A general form of relative recursion

Jaap van Oosten
Department of Mathematics
Utrecht University
P.O.Box 80.010, 3508 TA Utrecht, The Netherlands
jvoosten@math.uu.nl

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Abstract

The purpose of this note is to observe a generalization of the concept "computable in..." to arbitrary partial combinatory algebras. For every partial combinatory algebra (pca) A and every partial endofunction on A, a pca $A[f]$ is constructed such that in $A[f]$, the function f is representable by an element; a universal property of the construction is formulated in terms of Longley's 2-category of pcas and decidable applicative morphisms.

It is proved that there is always a geometric inclusion from the realizability topos on $A[f]$ into the one on A, and that there is a meaningful preorder on the partial endofunctions on A which generalizes Turing reducibility.

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Introduction

In [5], John Longley defined a 2-category of partial combinatory algebras (see 0.1.1 and 0.1.2 for definitions). The morphisms are different from what one might expect: rather than 'algebraic' maps, they are more like simulations (of one world of computation in another). Accordingly, a morphism from A to B is a total relation between the underlying sets.

Longley's definition made a lot of sense since there are nice functorial connections between pcas and their corresponding realizability categories (realizability toposes and categories of assemblies).

However, the 2-category has not been studied in great detail. It does not appear to have a lot of categorical structure, and not much is known.

Fundamental questions, such as: which properties of partial combinatory algebras are stable under isomorphism, or equivalence?, have not been answered (indeed, such questions have hardly been posed).

In this paper, I present a simple construction which is available in this category: *adjoin a partial function*. That is, given a pca A and a partial endofunction f on A, construct a pca $A[f]$ in which the function f is 'computable'. $A[f]$ should, of course, possess a universal property, and this property is formulated with respect to what Longley calls 'decidable' morphisms.

Characteristically for the non-algebraic flavour of the 2-category, $A[f]$ is not constructed by adding elements, but by modifying the application function. We obtain results generalizing the situation of computing relative to an oracle: a preorder, similar to (and generalizing) Turing reducibility, can be defined on the partial endofunctions on A; and there is always a geometric inclusion from the realizability topos on $A[f]$ into the one on A.

It is also a surprising corollary of this work that every total pca is isomorphic to a nontotal one.

0.1 Basic notions and notations

0.1.1 Partial combinatory algebras

A *partial combinatory algebra* (pca) is a set A together with a partial function $A \times A \rightharpoonup A$ called *application*, which satisfies a few conditions. We write the application as $(a, b) \mapsto ab$ or $a{\cdot}b$. $ab{\downarrow}$ means that the application ab is defined. When dealing with compound terms like $(ac)(bc)$, the definedness of the term is meant to imply the definedness of every subterm. For terms t and s, the notation $t \simeq s$ means that t is defined exactly when s is defined; and that they denote the same element when defined. $t = s$ will mean $t \simeq s$ and $t{\downarrow}$. As usual, we associate to the left, that is: abc means $(ab)c$. Elements of A are usually called *combinators*.

With these conventions, $(A, \cdot)$ is a pca iff there are combinators K and S in A satisfying, for all $a, b, c \in A$:

- $Kab = a$
- $Sab{\downarrow}$
- $Sabc \simeq ac(bc)$

For a careful account of the theory of pcas, see [1] or [5]. We recall a few properties.

In a pca A there is a choice of Booleans $\top$ and $\bot$, and a 'definition by cases' combinator C such that for all $a, b \in A$, $C\top ab = a$ and $C\bot ab = b$; C is pronounced (and written) as $\mathsf{If}\ldots\mathsf{then}\ldots\mathsf{else}\ldots$.

