mathbb{Z}_{\mathcal{X}'} \cong R^2\pi'_*\mathbb{Z}_{\mathcal{X}}$. If for some $o \in U$, ϕ_o is induced by isomorphism $f_o : X_o \cong X'_o$, then there exists a proper bimeromorphic morphism $\hat{U} \to U$, a closed analytic subspace $\mathcal{Z} \subset \mathcal{X} \times_{\hat{U}} \mathcal{X}'$ flat over $\hat{U}$, and a closed proper analytic subset $K \subsetneq \hat{U}$, such that*

(i) *if $\hat{u} \in \hat{U} \smallsetminus K$ lies over $u \in U$, then $Z_{\hat{u}}$ is the graph of an isomorphism $f_{\hat{u}} : X_u \cong X'_u$ which is isotopic to f_o and induces ϕ_u; moreover f_o appears in this manner: for some $\hat{o} \in \hat{U} \smallsetminus K$ over o, we have $f_{\hat{o}} = f_o$,*
(ii) *for every $u \in U$, X_u and X'_u are bimeromorphically equivalent,*

(iii) *if there exist* $\kappa \in H^0(U, R^2\pi_*\mathbb{R})$ *and* $\kappa' \in H^0(U, R^2\pi'_*\mathbb{R})$ *which restrict to a Kähler class in every fiber of* π *resp.* π' *and* $\phi_o(\kappa'(o)) = \kappa(o)$*, then we can take* $\hat{U} = U$ *and* $\mathcal{Z}$ *will be the graph of an* U*-isomorphism* $\mathcal{X} \cong \mathcal{X}'$*,*

(iv) *the group* $\mathrm{Aut}_0(\mathcal{X}/U)$ *of automorphisms of* $\mathcal{X}/U$ *that are fiberwise isotopic to the identity is finite, specializes for every* $u \in U$ *to the group* $\mathrm{Aut}_0(X_u)$ *of automorphisms of* X_u *isotopic to the identity, and is via* f_o *naturally identified with* $\mathrm{Aut}_0(\mathcal{X}'/U)$*.*

Proof. Let $D := D_{\mathcal{X}\times_U \mathcal{X}'/U}$ be the relative Douady space which parametrizes the compact analytic subspaces of $\mathcal{X} \times_U \mathcal{X}'$ contained in a fiber of $\mathcal{X} \times_U \mathcal{X}'/U$. This exists as an analytic space by a theorem of Pourcin [17], and comes with a universal family $\mathcal{Z}_D \subset \mathcal{X} \times_U \mathcal{X}' \times_U D$ that is proper and flat over D. Let $\hat{U}$ be the irreducible component of D which contains the graph of $f_o : X_o \cong X'_o$ and put $\mathcal{Z} := \mathcal{Z}_{\hat{U}}$. Since $\mathcal{X} \times_U \mathcal{X}' \to U$ is a Kähler morphism, it follows from work of Fujiki that the projection $r : \hat{U} \to U$ is proper (see the last paragraph of §1 of [4]). In particular, $r(\hat{U})$ is a closed subvariety of U. We show that it is all of U.

The local Torelli theorem implies that there exists a neighborhood V of o in U such that f_o extends to V-isomorphism $\mathcal{X}_V \to \mathcal{X}'_V$. The graph of this isomorphism appears in $\mathcal{Z}$ and so V lies in the image of r. A closed subvariety of U which contains a nonempty open set equals U and so $r(\hat{U}) = U$. It also follows that the locus K of $\hat{u} \in \hat{U}$ for which $Z_{\hat{u}}$ is not the graph of an isomorphism is a proper closed analytic subset of $\hat{U}$ and that $r(K)$ is a proper closed analytic subset of U.

Before we show that $\hat{U} \to U$ has degree one, we first address the other assertions. The proof of (ii) follows a standard argument [2]: if $\hat{u} \in \hat{U}$ lies over u, then the algebraic cycle $Z_{\hat{u}}$ on $X_u \times X'_u$ is of pure complex dimension $2n$ and contains a unique irreducible component with multiplicity one which projects with degree one on both X_u and X'_u. That component therefore establishes a bimeromorphic equivalence between the two factors. In the situation of (iii), each fiber of either family comes with a the Kähler class. So is we give each fiber the associated Einstein metric, then $Z_{\hat{u}}$ will be the graph of an isometry whenever it is the graph of an isomorphism. But the fiber metric depends continuously on the base point and so even when $\hat{u} \in K$, the correspondence $Z_{\hat{u}}$ will then implement an isometry of an open-dense subsets of X_u onto one of X'_u. As it will take a Cauchy sequence to a Cauchy sequence, this implies that $Z_{\hat{u}}$ is in fact the graph of an isometry and hence of an isomorphism. In other words, $K = \emptyset$. So $\hat{U}$ now parametrizes isometries in the same isotopy class. The local Torelli theorem then implies that $\hat{U} \to U$ is an unramified covering. But as U is simply connected and $\hat{U}$ irreducible, $\hat{U} \to U$ must be an isomorphism.

