*Math. Struct. in Comp. Science* (2001), *vol.* ?, *pp.* ?–?. Printed in the United Kingdom


# Tripos theory in retrospect

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*Received December 1999*

The notion of *tripos* (Hyland, Johnstone, and Pitts 1980; Pitts 1981) was motivated by the desire to explain in what sense Higg's description of sheaf toposes as $H$-valued sets and Hyland's realizability toposes are instances of the same construction. The construction itself can be seen as the universal solution to the problem of realizing the predicates of a first order hyperdoctrine as subobjects in a logos with effective equivalence relations. In this note it is shown that the resulting logos is actually a topos if and only if the original hyperdoctrine satisfies a certain comprehension property. Triposes satisfy this property, but there are examples of non-triposes satisfying this form of comprehension.

## 1. Introduction

In 1979 I was fortunate enough to attend some lectures in which Martin Hyland described, for the first time in public, how to use Kleene's notion of recursive realizability (Kleene 1945) to build what subsequently came to be known as the *effective topos* (Hyland 1982). Although motivated by applications in constructive analysis, this topos turned out to have some intriguing properties (Hyland 1988; Rosolini 1990) of use to the related fields of type theory and programming language semantics; see (Phoa 1990) and (Reus and Streicher 1999), for example. But back in 1979, the personal significance of Hyland's lectures was that they led me to formulate the notion of 'tripos' and were the catalyst for the research that formed my PhD thesis. The description Hyland gave of his topos was analogous to Higg's version of the category of sheaves on a complete Heyting algebra $H$, in terms of '$H$-valued sets' (see Fourman and Scott 1979, Section 4). Yet the properties of the effective topos are in many respects quite different from those of a category of sheaves. For example, it is not a Grothendieck topos (see Hyland, Johnstone, and Pitts 1980, p 222). Thus the following question naturally arose:

**Question.** Is there a common generalisation, with useful properties, of the constructions of $H$-valued sets and of the effective topos?

Drawing upon Lawvere's treatment of logic in terms of *hyperdoctrines* (Lawvere 1969; Lawvere 1970), I came up with an answer to this question based on a structure of indexed collections of posets with certain properties. Peter Johnstone (my PhD supervisor) suggested naming these structures with the acronym *tripos*—standing for Topos Representing Indexed Partially Ordered

Set[†]— and the rest, as they say, is history. Well in any case, the three of us developed the initial properties of set-based triposes in (Hyland, Johnstone, and Pitts 1980)[‡] and I went on in my thesis (Pitts 1981) to develop and apply the theory of triposes over an arbitrary base.

The purpose of this note is to point out that there is a slightly more general class of hyperdoctrines than triposes answering the above **Question**. The generalisation hinges upon a careful analysis of the comprehension properties that a hyperdoctrine may possess (different from the ones in the classic paper by Lawvere (1970) to do with reflecting predicates into subobjects). Thus there are hyperdoctrines that generate toposes in just the same way that triposes do, yet whose 'powerobject' structure is weaker than that required of triposes. This is explained, and examples given, in Section 4. The scene is set by recalling material on hyperdoctrines in Section 2 and the category of partial equivalence relations of a hyperdoctrine (i.e. the 'tripos to topos' construction) in Section 3.

I have been aware of this generalisation of triposes since about 1982, but never found a good excuse to air it in print. I'm grateful to the *Tutorial Workshop on Realizability Semantics* held as part of FLoC'99 for providing the opportunity to do so. A preliminary version of this paper appears in the proceedings of that workshop (Birkedal and Rosolini 1999).

## 2. First order hyperdoctrines

We will be concerned with categorical structures that are based on the notion of *hyperdoctrine* (Lawvere 1969) and that are tailored to modelling theories in first order intuitionistic predicate logic with equality. Such a structure has a 'base' category $\mathbf{C}$ (with finite products) for modelling the sorts and terms of a first order theory; and a $\mathbf{C}$-indexed category (Johnstone and Paré 1978) $\mathcal{P}$ for modelling its formulas. Since we will only be concerned with provability rather than proofs, we restrict attention to indexed partially ordered sets rather than indexed categories. The following definition recalls the properties of $(\mathbf{C}, \mathcal{P})$ needed to soundly model first order intuitionistic predicate logic with equality. The fact that we are dealing with full first order logic, rather than a fragment of it, masks some properties ('Frobenius reciprocity', stability of the equality predicate under re-indexing, etc) which the definition would otherwise have to contain: see (Pitts 2000, Section 5) for more details.

**Definition 2.1.** Let $\mathbf{C}$ be a category with finite products. A *first order hyperdoctrine* $\mathcal{P}$ over $\mathbf{C}$ is specified by a contravariant functor $\mathcal{P} : \mathbf{C}^{op} \longrightarrow \mathbf{Poset}$ from $\mathbf{C}$ into the category $\mathbf{Poset}$ of partially ordered sets and monotone functions, with the following properties.