In A there is a choice of elements $\overline{n}$ for every natural number n, such that for every partial recursive function F of k variables there is a combinator a_F such that for every k-tuple $(n_1, \ldots, n_k)$, $a_F\overline{n_1}\cdots\overline{n_k}{\downarrow}$ precisely when $F(n_1, \ldots, n_k)$ is defined, and $a_F\overline{n_1}\cdots\overline{n_k} = \overline{F(n_1, \ldots, n_k)}$ if this is the case. There is a coding of finite sequences of elements of A, together with combinators which allow us to manipulate them: if we write $[u_0, \ldots, u_{n-1}]$ for the code of the sequence $(u_0, \ldots, u_{n-1})$, there is a combinator lh which gives the *length* of the coded sequence (i.e. $\mathsf{lh}[u_0, \ldots, u_{n-1}] = \overline{n}$), there are combinators picking the i-th element of the coded sequence (we simply write u_i for its effect) and a concatenation operator; we write $[u_0, \ldots u_{n-1}] * [v_0, \ldots, v_{m-1}]$ for the effect of this last combinator.

All these facts follow from the existence, in A, of a combinator for primitive recursion. Moreover, in every pca A there is a fixpoint combinator Y satisfying: $Yf{\downarrow}$ for all $f \in A$, and $Yfa \simeq f(Yf)a$. We shall refer to this fact as 'the recursion theorem in A'.

Every pca A is 'combinatory complete': for every term t (constructed from variables, constants from A, and the application function) and every sequence of variables $x_1, \ldots x_{n+1}$ which contains all variables in t, there is an element $\Lambda^* x_1 \cdots x_{n+1}.t$ in A which satisfies for all $a_1, \ldots a_{n+1}$ in A:

- $(\Lambda^* x_1 \cdots x_{n+1}.t)a_1 \cdots a_n{\downarrow}$
- $(\Lambda^* x_1 \cdots x_{n+1}.t)a_1 \cdots a_{n+1} \simeq t(a_1, \ldots, a_{n+1})$

0.1.2 Longley's 2-category of pcas; assemblies; decidable maps

The following definition is due to John Longley ([5]).

Definition 0.1 Let A and B be pcas. An *applicative morphism* from A to B is a function γ from A to the set $\mathcal{P}^*(B)$ of nonempty subsets of B, such that there exists an element $r \in B$ with the property that if $aa'{\downarrow}$ in A, $b \in \gamma(a)$ and $b' \in \gamma(a')$, then $rbb'{\downarrow}$ and $rbb' \in \gamma(aa')$. The element r is said to be a *realizer* for γ.

Given two applicative morphisms $\gamma : A \to B$ and $\delta : B \to C$, the composition $\delta\gamma : A \to C$ is the function $a \mapsto \bigcup_{b \in \gamma(a)} \delta(b)$ from A to $\mathcal{P}^*(C)$. It is easy, using combinatory completeness, to find a realizer for $\delta\gamma$ in terms of realizers for γ and δ.

This composition is evidently associative and has identities $a \mapsto \{a\}$, so we have a category of pcas.

This category is preorder-enriched: given two applicative morphisms $\gamma, \delta : A \to B$, we say $\gamma \preceq \delta$ if there is an $s \in B$ such that for all $a \in A$ and all $b \in \gamma(a)$, $sb \in \delta(a)$. We say that γ and δ are isomorphic if $\gamma \preceq \delta$ and $\delta \preceq \gamma$ both hold.

Two pcas are *equivalent* if there are $\gamma : A \to B$ and $\delta : B \to A$ such that both composites are isomorphic to identities.

An *assembly* on a pca A is a set X together with a map $E_X : X \to \mathcal{P}^*(A)$. If (X, E_X) and (Y, E_Y) are assemblies on A, a map of assemblies is a function $f : X \to Y$ such that there is an element $r \in A$ such that for all $x \in X$ and all $a \in E_X(x)$, $ra\!\downarrow$ and $ra \in E_Y(f(x))$. One says that the element r *tracks* the function f. Assemblies on A and maps of assemblies form a category $\mathrm{Asm}(A)$. This category is regular and comes equipped with an adjunction to the category **Set** of Sets: the forgetful (or global sections) functor $\Gamma : \mathrm{Asm}(A) \to \mathbf{Set}$ is left adjoint to the functor $\nabla : \mathbf{Set} \to \mathrm{Asm}(A)$ which sends a set X to the pair (X, E_X) where $E_X(x) = A$ for all $x \in X$.