We now prove (iv). Let $u \in U$. Since $\mathrm{Aut}_0(X_u)$ acts as the identity on $H_2(X_u;\mathbb{R})$, it fixes a Kähler class. This class is uniquely represented by a Kähler-Einstein metric, which is then also preserved by the finite group $\mathrm{Aut}_0(X_u)$. Let $f_u \in \mathrm{Aut}_0(X_u)$, and assume it has finite order, d say. If we apply part (iii) to two copies of $\mathcal{X}/U$ with ϕ the identity and (o, f_o) replaced by (u, f_u), then we find that f_u extends to an automorphism F of $\mathcal{X}/U$ which will will have that same order d in every fiber. This also applies to $\mathcal{X}'/U$, and as their restrictions over $\hat{U} \smallsetminus K$ are the same, (iv) follows.

In order to show that $\hat{U} \to U$ has degree one, assume the contrary. Then there exist distinct $\hat{u}_1, \hat{u}_2 \in \hat{U} \smallsetminus K$ which lie over the same point $u \in U$. So $f_{\hat{u}_1}$ and $f_{\hat{u}_2}$ differ by an automorphism f_u of X_u. Since $\hat{U} \smallsetminus K$ is connected, it follows that f_u must be isotopic to the identity. By (iv), f_u is then the specialization of an $F \in \mathrm{Aut}_0(\mathcal{X}/\hat{U})$ which has in every fiber the *same order* d. So precomposition with F defines an automorphism of $\hat{U}/U$ which takes $\hat{u}_1$ to $\hat{u}_2$. It is clear that this automorphism cannot have a fixed point. Since U is simply connected, and $\hat{U}$ is irreducible, this can only happen when $\hat{u}_1$ and $\hat{u}_2$ belong to distinct irreducible components of $\hat{U}$ with F taking the one containing $\hat{u}_1$ to the one containing $\hat{u}_2$. This however contradicts the irreducibility of $\hat{U}$. □

We do not know whether the morphism $\hat{U} \to U$ appearing in this proposition is always an isomorphism.

Remark 2.4. A theorem of Huybrechts (Thm. 2.5 in [12]) asserts that every bimeromorphic equivalence between two compact hyperkählerian manifolds X and X' can arise as the specialization for a situation as in Proposition 2.3, with U the complex unit disk and $\mathcal{Z}$ being over $U \smallsetminus \{0\}$ the graph of an $(U \smallsetminus \{0\})$-isomorphism. This implies that such a bimeromorphic equivalence determines an isotopy class of diffeomorphisms between X and X' for which the associated map $H^2(X';\mathbb{Z}) \to H^2(X;\mathbb{Z})$ is an isomorphism of Hodge structures.

We endow $\mathcal{T}$ with an atlas whose charts are of the following type. Given an open subset U of $\mathcal{D}$, then let us agree that a *basic chart for* $\mathcal{T}$ *with domain* U is given by a complex structure on $M \times U$ for which the resulting complex manifold $\mathcal{X}$ has the property that

(i) the projection $\mathcal{X} \to U$ is holomorphic,
(ii) the fibers of $\mathcal{X} \to U$ are hyperkählerian manifolds and
(iii) its period map is given by the inclusion of U in $\mathcal{D}$.

It is clear that such an object defines an injection of U in $\mathcal{T}$. By the local Torelli theorem, every hyperkählerian complex structure on M appears as a member of such a family. In other words, the basic charts cover all of $\mathcal{T}$. We give $\mathcal{T}$ the quotient topology, that is, the finest topology, for which all the basic charts are continuous. It follows from Proposition 2.3 (with ϕ the identity and f_o isotopic to the identity) that the locus where two basic charts with domains U and U' of $\mathcal{D}$ agree, is the complement of a *closed* (analytic) subset of $U \cap U'$. This implies that each basic chart is an open map. It is now obvious that our atlas is complex-analytic and that it gives $\mathcal{T}$ the structure of a (non-separated) complex manifold for which $\mathcal{P}$ is a local isomorphism.

This also suggests that we define *separated Teichmüller space* $\mathcal{T}_s$ as follows: identify two members of our atlas with the same domain if the hypotheses of Proposition 2.3 are satisfied with ϕ the identity and f_o isotopic to the identity. In other words, two hyperkählerian complex structures on M which give complex manifolds X and X', define the same point of $\mathcal{T}_s$ if and only if there exist basic charts $\mathcal{X}/U$, $\mathcal{X}'/U$ containing X resp. X' over the same open subset $U \subset \mathcal{D}$, and a sequence $(z_i \in U)_{i=1}^{\infty}$ converging to some $o \in U$ such that X_{z_i} and X'_{z_i} differ by a C^∞-isotopy and $X_o = X$ and $X'_o = X'$. So X and X' then differ by a bimeromorphic equivalence whose graph is a limit of graphs of C^∞-isotopies of M. The space $\mathcal{T}_s$ is indeed a separated complex manifold and the period map factors through the *separated period map*

$$\mathcal{P}_s : \mathcal{T}_s \to \mathcal{D},$$

which is of course still a local isomorphism. Remark 2.4 tells us that a fiber of $\mathcal{T} \to \mathcal{T}_s$ represents a complete equivalence class of compact hyperkähler manifolds for bimeromorphic equivalence. This implies that our $\mathcal{T}_s$ is the same as what Verbitsky denotes in [20] by Teich_b.