(i) For each $\mathbf{C}$-object $X$, the partially ordered set $\mathcal{P}(X)$ is a Heyting algebra, i.e. has a greatest element ($\top$), binary meets ($\wedge$), a least element ($\bot$), binary joins ($\vee$), and relative pseudo-complements ($\rightarrow$).

(ii) For each $\mathbf{C}$-morphism $f : X \longrightarrow Y$, the monotone function $\mathcal{P}(f) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ is a homomorphism of Heyting algebras.

† It was partly a joke: the Tripos is the name Cambridge University gives to its examinations; for example, Martin's lectures were a graduate-level course on *Constructive Analysis* for Part III of that year's Mathematical Tripos. Maybe Peter was not making a serious suggestion, but being an obedient pupil, I adopted it.

‡ How appropriate that the first paper on triposes should have three authors.

(iii) For each diagonal morphism $\Delta_X : X \longrightarrow X \times X$ in $\mathbf{C}$, the left adjoint to $\mathcal{P}(\Delta_X)$ at the top element $\top \in \mathcal{P}(X)$ exists. In other words there is an element $=_X$ of $\mathcal{P}(X \times X)$ satisfying for all $A \in \mathcal{P}(X \times X)$ that

$$\top \leq \mathcal{P}(\Delta_X)(A) \qquad \text{if and only if} \qquad =_X \leq A.$$

(iv) For each product projection $\pi : \Gamma \times X \longrightarrow \Gamma$ in $\mathbf{C}$, the monotone function $\mathcal{P}(\pi) : \mathcal{P}(\Gamma) \longrightarrow \mathcal{P}(\Gamma \times X)$ has both a left adjoint $(\exists X)_\Gamma$ and a right adjoint $(\forall X)_\Gamma$:

$$\begin{array}{lll} A \leq \mathcal{P}(\pi)(A') & \text{if and only if} & (\exists X)_\Gamma(A) \leq A' \\ \mathcal{P}(\pi)(A') \leq A & \text{if and only if} & A' \leq (\forall X)_\Gamma(A). \end{array}$$

Moreover, these adjoints are natural in $\Gamma$, i.e. given $s : \Gamma \longrightarrow \Gamma'$ in $\mathbf{C}$, we have

$$\begin{tikzcd} \mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s \times id_X)"] \arrow[d, "(\exists X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\exists X)_\Gamma"] \\ \mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(\Gamma' \times X) \arrow[r, "\mathcal{P}(s \times id_X)"] \arrow[d, "(\forall X)_{\Gamma'}"'] & \mathcal{P}(\Gamma \times X) \arrow[d, "(\forall X)_\Gamma"] \\ \mathcal{P}(\Gamma') \arrow[r, "\mathcal{P}(s)"'] & \mathcal{P}(\Gamma). \end{tikzcd}$$

The elements of $\mathcal{P}(X)$, as $X$ ranges over $\mathbf{C}$-objects, will be referred to as $\mathcal{P}$*-predicates*.

Here are two examples of first order hyperdoctrines that are relevant to the development of tripos theory.

**Example 2.2 (Hyperdoctrine of a complete Heyting algebra).** Let $H$ be a complete Heyting algebra. It determines a first order hyperdoctrine over the category $\mathbf{Set}$ of sets and functions as follows. For each set $X$ we take $\mathcal{P}(X) = H^X$, the $X$-fold product of $H$ in the category of Heyting algebras; so the $\mathcal{P}$-predicates are indexed families of elements of $H$, ordered componentwise. Given $f : X \longrightarrow Y$, $\mathcal{P}(f) : H^Y \longrightarrow H^X$ is the Heyting algebra homomorphism given by re-indexing along $f$. Equality predicates $=_X$ in $H^{X \times X}$ are given by

$$=_X(x, x') \stackrel{\text{def}}{=} \begin{cases} \top & \text{if } x = x' \\ \bot & \text{if } x \neq x' \end{cases}$$

where of course $\top$ and $\bot$ are respectively the greatest and least elements of $H$. The quantifiers use set-indexed joins ($\bigvee$) and meets ($\bigwedge$), which $H$ possesses because it is complete: given $A \in H^{\Gamma \times X}$ one has

$$(\exists X)_\Gamma(A) \stackrel{\text{def}}{=} \lambda i \in \Gamma \,.\, \textstyle\bigvee_{x \in X} A(i, x) \qquad (\forall X)_\Gamma(A) \stackrel{\text{def}}{=} \lambda i \in \Gamma \,.\, \textstyle\bigwedge_{x \in X} A(i, x)$$

in $H^\Gamma$.