An important justification for definition 0.1 is the following theorem by Longley: every applicative morphism $\gamma : A \to B$ determines a regular functor $\gamma^* : \mathrm{Asm}(A) \to \mathrm{Asm}(B)$ which commutes with the functors Γ; conversely, every such functor is induced by an applicative morphism which is unique up to isomorphism.

Note that $\gamma : A \to B$ establishes A as an assembly on B.

Definition 0.2 A morphism $\gamma : A \to B$ is *decidable* if there is an element $d \in B$ (the *decider* for γ) such that if $\top_A, \bot_A$ are the Booleans in A and $\top_B, \bot_B$ the Booleans in B, for every $b \in \gamma(\top_A)$ we have $db = \top_B$ and for every $b \in \gamma(\bot_A)$, $db = \bot_B$.

In [5], Longley proved

Proposition 0.3 *An applicative morphism $\gamma : A \to B$ is decidable if and only if the corresponding functor $\gamma^* : \mathrm{Asm}(A) \to \mathrm{Asm}(B)$ preserves finite coproducts. Moreover this is equivalent to: γ^* preserves the natural numbers object.*

Corollary 0.4 *If $\delta = \gamma\zeta$ is a commutative triangle of applicative morphisms such that δ and ζ are decidable, then so is γ.*

1 Definition of $A[f]$ and basic properties

Definition 1.1 Let $\gamma : A \to B$ be an applicative morphism of pcas and $f : A \rightharpoonup A$ a partial function. We say that f is *representable* w.r.t. γ if there is an element $r_f \in B$ such that for every $a \in \mathrm{dom}(f)$ and every $b \in \gamma(a)$, $r_f b\downarrow$ and $r_f b \in \gamma(f(a))$. We say that f is representable in A if f is representable w.r.t. the identity morphism on A.

The representability of f with respect to γ can also be seen as follows: let $(\mathrm{dom}(f), \gamma)$ be the regular sub-assembly of (A, γ) (as assemblies on B). Then f is representable with respect to γ if and only if f is a map of assemblies: $(\mathrm{dom}(f), \gamma) \to (A, \gamma)$.

Theorem 1.2 *For every pca A and every partial endofunction f on A there exist a pca $A[f]$ and a decidable applicative morphism $\iota_f : A \to A[f]$ with the following properties:*

i) f is representable w.r.t. ι_f;

ii) for every decidable applicative morphism $\gamma : A \to B$ such that f is representable w.r.t. γ, there is a decidable applicative morphism $\gamma_f : A[f] \to B$ such that $\gamma_f \iota_f = \gamma$, and γ_f is unique with this property. Moreover, if $\delta : A[f] \to B$ is such that $\delta\iota_f \cong \gamma$, then $\delta \cong \gamma_f$

Proof. For the construction of $A[f]$, let's agree on some notation for codes of finite sequences: if $u = [u_0, \ldots, u_{n-1}]$ and $i < n$, $u^{<i}$ denotes $[u_0, \ldots, u_{i-1}]$ and $u^{\geq i}$ denotes $[u_i, \ldots, u_{n-1}]$; for $i \leq j < n$, ${}^{i\leq}u^{<j}$ denotes $[u_i, \ldots, u_{j-1}]$. Let p, p_0, p_1 be pairing and projection combinators in A, i.e. satisfying for all $a, b \in A$: $p_0(pab) = a$ and $p_1(pab) = b$. Let Not be a combinator such that $\mathsf{Not}\top = \bot$ and $\mathsf{Not}\bot = \top$.

The underlying set of $A[f]$ will be A. We define a new application $\cdot^f$ on A as follows. For $a, b \in A$, an f-*dialogue* between a and b is a code of a sequence $u = [u_0, \ldots, u_{n-1}]$ such that for all $i < n$ there is a $v_i \in A$ such that

$$a\cdot([b] * u^{<i}) = p\bot v_i \text{ and } f(v_i) = u_i$$

We say that $a\cdot^f b$ is defined with value c, if there is an f-dialogue u between a and b such that

$$a\cdot([b] * u) = p\top c$$

We show first, that $(A, \cdot^f)$ is a pca.