Other moduli spaces. There is a good reason to consider also two related Teichmüller spaces, if only to better understand the formation of the separated quotient above. One is the space $\mathcal{T}_{\mathrm{HK}}$ of hyperkähler structures on M given up to C^∞-isotopy and with the metric given up to scalar (or normalized such that M has unit volume). In view of the discussion above this amounts to specifying in addition a ray in $H_{\mathbb{R}}$ (or rather, in $H^{1,1}(X;\mathbb{R})$) spanned by a Kähler class. In particular, if $\mathcal{D}_{\mathrm{HK}}$ denotes the space of pairs (z,r) with $z \in \mathcal{P}$ and r a ray in the positive cone of Π_z, then in an evident manner we have defined a *hyperkähler period map* $\mathcal{P}_{\mathrm{HK}} : \mathcal{T}_{\mathrm{HK}} \to \mathcal{D}_{\mathrm{HK}}$. Note that the projection $\mathcal{D}_{\mathrm{HK}} \to \mathcal{D}$ is a locally trivial fiber bundle with fibers having the structure of a hyperbolic n-space.

Corollary 2.5. *The moduli space $\mathcal{T}_{\mathrm{HK}}$ is a separated manifold of dimension $3n+2$ such that $\mathcal{P}_{\mathrm{HK}}$ is a local diffeomorphism. The natural map $\pi_{\mathrm{HK}} : \mathcal{T}_{\mathrm{HK}} \to \mathcal{T}$ is open, with each fiber having the structure of a convex open set in an n-dimensional hyperbolic space.*

Proof. The first assertion follows from Property (iii) of Proposition 2.3. The openness and convexity properties are general facts, which hark back to Kodaira. In our case, the rays in the positive cone of X make up a real hyperbolic space of dimension n, and so the space of rays spanned by Kähler classes make up an open convex subset this space. □

So the composite $\mathcal{T}_{\mathrm{HK}} \to \mathcal{T} \to \mathcal{T}_s$ is a submersion of separable manifolds. Its fibers are disjoint unions of convex open sets in a hyperbolic n-space and the factorization can be understood as a topological Stein factorization. Perhaps $\mathcal{T}$ is best understood via the following characterization.

Corollary 2.6. *A section of $\mathcal{T} \to \mathcal{T}_s$ over an open subset $U \subset \mathcal{T}_s$ is given by a section of $\mathcal{T}_{\mathrm{HK}} \to \mathcal{T}_s$ given up to homotopy.*

Proof. This is merely the observation that each homotopy class of sections over U has a natural convex structure, hence is canonically contractible. □

If we only retain the Einstein metric (so do not wish to single out a complex structure for which the metric is Kähler) and the associated spin structure, then we obtain another Teichmüller space $\mathcal{T}_{\mathbb{H}}$ ([2]) of Einstein metrics on M for which M has unit volume, again given up to isotopy. We have a natural projection $\mathcal{T}_{\mathrm{HK}} \to \mathcal{T}_{\mathbb{H}}$ and we give $\mathcal{T}_{\mathbb{H}}$ the quotient topology. The twistor construction makes it clear that the evident projection $\mathcal{T}_{\mathrm{HK}} \to \mathcal{T}_{\mathbb{H}}$ is a locally trivial S^2-bundle and that the "period map"

$$\mathcal{P}_{\mathbb{H}} : \mathcal{T}_{\mathbb{H}} \to \mathrm{Gr}_3^+(H_{\mathbb{R}}),$$

which assigns to an Einstein metric g on M the subspace $P_{(M,g)}$, is a local diffeomorphism. Note that its target $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ is the symmetric space of $\mathrm{O}(q_{\mathbb{R}})$, so that the arithmetic group $\mathrm{O}(q)$ acts properly discretely on it.

[2] This subscript is intended to honor Hamilton and does not stand for anything *hyper*.

3. A Torelli type theorem

The *mapping class group* $\mathrm{Mod}(M)$ of M will (for us) be the connected component group of the group of diffeomorphisms of M which preserve the initial data, that is, the orientation and the spin structure on $H^2(M;\mathbb{R})$. It is clear that $\mathrm{Mod}(M)$ acts naturally on all the Teichmüller spaces which we introduced.

Let ρ be the (orthogonal) representation of $\mathrm{Mod}(M)$ on $H^2(M)$ and denote by $\Gamma_M \subset \mathrm{GL}(H)$ its image. As we have seen, we have $\Gamma_M \subset \mathrm{O}(q)$ (with our definition we land in fact in an index 2 subgroup of $\mathrm{O}(q)$, namely the kernel of the spinor norm for $-q$). As Verbitsky had noticed, a theorem of Sullivan implies that Γ_M is an arithmetic subgroup of $\mathrm{O}(q)$ (i.e., it contains the kernel of a reduction map $\mathrm{O}(q) \to \mathrm{GL}(H/\ell H)$ for some $\ell > 0$).

Theorem 3.1 (A Torelli theorem for hyperkählerian manifolds). *The period map $\mathcal{P}_s : \mathcal{T}_s \to \mathcal{D}$ maps every connected component of $\mathcal{T}_s$ isomorphically onto $\mathcal{D}$. In particular, the $\mathrm{Mod}(M)$-stabilizer of a component acts with finite kernel on $H^2(M;\mathbb{Z})$.*

Remark 3.2. Verbitsky [20] claimed in addition that $\mathcal{P}_s$ is a finite covering, but as Kreck and Yang Su [13] have shown, this is not always true. In fact, it follows from their work (and the Torelli theorem above) that for certain M, there exist elements in $\mathrm{Ker}(\rho)$ of which no nontrivial power can appear in the monodromy group of a connected (holomorphic) family of hyperkähler manifolds.