**Example 2.3 (Realizability hyperdoctrines).** A *partial combinatory algebra* (PCA) is specified by a set $\mathbb{A}$ together with a partial binary operation $(-) \cdot (-) : \mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$ for which there exist elements $k, s \in \mathbb{A}$ satisfying for all $a, a', a'' \in \mathbb{A}$ that

$$\begin{array}{l} k \cdot a \downarrow \quad \text{and} \quad (k \cdot a) \cdot a' \equiv a \\ s \cdot a \downarrow, \quad (s \cdot a) \cdot a' \downarrow, \quad \text{and} \quad ((s \cdot a) \cdot a') \cdot a'' \equiv (a \cdot a'') \cdot (a' \cdot a'') \end{array}$$

where in general $e\downarrow$ means '$e$ is defined' and $e \equiv e'$ is Kleene equivalence, i.e. '$e$ is defined if and only $e'$ is, and in that case they are equal'. For example the set of natural numbers $\mathbb{N}$ is a partial combinatory algebra if we define $m \cdot n$ to be the value at $n$ (if any) of the $m$th partial recursive function, for some suitable enumeration. Another important example, in which the application function $(-)\cdot(-)$ is total, is given by the untyped lambda terms modulo $\alpha\beta$-conversion.

Given a PCA $\mathbb{A}$, we can form a first order hyperdoctrine $\mathcal{P}$ over $\mathbf{Set}$. For each set $X$, the partially ordered set $\mathcal{P}(X)$ is defined as follows. Let $P(\mathbb{A})^X$ denote the set of functions from $X$ to the powerset of $\mathbb{A}$. Let $\leq$ denote the binary relation on this set defined by: $\Phi \leq \Phi'$ if and only if there is some $a' \in \mathbb{A}$ such that for all $x \in X$ and $a \in \Phi(x)$, $a' \cdot a$ is defined and in $\Phi'(x)$. Standard properties of PCAs imply that this relation is reflexive and transitive, i.e. is a preorder. Then define $\mathcal{P}(X)$ to be the quotient of $P(\mathbb{A})^X$ by the equivalence relation generated by $\leq$; the partial order between equivalence classes $[\Phi]$ is that induced by $\leq$. Given a function $f : X \longrightarrow Y$, the function $\mathcal{P}(f) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ sends $[\Phi]$ to $[\Phi \circ f]$; it is easily seen to be well-defined, monotone and functorial.

As is well known, PCAs are functionally complete. In particular, from $k$ and $s$ one can construct elements $p, p_1, p_2$ so that $(a, a') \mapsto (p \cdot a) \cdot a'$ is an injection of $\mathbb{A} \times \mathbb{A}$ into $\mathbb{A}$ with left inverse $a \mapsto (p_1 \cdot a, p_2 \cdot a)$. From this it follows that each $\mathcal{P}(X)$ is a Heyting algebra with the Heyting operations given as follows.

$$\begin{aligned}
\top &\stackrel{\text{def}}{=} [\lambda x \in X \,.\, \mathbb{A}] \\
[\Phi] \wedge [\Phi'] &\stackrel{\text{def}}{=} [\lambda x \in X \,.\, \{(p \cdot a) \cdot a' \mid a \in \Phi(x) \;\&\; a' \in \Phi'(x)\}] \\
\bot &\stackrel{\text{def}}{=} [\lambda x \in X \,.\, \emptyset] \\
[\Phi] \vee [\Phi'] &\stackrel{\text{def}}{=} [\lambda x \in X \,.\, \{(p \cdot p_1) \cdot a \mid a \in \Phi(x)\} \cup \{(p \cdot p_2) \cdot a' \mid a' \in \Phi'(x)\}] \\
[\Phi] \rightarrow [\Phi'] &\stackrel{\text{def}}{=} [\lambda x \in X \,.\, \{a' \mid \forall a \in \Phi(x) \,.\, a' \cdot a \text{ is defined and in } \Phi'(x)\}].
\end{aligned}$$

The equality $\mathcal{P}$-predicate for $X$ is given by

$$=_X \stackrel{\text{def}}{=} [\lambda(x, x') \in X \times X \,.\, \textit{if } x = x' \textit{ then } \mathbb{A} \textit{ else } \emptyset]$$

and the quantifier operations on any $[\Phi] \in \mathcal{P}(\Gamma \times X)$ are given by set-theoretic union and intersection:

$$\begin{aligned}
(\exists X)_\Gamma([\Phi]) &\stackrel{\text{def}}{=} [\lambda i \in \Gamma \,.\, \textstyle\bigcup_{x \in X} \Phi(i, x)] \\
(\forall X)_\Gamma([\Phi]) &\stackrel{\text{def}}{=} [\lambda i \in \Gamma \,.\, \textstyle\bigcap_{x \in X} \Phi(i, x)].
\end{aligned}$$

We will call this first order hyperdoctrine the *realizability hyperdoctrine* determined by the partial combinatory algebra $\mathbb{A}$.