Let $K_f = \Lambda^* x.p\top(\Lambda^* y.p\top x_0)$. Then clearly $K_f \cdot^f a = \Lambda^* y.p\top a$ for all $a \in A$, so $(K_f \cdot^f a)\cdot^f b = a$ for all $a, b \in A$.

For the combinator S_f, by primitive recursion it is possible to construct a term $t(x,y)$ of A such that for all u, the application $t(x,y)\cdot u$ is given by the following instructions:
$t(x,y)\cdot u =$

xu if $\forall i \leq \mathsf{lh}u\ \mathsf{Not}(p_0(xu^{<i}))$.

If i is minimal such that $p_0(xu^{<i})$, let $\alpha = p_1(xu^{<i})$ and output $y([u_0] * u^{\geq i})$ if $\forall j(i \leq j < \mathsf{lh}u \to \mathsf{Not}p_0(y([u_0] *^{i\leq} u^{<j}))$.

If j is minimal such that $p_0(y([u_0] *^{i\leq} u^{<j}))$, let $\beta = p_1(y([u_0] *^{i\leq} u^{<j}))$ and output $\alpha([\beta] * u^{\geq j})$ if $\forall k(j \leq k < \mathsf{lh}u \to \mathsf{Not}(p_0(\alpha([\beta] *^{j\leq} u^{<k}))))$.

If k is minimal such that $(p_0(\alpha([\beta] *^{j\leq} u^{<k})))$, output $(p_1(\alpha([\beta] *^{j\leq} u^{<k})))$.

Note, that $t(a,b)\cdot^f c \simeq (a\cdot^f c)\cdot^f (b\cdot^f c)$ for all a,b,c. Therefore, let

$$S_f = \Lambda^* x.p\top(\Lambda^* y.p\top t(x_0, y_0))$$

Then $(S_f \cdot^f a)\cdot^f b = t(a,b)$ for all a and b. This establishes $A[f]$ as a pca.

Note that the combinators K_f and S_f don't really depend on f. This is analogous to the fact that for a coding of Turing machine computations with oracle U, the S^m_n-functions are primitive recursive, and do not depend on U.

The map $\iota_f : A \to A[f]$ given by $a \mapsto \{a\}$ is an applicative morphism $A \to A[f]$. Indeed, if $ab = c$ then $(\Lambda^* x.p\top(ax_0))\cdot^f b = c$; so if $r = \Lambda^* yx.p\top(y_0 x_0)$ then r realizes ι_f.

The decidability of ι_f is left to the reader.

For the universal property, suppose $\gamma : A \to B$ is a decidable applicative morphism which is realized by r and let d be a decider for γ. Moreover suppose that $\overline{f}$ represents f w.r.t. γ.

Let $\pi_0, \pi_1 \in B$ be such that if $b \in \gamma(a)$ then $\pi_i b \in \gamma(p_i a)$. Similarly, let C and C' in B be such that if $b \in \gamma(a)$ and $v \in \gamma(u)$ then $Cbv \in \gamma([a] * u)$ and $C'bv \in \gamma(u * [a])$.

Now use the recursion theorem in B to find an element U such that for all b, b', v:

$$\begin{array}{rcl} Ubb'v & \simeq & \mathsf{If}\ d(\pi_0(rb(Cb'v))) \\ & & \mathsf{then}\ \pi_1(rb(Cb'v)) \\ & & \mathsf{else}\ Ubb'(C'(\overline{f}(\pi_1(rb(Cb'v))))v) \end{array}$$

The reader can check the following: suppose u is an f-dialogue between a and a' in A, $b \in \gamma(a), b' \in \gamma(a')$, $i < \mathsf{lh}u$, $v \in \gamma(u^{<i})$ and $w = C'(\overline{f}(\pi_1(rb(Cb'v))))v$.

Then $w \in \gamma(u^{\leq i})$ and $Ubb'v = Ubb'w$. Furthermore, if u is such that $a([a'] * u) = p\top c$, then $Ubb'v \in \gamma(c)$.