Theorem 3.1 can be considered as a global Torelli theorem for a single component of the Teichmüller space of M. A (weak) version of a global Torelli theorem for the full Teichmüller space is then obtained as follows. A finiteness result of Huybrechts [11] implies that $\mathrm{Ker}(\rho)$ acts properly on the connected component set $\pi_0\mathcal{T}_s$ of $\mathcal{T}_s$ and has in $\pi_0\mathcal{T}_s$ only finitely many orbits. Since the period map factors through the orbit space $\mathrm{Ker}(\rho)\backslash\mathcal{T}_s$, it then follows that the induced map $\mathrm{Ker}(\rho)\backslash\mathcal{T}_s \to \mathcal{D}$ is a *finite* (trivial) covering map. By construction, this covering map comes with an action of Γ_M.

Remark 3.3 (Comparison with moduli spaces of marked hyperkählerian manifolds). Some authors consider instead of Teichmüller spaces, moduli spaces of *marked* hyperkählerian manifolds. This amounts to starting with an abstract a lattice Λ endowed with a nondegenerate quadratic form and to consider hyperkählerian manifolds X endowed with an isomorphism of lattices $H^2(X;\mathbb{Z}) \cong \Lambda$. The difference is essentially in the way we count components, for it is clear that a connected component of $\mathrm{Ker}(\rho)\backslash\mathcal{T}$ is a connected component of such a moduli space and that all such connected components are so obtained. According to Sullivan, the kernel of the representation of the diffeomorphism group on the full cohomology $H^\bullet(M)$ has a finitely generated torsion free unipotent group as a subgroup of finite index. By the finiteness property mentioned above, this remains true if we replace $H^\bullet(M)$ by $H^2(M)$. As Kreck and Su have shown, this unipotent group is nontrivial for the hyperkähler 4-fold defined by an abelian surface S (namely $\mathrm{Hilb}^3(S)/S$), so that the passage to $\mathrm{Ker}(\rho)\backslash\mathcal{T}$ may mean that we identify infinitely many connected components. Since $\mathrm{Ker}(\rho)\backslash\mathcal{T}_s$ is the separated quotient of $\mathrm{Ker}(\rho)\backslash\mathcal{T}$, we have a similar description for the separated quotients of moduli spaces of marked hyperkählerian manifolds.

If we combine these assertions, we get:

Corollary 3.4 (A weak global Torelli theorem)**.** *The set of hyperkählerian complex structures on M with a prescribed Hodge structure on $H^2(M;\mathbb{Z})$ is nonempty and decomposes into a finite number of complete bimeromorphic equivalence classes.* □

Problem 3.5. Find a concrete (discrete) invariant for hyperkählerian metrics on M, which allows us to separate the connected components of $\mathcal{T}_{\mathbb{H}}$, at least up to finite ambiguity.

Since $\mathcal{D}$ is simply connected, Theorem 3.1 is equivalent to saying that $\mathcal{P}_s$ is a covering map. This is in fact what we will prove and indeed, it is implied by:

Lemma 3.6. *Let $t \in \mathcal{T}_s$ and let (U, ϕ) be a holomorphic coordinate chart for $\mathcal{D}$ which maps U onto the unit ball in $\mathbb{C}^{n+1}$ and takes $\mathcal{P}_s(t)$ to 0. Then we have a unique section σ over U which takes $\mathcal{P}_s(t)$ to t.*

The proof of this lemma involves little more than twistor deformations and the following theorem of Huybrechts (that is based on work of Demailly-Paun [3]) which ensures that there are enough of these.

Proposition 3.7 (Huybrechts [10])**.** *If $H^{1,1}(X) \cap H^2(X;\mathbb{Z}) = \{0\}$ (in other words, if $\Pi_z^\perp \cap H = \{0\}$), then every element of the positive cone of X represents a Kähler class.* □

Let V be a real vector space defined over $\mathbb{Q}$. For a linear subspace W of V, we define its *rational closure* to be the smallest linear subspace of V defined over $\mathbb{Q}$ which contains W. If this is all of V, then we say that W is *transcendental*. It is clear that in the Grassmannian of all linear subspaces of V the non-transcendental ones form a countable union of proper subvarieties defined over $\mathbb{Q}$. In particular, the transcendental subspaces are dense.

Proposition 3.7 and Lemma 2.2 imply:

Corollary 3.8. *Let P be a transcendental positive 3-plane in $H_{\mathbb{R}}$. Then $\mathcal{P}_s$ maps every connected component of $\mathcal{P}_s^{-1}\mathcal{D}(P)$ isomorphically onto $\mathcal{D}(P)$.*

The proof below both re-arranges and simplifies some of the material in [20].