Let us recall briefly the connection between first order hyperdoctrines and first order logic (see Makkai 1993; or Pitts 2000, Section 5 for an overview). Given a first order signature of sorts $X$, function symbols $f : X_1, \ldots, X_n \to X$, and relation symbols $R \subset X_1, \ldots, X_n$, a *structure* $[\![-]\!]$ for the signature in a first order hyperdoctrine $(\mathbf{C}, \mathcal{P})$ assigns a $\mathbf{C}$-object $[\![X]\!]$ to each sort, a $\mathbf{C}$-morphism $[\![f]\!] : [\![X_1]\!] \times \cdots \times [\![X_n]\!] \longrightarrow [\![X]\!]$ to each function symbol, and a $\mathcal{P}$-predicate

$[\![R]\!] \in \mathcal{P}([\![X_1]\!] \times \cdots \times [\![X_n]\!])$ to each relation symbol. Then each term $t$ over the signature, with variables in $\Gamma = [x_1 : X_1, \ldots, x_n : X_n]$ and of sort $X$ say, can be interpreted as a $\mathbf{C}$-morphism $[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![X]\!]$, where $[\![\Gamma]\!] = [\![X_1]\!] \times \cdots \times [\![X_n]\!]$; and each first order formula $A$, with free variables in $\Gamma$ say, can be interpreted as a $\mathcal{P}$-predicate $[\![A]\!] \in \mathcal{P}([\![\Gamma]\!])$. The definitions of $[\![t]\!]$ and $[\![A]\!]$ proceed by induction on the structure of those expressions, using the various properties given in Definition 2.1 to interpret the logical symbols. For example, the atomic formula $t =_X t'$ asserting the equality of two terms of sort $X$ is mapped to the $\mathcal{P}$-predicate $\mathcal{P}(\langle [\![t]\!], [\![t']\!] \rangle)(=_{[\![X]\!]})$; and a universally quantified formula $\forall x : X \,.\, A$ is mapped to $(\forall [\![X]\!])_{[\![\Gamma]\!]}([\![A]\!])$. Note in particular that a first order sentence (i.e. a formula with no free variables) gets interpreted as an element of $\mathcal{P}(1)$, where 1 is the terminal object in $\mathbf{C}$. We say that the structure *satisfies* a sentence $A$ if $[\![A]\!]$ is the top element of $\mathcal{P}(1)$. This notion of satisfaction is sound for first order intuitionistic logic, in the sense that all provable sentences are satisfied. It is also complete, in the sense that a sentence is provable if it is satisfied by all structures in first order hyperdoctrines. This completeness result is not very informative because the collection of such structures includes one (in a 'Lindenbaum-Tarski' hyperdoctrine constructed from syntax) in which satisfaction coincides with provability. A more useful[§] consequence of this connection between first order logic and first order hyperdoctrines is the ability to use the familiar language of first order logic to give constructions in a hyperdoctrine that would otherwise involve complicated, order-enriched commutative diagrams. To do this one uses the following language.

**Definition 2.4 (Internal language of a hyperdoctrine).** One can associate to each first order hyperdoctrine $(\mathbf{C}, \mathcal{P})$ a signature having a sort for each $\mathbf{C}$-object, an $n$-ary function symbol for each $\mathbf{C}$-morphism of the form $X_1 \times \cdots \times X_n \longrightarrow X$ and an $n$-ary relation symbol for each $\mathcal{P}$-predicate in $\mathcal{P}(X_1 \times \cdots \times X_n)$ (for each list $X_1, \ldots, X_n$ of objects and each object $X$). The terms and first order formulas over this signature form the *internal language* of the hyperdoctrine.

There is an obvious structure in $(\mathbf{C}, \mathcal{P})$ for this signature and this enables one to use the internal language to name various $\mathbf{C}$-objects, $\mathbf{C}$-morphisms and $\mathcal{P}$-predicates; and satisfaction by this structure of sentences in the internal language can be used to express conditions on the hyperdoctrine. We make extensive use of this in the rest of the paper.

## 3. The category of partial equivalence relations of a hyperdoctrine

Higg's version of the topos of sheaves on a complete Heyting algebra and Hyland's realizability topos on a partial combinatory algebra can be obtained by applying the same construction to the indexed partially ordered sets in Examples 2.2 and 2.3 respectively. The construction only relies upon the fact that these indexed posets are first order hyperdoctrines in the sense of Definition 2.1. (In fact it only relies upon the $=\&\exists$ part of first order logic/hyperdoctrines, but the considerations in the next section need full first order logic.) Here is the construction.