Therefore, choose $e \in \gamma([\,])$ and let

$$\rho = \Lambda^* xx'.Uxx'e$$

Then ρ realizes γ as applicative morphism: $A[f] \to B$. We denote this last morphism by γ_f.

Obviously, the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_f"] \arrow[dr, "\gamma"'] & A[f] \arrow[d, "\gamma_f"] \\ & B \end{tikzcd}$$

commutes on the nose. Moreover, since $\iota_f(a) = \{a\}$, if $\delta : A[f] \to B$ were such that $\delta\iota_f \cong \gamma_f\iota_f$, then $\delta \cong \gamma_f$. So γ_f is unique with respect to the property that the diagram commutes on the nose, and essentially unique with respect to the property that it commutes up to isomorphism. The decidability of γ_f is a direct consequence of Corollary 0.4 and can also be verified directly. ∎

Corollary 1.3

i) *If f is representable in A, then A and $A[f]$ are isomorphic pcas.*

ii) *If f and g are two partial endofunctions on A, the pcas $A[f][g]$ and $A[g][f]$ are isomorphic; we may therefore write $A[f, g]$.*

iii) *If $\mathcal{K}_1$ denotes Kleene's pca of partial recursive application, $f : \mathbb{N} \to \mathbb{N}$ is a partial function and $\mathcal{K}_1^f$ is the pca of partial recursive application with an oracle for f, then $\mathcal{K}_1^f$ is isomorphic to $\mathcal{K}_1[f]$.*

iv) *There exists a nontotal pca which is isomorphic to a total pca.*

Proof. The first two statements are immediate from the uniqueness statement in theorem 1.2. The third statement is easy. Finally, the fourth statement follows from the fact that $A[f]$ is never total (the element $a = \Lambda^* x.p\bot\bot$ is such that $a \cdot^f b$ is never defined), so if A is total and f is representable in A, then $A \cong A[f]$ by i). ∎

Example In [7], a total combinatory algebra $\mathcal{B}$ of partial functions on $\mathbb{N}$ is defined, and it is proved that the representable functions are those functions which are continuous for the Scott topology and satisfy some "sequentiality" condition. One might consider what happens if a Scott-continuous "parallel" function is adjoined to this: e.g. let $F : \mathcal{B} \to \mathcal{B}$ be the function such that for all $\alpha \in \mathcal{B}$, $F(\alpha)(0) = 0$ if and only if $\mathrm{dom}(\alpha) \neq \emptyset$ (and undefined else), and $F(\alpha)(n)$ is undefined for all $n > 0$. What would the representable functions of $\mathcal{B}[F]$ be? It is not hard to see that for every Scott-open subset U of $\mathcal{B}$, the function G_U, defined by

$$G_U(\alpha)(n) = \begin{cases} 0 & \text{if } n = 0 \text{ and } \alpha \in U \\ \text{undefined} & \text{else} \end{cases}$$

is representable in $\mathcal{B}[F]$. An interesting question is: are there finitely many Scott-continuous functions $G_1, \ldots, G_n$ such that in $\mathcal{B}[G_1, \ldots, G_n]$ all Scott-continuous functions from $\mathcal{B}$ to $\mathcal{B}$ are representable? My conjecture would be *no*.

Remarks

1. The construction of $A[f]$ induces a preorder on the set of partial endofunctions of A, which generalizes Turing degrees: let $f \leq_A g$ if and only if f is representable in $A[g]$ (with respect to ι_g). Since the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[g] \arrow[d] \\ A[h] \arrow[r] & A[g,h] \end{tikzcd}$$

commutes, it is easy to see that $\leq_A$ is a transitive relation (it is reflexive by Theorem 1.2(i)): suppose $f \leq_A g$ and $g \leq_A h$. Then the bottom arrow in the diagram is an isomorphism and the top arrow factors through $\iota_f : A \to A[f]$. It follows that also the map $A \to A[h]$ factors through ι_f; that is, $f \leq_A h$.