Proof of Lemma 3.6. Let r be the supremum of the $a \in (0, 1]$ for which there exists a section over the open ball $B_{<a}$ defined by $\rho < a$. Then $r > 0$, because $\mathcal{P}_s$ is open. We must show that $r = 1$. Suppose $r < 1$. Since $\mathcal{P}_s$ is a local homeomorphism between separated spaces, two sections defined on the same connected subset of $\mathcal{D}$ are equal when they are equal at some point. So if B_r denotes the ball $\rho \le r$, then we have a section σ defined over its interior $B_{<r}$. Let $z \in \partial B_r$. A positive line ℓ in $\Pi_z^\perp$ determines a twistor conic $\mathcal{D}(\ell + \Pi_z)$. We can (and will) take ℓ such that $\mathcal{D}(\ell + \Pi_z)$ is transversal to the tangent space of ∂B_r at z (an open condition) and ℓ is transcendental (this condition is dense). Then $\ell + \Pi_z$ is transcendental and $B_r \cap \mathcal{D}(\ell + \Pi_z)$ is near z a manifold with boundary, with z being a boundary point. It follows from Corollary 3.8 that the restriction of σ to $B_{<r} \cap \mathcal{D}(\ell + \Pi_z)$ extends across z. So we have a section σ_z on an open ball neighborhood U_z of z in U such that σ and σ_z take the same value in some point of $U_z \cap B_{<r}$. Since $U_z \cap B_{<r}$ is connected, it follows that σ and σ_z coincide on $U_z \cap B_{<r}$. A useful feature of taking the U_z to be open balls is that if U_z and U'_z meet (with $z, z' \in \partial B_r$), then both $U_z \cap U'_z$ and $U_z \cap U'_z \cap B_{<r}$ are connected. For it then follows that σ and the collection $\{\sigma_z\}_{z \in \partial B_r}$ together define a section of $\mathcal{P}_s$ on a neighborhood of B_r. Since such a neighborhood contains an open ball of radius $> r$, we get a contradiction. □

4. Refinements and other consequences

The other two period maps. Let $\mathcal{C}$ be a connected component of $\mathcal{T}_{\mathbb{H}}$. Its preimage $\mathcal{C}_{\mathrm{HK}}$ under the projection $\mathcal{T}_{\mathrm{HK}} \to \mathcal{T}_{\mathbb{H}}$ is then a connected component of $\mathcal{T}_{\mathrm{HK}}$ and the image of $\mathcal{C}_{\mathrm{HK}}$ in $\mathcal{T}$ is a connected component of $\mathcal{T}$. Since the Torelli theorem asserts that the period map identifies the separated quotient of the latter with $\mathcal{D}$, it follows that the other period maps define open embeddings $\mathcal{C}_{\mathrm{HK}} \hookrightarrow \mathcal{D}_{\mathrm{HK}}$ and $\mathcal{C} \hookrightarrow \mathcal{D}_{\mathbb{H}}$. Our goal is to say something about their images and to prove that these components come with universal families.

We begin with stating a stronger form of Proposition 3.7. Let us say that a linear form $H \to \mathbb{Z}$ is *negative* if its kernel has signature $(3, n-1)$. This of course amounts to identifying such a linear form with a vector in $H_{\mathbb{Q}}$ and then asking that this vector is negative for q, but in the present setting it is more natural to use linear forms. We need the following the two theorems.

Proposition 4.1 (Boucksom, [1], Thme. 1.2)**.** *Let X be hyperkählerian manifold. Then the set of Kähler classes in the positive cone in $H^{1,1}(X;\mathbb{R})$ is the intersection of the positive cone with the open half spaces defined by the fundamental classes of the irreducible rational curves C on X that are negative (defined by $\int_C > 0$).* $\square$

We also need the following result, due to Mongardi.

Proposition 4.2 (Mongardi [16], Thm. 1.3)**.** *For an Einstein metric g on M, the set of negative vectors in $H^\vee$ whose kernel does* not *contain the associated 3-plane P_g only depends on the connected component of the image of (M, g) in $\mathcal{T}_{\mathbb{H}}$.* $\square$

This suggests that we define $\Delta_{\mathcal{C}} \subset H^\vee$ as the set of indivisible negative linear forms $H \to \mathbb{Z}$, which for some Kähler-Einstein metric associated with an element of $\mathcal{C}$, are representable by an irreducible rational curve.

We now need some elementary properties of arrangements on the Grassmannian $\mathrm{Gr}_3^+(H_{\mathbb{R}})$. If $\delta \in H^\vee$ is negative, then $\mathrm{Gr}_3^+(\ker(\delta_{\mathbb{R}}))$ is a codimension 3-submanifold of $\mathrm{Gr}_3^+(H_{\mathbb{R}})$. In particular, $\mathrm{Gr}_3^+(H_{\mathbb{R}}) \smallsetminus \mathrm{Gr}_3^+(\ker(\delta_{\mathbb{R}}))$ is simply connected.

Lemma 4.3. *Let Δ be a subset of $H^\vee$ which consists of indivisible* negative *vectors, and is invariant under a subgroup of finite index of Γ of $\mathrm{O}(q)$. Then*

$$\mathrm{Gr}_3^+(H_{\mathbb{R}})_\Delta := \mathrm{Gr}_3^+(H_{\mathbb{R}}) \smallsetminus \cup_{\delta\in\Delta} \mathrm{Gr}_3^+(\ker(\delta_{\mathbb{R}}))$$

is open in $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ if and only if Γ has finitely many orbits in Δ. If these equivalent conditions are fulfilled, then $\{\mathrm{Gr}_3^+(\ker(\delta_{\mathbb{R}}))\}_{\delta\in\Delta}$ is locally finite on $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ so that any open subset of $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ which contains $\mathrm{Gr}_3^+(H_{\mathbb{R}})_\Delta$ is simply connected, but if they are not, then $\mathrm{Gr}_3^+(H_{\mathbb{R}})_\Delta$ has empty interior.