**Definition 3.1 (The category $\mathbf{C}[\mathcal{P}]$).** Let $\mathbf{C}$ be a category with finite products and $\mathcal{P}$ a first order hyperdoctrine over $\mathbf{C}$. Define a category $\mathbf{C}[\mathcal{P}]$ as follows.

[§] More useful, of course, only for those people who prefer the 'element-centric' language of predicate logic; others prefer to stick with the 'arrow-centric' language of category theory.

(i) An object is a pair $(X, E)$ with $X$ a $\mathbf{C}$-object and $E \in \mathcal{P}(X \times X)$ a $\mathcal{P}$-predicate satisfying the following sentences of the internal language of $(\mathbf{C}, \mathcal{P})$ expressing that it is a partial equivalence relation (i.e. symmetric and transitive, but not necessarily reflexive).

$$\forall x, x' : X \, . \, E(x, x') \;\Rightarrow\; E(x', x) \tag{1}$$

$$\forall x, x', x'' : X \, . \, E(x, x') \;\&\; E(x', x'') \;\Rightarrow\; E(x, x''). \tag{2}$$

(ii) A morphism from $(X_1, E_1)$ to $(X_2, E_2)$ is given by a $\mathcal{P}$-predicate $F \in \mathcal{P}(X_1 \times X_2)$ satisfying the following sentences of the internal language of $(\mathbf{C}, \mathcal{P})$ expressing that it respects the partial equivalence relations $E_1$ and $E_2$, and is single-valued and total with respect to them.

$$\forall x_1 : X_1, \, x_2 : X_2 \, . \, F(x_1, x_2) \;\Rightarrow\; E_1(x_1, x_1) \;\&\; E_2(x_2, x_2) \tag{3}$$

$$\forall x_1, x_1' : X_1, \, x_2, x_2' : X_2 \, . \, E_1(x_1, x_1') \;\&\; E_2(x_2, x_2') \;\&\; F(x_1, x_2) \;\Rightarrow\; F(x_1', x_2') \tag{4}$$

$$\forall x_1 : X_1, \, x_2, x_2' : X_2 \, . \, F(x_1, x_2) \;\&\; F(x_1, x_2') \;\Rightarrow\; E_2(x_2, x_2') \tag{5}$$

$$\forall x_1 : X_1 \, . \, E_1(x_1, x_1) \;\Rightarrow\; \exists x_2 : X_2 \, . \, F(x_1, x_2). \tag{6}$$

(iii) The identity morphism on $(X, E)$ is given by $E$ itself.

(iv) Composition of $F : (X_1, E_1) \longrightarrow (X_2, E_2)$ and $G : (X_2, E_2) \longrightarrow (X_3, E_3)$ is the $\mathcal{P}$-predicate in $\mathcal{P}(X_1 \times X_3)$ determined by the formula $\exists x_2 : X_2 \, . \, F(x_1, x_2) \;\&\; G(x_2, x_3)$ in the internal language of $(\mathbf{C}, \mathcal{P})$.

That composition in $\mathbf{C}[\mathcal{P}]$ is well defined, associative and has the indicated morphisms as identities all follows from the soundness of first order hyperdoctrines for first order intuitionistic logic. The same is true for the following characterisation of finite products and subobjects in $\mathbf{C}[\mathcal{P}]$.

**Lemma 3.2 (Finite products in $\mathbf{C}[\mathcal{P}]$).**

(i) $\mathbf{C}[\mathcal{P}]$ has a terminal object: it is $(1, =_1)$, where 1 is terminal in $\mathbf{C}$.

(ii) The product of $\mathbf{C}[\mathcal{P}]$-objects $(X_1, E_1)$ and $(X_2, E_2)$ is

$$(X_1, E_1) \xleftarrow{P_1} (X_1 \times X_2, E_1 \times E_2) \xrightarrow{P_2} (X_2, E_2)$$

where $X_1 \xleftarrow{\pi_1} X_1 \times X_2 \xrightarrow{\pi_2} X_2$ is the product in $\mathbf{C}$, and $E_1 \times E_2 \in \mathcal{P}((X_1 \times X_2) \times (X_1 \times X_2))$ and $P_i \in \mathcal{P}((X_1 \times X_2) \times X_i)$ are defined by:

$$(E_1 \times E_2)(y, y') \overset{\text{def}}{\Leftrightarrow} E_1(\pi_1(y), \pi_1(y')) \;\&\; E_2(\pi_2(y), \pi_2(y'))$$

$$P_i(y, x_i) \overset{\text{def}}{\Leftrightarrow} E_i(\pi_i(y), x_i).$$

□

**Lemma 3.3 (Subobjects in $\mathbf{C}[\mathcal{P}]$).**

(i) Every subobject of a $\mathbf{C}[\mathcal{P}]$-object $(X, E)$ can be represented by a monomorphism of the form $E|A : (X, E|A) \longrightarrow (X, E)$ where $A \in \mathcal{P}(X)$ satisfies