2. There is a universal solution to the problem of "making A decidable"; adjoin a function f to A where

$$f(x) = \begin{cases} \top & \text{if } p_0 x = p_1 x \\ \bot & \text{else} \end{cases}$$

3. This seems to be a good point to correct a claim made in [2], lemma 5.4. It is claimed that no total pca can be equivalent to a pca A in which there is an element z such that for all x, $zx\!\downarrow$ and $zx \neq x$. However,

this is established only if "equivalent" is replaced by "isomorphic". Therefore the original claim remains an open problem. Another open problem, as far as I know, is: give an example of two pcas which are equivalent, but not isomorphic.

2 A geometric inclusion of realizability toposes

The construction of $A[f]$ generalizes another aspect of relative recursion, known from the theory of realizability toposes. It is well known that for every pca A there exists a *realizability topos* $\mathrm{RT}(A)$. The best studied example is $\mathrm{RT}(\mathcal{K}_1)$, the *effective topos*([4]). In [4] and [6] it is explained that $\mathrm{RT}(\mathcal{K}_1^f)$ is a *subtopos* of $\mathrm{RT}(\mathcal{K}_1)$, in the topos-theoretic sense. Here we shall see that this generalizes to geometric inclusions $\mathrm{RT}(A[f]) \to \mathrm{RT}(A)$.

In [2], the authors analyze a generalization of Longley's 2-category of pcas, and characterize which applicative morphisms give rise to geometric morphisms between realizability toposes. The key concept is that of a *computationally dense* morphism. Unfortunately, the definition given in l.c. is not quite adequate; see also [3]. I state the correct definition here for pcas.

Definition 2.1 Suppose that $F : A \to B$ is a function between pcas such that the map $a \mapsto \{F(a)\}$ is an applicative morphism. F is *computationally dense* if there is an $m \in B$ with the property that for every $b \in B$ one can find an $a \in A$ such that for all $a' \in A$:

If $bF(a')\downarrow$ in B, then $aa'\downarrow$ in A, and $mF(aa') = bF(a')$

Let $P(A)$ and $P(B)$ denote the realizability triposes on A and B. Then in [2] it is shown that the map of indexed preorders induced by F^* (where $F^* : \mathcal{P}(A) \to \mathcal{P}(B)$ sends α to $F[\alpha]$) has an indexed right adjoint if and only if F is computationally dense.

In that case, the right adjoint is induced by the map $\hat{F} : \mathcal{P}(B) \to \mathcal{P}(A)$, given by

$$\hat{F}(\beta) = \{a \in A \mid mF(a) \in \beta\}$$

where $m \in B$ witnesses the computational density of F.

It is easily verified then, that if F is computationally dense and m is as in definition 2.1, then the geometric morphism $(\hat{F}, F^*)$ is an inclusion precisely when the following condition holds:

(in) There is a $c \in B$ such that for every $b \in B$ there is an $a \in A$ such that $cb = F(a)$ and $m(cb) = b$

Proposition 2.2 *The identity function $A \to A[f]$ is computationally dense and satisfies the condition* (in).

Proof. This is quite simple. Let m be an element of A such that for every $y \in A$ and every code of a sequence v, $m([y] * v) \simeq yv$.

Given $b \in A$, let $a \in A$ be such that for all $a' \in A$, $aa' \simeq \Lambda^* v.b([a'] * v)$. Then aa' is always defined. Moreover,

$$m([aa'] * v) \simeq (aa')v \simeq b([a'] * v)$$

It follows that $m\cdot^f(aa') \simeq b\cdot^f a'$ in $A[f]$. This proves that the identity function is computationally dense.

Moreover, if $c = \Lambda^* x.p\top(\Lambda^* v.p\top x_0)$ then for all a, $c[a] = p\top(\Lambda^* v.p\top[a]_0)$; hence $c\cdot^f a = \Lambda^* v.p\top[a]_0$ and

$$m([c\cdot^f a]) = (c\cdot^f a)[\,] = p\top a$$

so $m\cdot^f(c\cdot^f a) = a$, which proves (in). ∎

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