Proof. If P is a positive 3-plane, then it is clear that for every $0 < \varepsilon < 1$, the open subset $U_\varepsilon(P^\perp)$ of $H_{\mathbb{R}}$ consisting of $(v, w) \in P \oplus P^\perp = H_{\mathbb{R}}$ with $q(v) < -\varepsilon q(w)$ is an open neighborhood of $P^\perp \smallsetminus \{0\}$ which consists of negative vectors. The lemma follows from the assertion that such an open subset meets every Γ-orbit in H is finite, and meets every infinite union of Γ-orbits consisting of negative indivisible vectors in an infinite set. The proof of this last property is left to the reader. $\square$

Corollary 4.4. *Let $\mathcal{C}$ be a connected component of $\mathcal{T}_{\mathbb{H}}$. Then $\Delta_{\mathcal{C}}$ is a finite union of Γ_N-orbits and $\mathcal{P}_{\mathbb{H}}$ maps $\mathcal{C}$ diffeomorphically onto $\mathrm{Gr}^+(H_{\mathbb{R}})_{\Delta_{\mathcal{C}}}$. In particular, $\mathcal{C}$ is simply connected.*

Proof. The Torelli theorem 3.1 implies that $\mathcal{P}_{\mathbb{H}}$ defines an open embedding of $\mathcal{C}$ in $\mathrm{Gr}^+(H_{\mathbb{R}})$. This image is of course Γ_N-invariant. Propositions 4.1 and 4.2 imply that this image in $\mathcal{P}_{\mathbb{H}}$ must be contained in $\mathrm{Gr}^+(H_{\mathbb{R}})_{\Delta_{\mathcal{C}}}$. Lemma 4.3 then tells us that $\Delta_{\mathcal{C}}$ must be a finite union of Γ_N-orbits and defines a locally finite arrangement on $\mathrm{Gr}^+(H_{\mathbb{R}})$. Then turning back to Proposition 4.1, we see that this implies that the image of $\mathcal{C}$ is exactly $\mathrm{Gr}^+(H_{\mathbb{R}})_{\Delta_{\mathcal{C}}}$. □

The Torelli theorem asserts among other things that the $\mathrm{Mod}(M)$-stabilizer of $\mathcal{C}$, $\mathrm{Mod}(M)_{\mathcal{C}}$, acts with finite kernel on $H = H^2(M;\mathbb{Z})$. Property (iv) of Proposition 2.3 implies that for every Einstein metric g on M which represents a point of $\mathcal{C}$, the group of isometries of (M,g) that are isotopic to the identity only depends on $\mathcal{C}$ and hence can be identified with the kernel of this action. We therefore denote this kernel by $\mathrm{Aut}_0(\mathcal{C})$.

Corollary 4.5. *The Teichmüller space of Einstein metrics on M, $\mathcal{T}_{\mathbb{H}}$, carries a family of Einstein manifolds $\mathcal{N}_{\mathbb{H}}/\mathcal{T}_{\mathbb{H}}$ which is endowed with a faithful action of $\mathrm{Mod}(M)$. It is almost-universal in the sense that every family of Einstein metrics on M is a pull-back of this one, but can be so in more than one way, with the ambiguity residing in a finite group which is constant on every connected component $\mathcal{T}_{\mathbb{H}}$.*

In somewhat fancier language: $\mathcal{T}_{\mathbb{H}}$ underlies a (Deligne-Mumford) stack and this stack is a constant gerbe on every connected component.

As it suffices to prove this per connected component of $\mathcal{T}_{\mathbb{H}}$, we check this for $\mathcal{C}$. At issue is then the possible non-triviality of $\mathrm{Aut}_0(\mathcal{C})$: we need to glue the local universal deformations to a global object over $\mathcal{C}$ and this group prevents us, at least *a priori*, to do this in a canonical fashion. But as we shall see, the simply-connectivity of $\mathcal{C}$ saves us. In the argument below we will use the rigidity of finite group actions on compact manifolds: if G is a finite group and N a compact manifold, then every connected component of $\mathrm{Hom}(G, \mathrm{Diff}(N))$ is a $\mathrm{Diff}^0(N)$-orbit.

Proof of Corollary 4.5. We abbreviate $\mathrm{Aut}_0(\mathcal{C})$ by G. Choose an Einstein metric on M so that the resulting Einstein manifold N represents a point of $\mathcal{C}$. Then N comes with an action of G as isometry group. We regard the G-orbit space $\overline{N}$ as an orbifold in the metric sense, meaning that each of its points is represented by a G-orbit in N. The rigidity property just mentioned implies that the automorphism group $\overline{N}$ is trivial in the sense that it has no automorphisms that lift to an isometry isotopic to the identity. So the glueing of such local families is unique, which implies that $\mathcal{C}$ supports a family of such orbifolds, $\overline{\mathcal{N}}_{\mathcal{C}}/\mathcal{C}$. The regular part of $\overline{\mathcal{N}}^{\mathrm{reg}}_{\mathcal{C}}/\mathcal{C}$ of this family of orbifolds is (topologically) locally trivial over $\mathcal{C}$. Since $\mathcal{C}$ is simply connected, the inclusion of every fiber $\overline{N}^{\mathrm{reg}}_{\mathcal{C}}$ in $\overline{\mathcal{N}}^{\mathrm{reg}}_{\mathcal{C}}$ induces an isomorphism on fundamental groups. Hence the G-cover $N \to \overline{N}$ extends uniquely to a G-cover $\mathcal{N}_{\mathcal{C}} \to \overline{\mathcal{N}}_{\mathcal{C}}$. Then $\mathcal{N}_{\mathcal{C}}/\mathcal{C}$ is the desired family. □

It follows from Sullivan's theorem that $\mathrm{Mod}(M)$ has torsion free subgroups of finite index. If Γ is such a subgroup and normal, then Γ has trivial intersection with each $\mathrm{Aut}_0(\mathcal{C})$, so that if we pass to the orbit space of the universal family, $\Gamma\backslash\mathcal{T}_{\mathbb{H}}$ still underlies a stack with the property that over every connected component it is a constant gerbe.