$$\forall x : X \, . \, A(x) \;\Rightarrow\; E(x, x) \tag{7}$$

$$\forall x, x, : X \, . \, A(x) \;\&\; E(x, x') \;\Rightarrow\; A(x') \tag{8}$$

and where $E|A \in \mathcal{P}(X \times X)$ is defined from $E$ and $A$ by

$$(E|A)(x, x') \overset{\text{def}}{\Leftrightarrow} E(x, x') \,\&\, A(x).$$

This sets up an isomorphism between the sub-poset of $\mathcal{P}(X)$ consisting of those $A$ satisfying (7) and (8) and the usual poset of subobjects of $(X, E)$ in $\mathbf{C}[\mathcal{P}]$.

(ii) $\mathbf{C}[\mathcal{P}]$ has pullbacks of subobjects. The pullback of $E|A : (X, E|A) \longrightarrow (X, E)$ along a morphism $F : (X', E') \longrightarrow (X, E)$ is the subobject of $(X', E')$ determined, as in (i), by the element $A' \in \mathcal{P}(X')$ given by

$$A'(x') \overset{\text{def}}{\Leftrightarrow} \exists x : X \,.\, F(x', x) \,\&\, A(x).$$

$\square$

Recall that a category $\mathbf{E}$ is a *logos* if it has finite limits, pullback-stable images and dual images of subobjects along morphisms, and pullback-stable finite joins of subobjects (Makkai and Reyes 1977). Any category $\mathbf{E}$ with finite limits determines an $\mathbf{E}$-indexed poset $Sub_{\mathbf{E}} : \mathbf{E}^{op} \longrightarrow \mathbf{Poset}$ mapping $\mathbf{E}$-objects to their posets of subobjects and mapping $\mathbf{E}$-morphisms to pullback functions. (Well of course the posets involved may actually be po*classes* unless one assumes $\mathbf{E}$ is well-powered, but size is not an issue here.) Then we can give an alternative characterisation of logoses in terms of hyperdoctrines: *they are precisely the finitely complete categories* $\mathbf{E}$ *for which* $Sub_{\mathbf{E}}$ *is a first order hyperdoctrine over* $\mathbf{E}$. Using this fact combined with Lemmas 3.2 and 3.3, we can deduce some exactness properties of $\mathbf{C}[\mathcal{P}]$.

**Theorem 3.4.** The category $\mathbf{C}[\mathcal{P}]$ of partial equivalence relations of a first order hyperdoctrine is a logos. Moreover, all equivalence relations in $\mathbf{C}[\mathcal{P}]$ have a *quotient*, i.e. have a coequalizer whose kernel-pair is the equivalence relation (see Makkai and Reyes 1977, Definition 3.3.7); one says that a logos has *effective equivalence relations* in this case.

*Proof.* Since $\mathbf{C}[\mathcal{P}]$ has finite products (Lemma 3.2) and pullbacks of all monomorphisms (Lemma 3.3), it also has equalizers and hence all finite limits. Using the soundness of first order logic for the internal language of $(\mathbf{C}, \mathcal{P})$ and the characterisation of subobjects in Lemma 3.3, it is straightforward to deduce that $Sub_{\mathbf{C}[\mathcal{P}]}$ is a first order hyperdoctrine over $\mathbf{C}[\mathcal{P}]$ and hence that the latter is a logos. As for quotients of equivalence relations, if a monomorphism $(X \times X, (E \times E)|R) \longrightarrow (X \times X, E \times E)$ determines an equivalence relation on $(X, E)$ in $\mathbf{C}[\mathcal{P}]$, then it follows that $(X, R)$ is also a $\mathbf{C}[\mathcal{P}]$-object, and that $R$ determines a morphism from $(X, E)$ to $(X, R)$ which is the quotient of the equivalence relation. $\square$

**Definition 3.5 (Constant objects in $\mathbf{C}[\mathcal{P}]$).** We can define a functor $\Delta_{\mathcal{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{P}]$ as follows. On objects, $\Delta_{\mathcal{P}}$ maps $X$ to $(X, =_X)$; and on morphisms, $\Delta_{\mathcal{P}}$ maps $f : X_1 \longrightarrow X_2$ to the morphism from $(X_1, =_{X_1})$ to $(X_2, =_{X_2})$ given by the formula $f(x_1) =_{X_2} x_2$ in the internal language of $(\mathbf{C}, \mathcal{P})$. From Lemma 3.2 we have that $\Delta_{\mathcal{P}}$ preserves finite products; and from Lemma 3.3 it follows that $Sub_{\mathbf{C}[\mathcal{P}]}(\Delta_{\mathcal{P}}(X))$ is isomorphic to $\mathcal{P}(X)$, naturally in $X$. Objects of the form $\Delta_{\mathcal{P}}(X)$ in $\mathbf{C}[\mathcal{P}]$ are called *constant objects* in (Pitts 1981).

It is not hard to see that any $\mathbf{C}[\mathcal{P}]$-object $(X, E)$ can be presented as a quotient of the subobject of $\Delta_{\mathcal{P}}(X)$ determined by the $\mathcal{P}$-predicate $E(x,x)$, with the quotient morphism given by $E$ itself:

$$\begin{tikzcd} \llbracket E(x,x) \rrbracket \arrow[r, two heads, "E"] \arrow[d, tail] & (X, E) \\ \Delta_{\mathcal{P}}(X). & \end{tikzcd}$$

From this observation it is but a short step to the following (folklore?) characterisation of the category of partial equivalence relations of a first order hyperdoctrine.

**Theorem 3.6 (Universal property of $\Delta_{\mathcal{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{P}]$).** Let $\mathbf{C}$ be a category with finite products and let $\mathcal{P}$ be a first order hyperdoctrine over $\mathbf{C}$. Then $\Delta_{\mathcal{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{P}]$ gives the universal way of realizing $\mathcal{P}$-predicates as subobjects in a logos with effective equivalence relations. For if $\mathbf{E}$ is such a logos and $I : \mathbf{C} \longrightarrow \mathbf{E}$ is a functor preserving finite products, then there is a natural equivalence

$$\frac{\text{poset of first order hyperdoctrine morphisms:} \quad \mathcal{P}(-) \longrightarrow Sub_{\mathbf{E}}(I(-))}{\text{category of logos morphisms over } \mathbf{C}\text{:} \quad \begin{tikzcd}[column sep=small] \mathbf{C}[\mathcal{P}] \arrow[rr] & & E. \\ & \mathbf{C} \arrow[ul, "\Delta_{\mathcal{P}}"] \arrow[ur, "I"'] \arrow[u, phantom, "\cong"] & \end{tikzcd}}$$

Thus $\mathcal{P} \mapsto \mathbf{C}[\mathcal{P}]$ provides a left adjoint (qua bicategories) to the functor mapping $I : \mathbf{C} \longrightarrow \mathbf{E}$ to $Sub_{\mathbf{E}}(I(-))$. The logos morphism $\mathbf{C}[Sub_{\mathbf{E}}(I(-))] \longrightarrow \mathbf{E}$ which is the counit of this adjunction at $I : \mathbf{C} \longrightarrow \mathbf{E}$ is always full and faithful; moreover, it is also essentially surjective (and hence an equivalence) if and only if every $\mathbf{E}$-object is a quotient of a subobject of some object in the image of $I$. $\square$

In a sense the construction $(\mathbf{C}, \mathcal{P}) \mapsto \mathbf{C}[\mathcal{P}]$ falls between two stools. If one just wants to realize $\mathcal{P}$-predicates as subobjects in a logos, then the full subcategory of $\mathbf{C}[\mathcal{P}]$ consisting of subobjects of constant objects is the universal solution. On the other hand, as well as considering logoses with effective equivalence relations, it is very natural to consider ones with finite disjoint coproducts as well—i.e. Heyting pretoposes (cf. Pitts 1989). The universal solution to realizing $\mathcal{P}$-predicates in a Heyting pretopos is the mild generalisation of $\mathbf{C}[\mathcal{P}]$ (implicit in Makkai and Reyes 1977, Part II) whose objects are partial equivalence relations 'spread over a finite number of $\mathbf{C}$-objects': the definition is like that given on pp 45–46 of (Pitts 1989).

## 4. When is $\mathbf{C}[\mathcal{P}]$ a topos?

Let $\mathbf{C}$ be a category with finite products and $\mathcal{P}$ a first order hyperdoctrine over it. Suppose that $\mathbf{C}[\mathcal{P}]$ does happen to be a topos (Johnstone 1977). So for each object, and in particular for each constant object $\Delta_{\mathcal{P}}(X)$ there is a powerobject $\Omega^{\Delta_{\mathcal{P}}(X)}$ equipped with a membership relation $\in_{\Delta_{\mathcal{P}}(X)} \rightarrowtail \Delta_{\mathcal{P}}(X) \times \Omega^{\Delta_{\mathcal{P}}(X)}$ such that every subobject $\cdot \rightarrowtail \Delta_{\mathcal{P}}(X) \times Y$ arises via a pullback