We thus recover a recent theorem of Markman [15]:

Corollary 4.6. *The Teichmüller spaces $\mathcal{T}_{\mathrm{HK}}$ and $\mathcal{T}$ carry families of hyperkähler resp. hyperkählerian manifolds. These are endowed with a faithful action of $\mathrm{Mod}(M)$ and are almost-universal in the sense above.*

Proof. For $\mathcal{T}_{HK}$ this is immediate from Corollary 4.5. The corresponding result for $\mathcal{T}$ then follows from the fact that we have a descent along $\mathcal{T}_{HK} \to \mathcal{T}$ (where we note that the fibers have the structure of convex open subsets, over which we have canonical trivializations). □

A connected component $\mathcal{C}$ of $\mathcal{T}_{\mathbb{H}}$ was identified with an open subset of $\mathrm{Gr}_3^+(H_{\mathbb{R}})$, and so it inherits from this a locally symmetric metric and hence a notion of geodesic interval. The twistor construction singles out such intervals of a particular type: Let two elements of $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ be represented by the 3-planes P_0 and P_1 and assume that these have a 2-plane Π in common. Recall that by our convention, P_0 and P_1 are naturally oriented and so an orientation of Π determines a ray r_i in the orthogonal complement of Π in P_i. If we connect r_0 with r_1 in the orthogonal complement of Π in $P_0 + P_1$ (in the obvious manner) by a path $\{r_t\}_{t\in[0,1]}$, then the orthogonal complement P_t of r_t in $P_0 + P_1$ traverses a geodesic segment $[P_0, P_1]$ in $\mathrm{Gr}_3^+(H_{\mathbb{R}})$. Suppose now that P_0 represents an Einstein metric g_0 on M. Then Π defines a member of the associated twistor family and hence defines a complex structure on M for which g_0 is Kähler-Einstein. For the underlying complex manifold X, the family r_t defines an interval in its (projectivized) Kähler cone, hence gives a path of Einstein metrics *on* M that begins with g_0. If this is part of a piecewise geodesic loop $(P_0, P_1, \dots, P_k)$ with $\dim(P_{i-1} \cap P_i) \geq 2$ and $P_0 = P_k$, then we also get a loop in $\mathcal{C}$ (perhaps the most basic case is that of a small triangle with P_0, P_1, P_2 having a line in common). This means that the Einstein metric on M that we end up with must differ from g_0 by an isotopy of M.

Question 4.7. What is the subgroup of $\mathrm{Diff}^0(M)$ generated by such isotopies? Note that we are here essentially asking for a description of the structure group of the universal bundle over $\mathcal{C}$. A recent theorem of Giansiracusa-Kupers-Tshishiku [5] asserts that for a $K3$-surface M, the natural map $\mathrm{Diff}^+(M) \to \mathrm{Mod}(M)$ does not split, not even over a subgroup of finite index. So for such surfaces this must be an infinite group ([3]).

The period map for the full cohomology. In this subsection it is convenient to adopt the language of the theory of algebraic groups and in particular that of Shimura varieties.

The functor which assigns to any $\mathbb{Q}$-algebra R, the subgroup $\mathrm{SO}(q_R) \subset \mathrm{GL}(H_R)$, is represented by a $\mathbb{Q}$-algebraic group $\mathcal{SO}_q$, so that for example $\mathcal{SO}_q(\mathbb{R}) = \mathrm{SO}(q_{\mathbb{R}})$. Although $\mathrm{SO}(q_{\mathbb{R}})$ has two connected components when $n > 0$, as an algebraic group, $\mathcal{SO}_q$ is connected. We denote by $\mathcal{S}pin_q$ the algebraic universal cover $\mathcal{SO}_q$. This is a semi-simple algebraic group defined over $\mathbb{Q}$ and $\mathcal{S}pin_q(\mathbb{R})$ is the usual $\mathrm{Spin}(q_{\mathbb{R}})$ (which for $n \geq 3$ is the universal cover of $\mathrm{SO}(q_{\mathbb{R}})^\circ$ for the Hausdorff topology) and has $\mathrm{Gr}_3^+(H_{\mathbb{R}})$ as its symmetric space. We identify the kernel of $\mathcal{S}pin_q \to \mathcal{SO}_q$ with $\mu_2 = \{\pm 1\}$ and put $\mathcal{CS}pin_q := \mathcal{S}pin_q \times^{\mu_2} \mathbb{G}_m$. This is a reductive algebraic group over $\mathbb{Q}$ that can be regarded as an extension of $\mathcal{SO}_q$ by $\mathbb{G}_m$,

[3] There is a similar question for the usual Teichmüller space: if C is a closed Riemann surface of genus ≥ 2, then a closed loop in its Teichmüller space consisting of piecewise Teichmüller geodesics defines a diffeomorphism of C isotopic to the identity. What subgroup of $\mathrm{Diff}^0(C)$ do such diffeomorphisms generate? We have been asking around for a while, but no-one seems to know.

but whose commutator subgroup is $\mathcal{S}pin_q$. It is clear that the action of $\mathcal{S}pin_q \times \mathbb{G}_m$ on $H_\mathbb{Q}$ for which $\mathcal{S}pin_q$ acts via $\mathcal{SO}_q$ and $t \in \mathbb{G}_m$ as scalar multiplication with t^{-2}, factors through $\mathcal{CS}pin_q$ and makes $H_\mathbb{Q}$ a $\mathbb{Q}$-representation of $\mathcal{CS}pin$.

Any $z \in \mathcal{D}$ defines an embedding $\bar{j}_z : \mathrm{U}(1) \hookrightarrow \mathcal{SO}_q(\mathbb{R})$ that is given by rotation in the oriented plane Π_z and as the identity in $\Pi_z^\perp$. Its preimage in $\mathcal{S}pin_q(\mathbb{R})$ is a double (connected) cover in the sense that it yields a group homomorphism $j_z : \mathrm{U}(1) \to \mathcal{S}pin_q(\mathbb{R})$ whose square lifts $\bar{j}_z$. We may thus identify $\mathcal{D}$ with a distinguished conjugacy class of group monomorphisms $j_z : \mathrm{U}(1) \to \mathcal{S}pin_q(\mathbb{R})$. The preimage of the center in this new copy of $\mathrm{U}(1)$ is μ_2. The preimage of $\bar{j}_z$ under the projection $\mathcal{CS}pin(\mathbb{R}) \to \mathcal{O}_q^\circ(\mathbb{R})$ is of course a copy of $\mathrm{U}(1) \times^{\mu_2} \mathbb{R}^\times$, which is just a complicated way of writing $\mathbb{C}^\times$, but regarded as the group of real points of a group defined over $\mathbb{R}$. In other words, it is a copy of the Deligne torus $\mathcal{S}(\mathbb{R})$. Thus $z \in \mathcal{D}$ also determines a group homomorphism $J_z : \mathcal{S}(\mathbb{R}) \to \mathcal{S}pin_q(\mathbb{R})$. This identifies $\mathcal{D}$ with a conjugacy class of such homomorphisms (and endows $\mathcal{D}$ almost with the structure of a Shimura variety as any nonempty hyperplane section of $\mathcal{D}$ defined over $\mathbb{Q}$ then comes that structure).

Let g be an Einstein metric on M and denote the resulting Riemann manifold N as before. Then we have associated to N an algebra of quaternions $\mathbb{H}_N$ and a positive 3-plane $P_N \subset H_\mathbb{R}$ such that $\mathbb{H}_N^\times$ acts on P_N as the subgroup of $\mathcal{CS}pin(\mathbb{R})$ which leaves $P^\perp$ pointwise fixed. The embedding $\mathbb{H}_N^\times \hookrightarrow \mathcal{CS}pin(\mathbb{R})$ is then unique and takes the distinguished conjugacy class in $\mathrm{Hom}(\mathcal{S}(\mathbb{R}), \mathbb{H}_N^\times)$ to the distinguished conjugacy class in $\mathrm{Hom}(\mathcal{S}(\mathbb{R}), \mathcal{CS}pin(\mathbb{R}))$. Let us refer to the image of such an $\mathbb{H}_N^\times$ as *twistor subgroup* of $\mathcal{CS}pin(\mathbb{R})$. We thus recover a recent result due independently to Soldatenkov (Thm. 3.6 of [19]) and Green-Kim-Laza-Robles (Thm. 4.1 of [6]).

Corollary 4.8 (Soldatenkov, Green-Kim-Laza-Robles)**.** *Let $\mathcal{C}$ be a connected component of $\mathcal{T}$ and identify its separated quotient with $\mathcal{D}$. Then the associated variation of Hodge structure on the full cohomology $H^\bullet(M;\mathbb{Q})$ over $\mathcal{D}$ is defined by a representation of $\mathcal{CS}pin(\mathbb{R})$ on $H^\bullet(M;\mathbb{R})$.*

Proof. We have seen that this is true when we restrict to a twistor family and the corresponding twistor subgroup. Since the twistor subgroups generate $\mathcal{CS}pin(\mathbb{R})$ and the union of the twistor families make up a dense subset of $\mathcal{C}$, the assertion follows in general. □

Remark 4.9. We do not know whether this representation is defined over $\mathbb{Q}$. The isogeny space of an irreducible representation of $\mathcal{CS}pin(\mathbb{R})$ appearing in $H^\bullet(M;\mathbb{R})$ has a Hodge structure which only depends on the connected component of $\mathcal{T}$. If this Hodge structure is trivial, then the answer is yes. If this is not always the case, then we may have here an interesting invariant on $\pi_0(\mathcal{T})$. The presence of such locally constant Hodge structures is detected by the Mumford-Tate group of this variation of Hodge structure (a reductive $\mathbb{Q}$-group which contains $\mathcal{CS}pin$ or $\mathcal{CS}pin/\mu_2 \cong \mathcal{SO}_q \times \mathbb{G}_m$ as a normal $\mathbb{R}$-subgroup). The LLV-decomposition, which is the main tool in the proofs of the cited references, should help to bound the size of the quotient group.

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