$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] & \Delta_{\mathcal{P}}(X) \times Y \arrow[d, "id \times \chi"] \\ \in_{\Delta_{\mathcal{P}}(X)} \arrow[r, tail] & \Delta_{\mathcal{P}}(X) \times \Omega^{\Delta_{\mathcal{P}}(X)} \end{tikzcd} \tag{9}$$

from a unique morphism $\chi : Y \longrightarrow \Omega^{\Delta_{\mathcal{P}}(X)}$. Let us suppose that $\Omega^{\Delta_{\mathcal{P}}(X)}$ is $(PX, Eq_X)$, say. So the membership relation $\in_{\Delta_{\mathcal{P}}(X)}$ is given, as in Lemma 3.3, by a $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$. Amongst other things, $In_X$ must respect the partial equivalence relation $Eq_X$:

$$\forall x : X,\, s, s' : PX \,.\, In_X(x, s) \,\&\, Eq_X(s, s') \;\Rightarrow\; In_X(x, s'). \tag{10}$$

Specialising to the case when $Y$ is a constant object $\Delta_{\mathcal{P}}(\Gamma) = (\Gamma, =_\Gamma)$, for which subobjects

$$\cdot \rightarrowtail \Delta_{\mathcal{P}}(X) \times Y = \Delta_{\mathcal{P}}(X \times \Gamma)$$

are determined by arbitrary $\mathcal{P}$-predicates $R \in \mathcal{P}(X \times \Gamma)$, we find that the morphism $\chi$ is a $\mathcal{P}$-predicate in $\mathcal{P}(\Gamma \times PX)$ which, in order for (9) to be a pullback, satisfies

$$\forall x : X, i : \Gamma \,.\, R(x, i) \Leftrightarrow \exists s : PX \,.\, In_X(x, s) \,\&\, \chi(i, s). \tag{11}$$

Since $\chi$ does determine a morphism $(\Gamma, =_\Gamma) \longrightarrow (PX, Eq_X)$ it also satisfies

$$\forall i : \Gamma, s, s' : PX \,.\, \chi(i, s) \,\&\, \chi(i, s') \;\Rightarrow\; Eq_X(s, s') \tag{12}$$

and $\forall i : \Gamma \,.\, i =_X i \;\Rightarrow\; \exists s : PX \,.\, \chi(i, s)$, which since $i =_X i$ is $\top$ means that

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \chi(i, s). \tag{13}$$

From (10), (11) and (12) we deduce

$$\forall i : \Gamma,\, s : PX \,.\, \chi(i, s) \;\Rightarrow\; \forall x : X \,.\, In_X(x, s) \Leftrightarrow R(x, i)$$

which combined with (13) gives

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \forall x : X \,.\, In_X(x, s) \Leftrightarrow R(x, i). \tag{14}$$

So we have shown that if $\mathbf{C}[\mathcal{P}]$ is a topos, then $(\mathbf{C}, \mathcal{P})$ satisfies the following Comprehension Axiom.

**Axiom 4.1 (CA).** For all $\mathbf{C}$-objects $X$ there is a $\mathbf{C}$-object $PX$ and a $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$ such that for any $\mathbf{C}$-object $\Gamma$ and $\mathcal{P}$-predicate $R \in \mathcal{P}(X \times \Gamma)$, $\mathcal{P}$ satisfies the sentence (14) of its internal language.

**Theorem 4.2 (First order hyperdoctrine + CA = topos).** Suppose $\mathbf{C}$ is a category with finite products and $\mathcal{P}$ is a first order hyperdoctrine over $\mathbf{C}$. Then the associated category of partial equivalence relations $\mathbf{C}[\mathcal{P}]$ is a topos if and only if $(\mathbf{C}, \mathcal{P})$ satisfies (CA).

*Proof.* The argument above gives the 'only if' direction. Conversely suppose the hyperdoctrine does satisfy (CA). We will show how to construct the powerobject $\Omega^{(X,E)}$ of any object $(X, E)$ in $\mathbf{C}[\mathcal{P}]$. Define $Eq_X \in \mathcal{P}(PX \times PX)$ by

$$Eq_X(s, s') \overset{\text{def}}{\Leftrightarrow} Ex_X(s) \;\&\; \forall x : X \,.\, In_X(x, s) \Leftrightarrow In_X(x, s')$$

where

$$Ex_X(s) \stackrel{\text{def}}{\Leftrightarrow} \quad (\forall x : X \,.\, In_X(x,s) \Rightarrow E(x,x)) \mathbin{\&} \\ (\forall x, x' : X \,.\, In_X(x,s) \mathbin{\&} E(x,x') \Rightarrow In_X(x',s)).$$

One can show that $(PX, Eq_X)$ is a $\mathbf{C}[\mathcal{P}]$-object and that the formula $In_X(x,s) \mathbin{\&} Ex_X(s)$ determines (via Lemma 3.3) a subobject

$$\in \rightarrowtail (X,E) \times (PX, Eq_X). \tag{15}$$

For any other $\mathbf{C}[\mathcal{P}]$-object $(\Gamma, G)$ and subobject

$$\cdot \rightarrowtail (X,E) \times (\Gamma, G) \tag{16}$$

determined by $R \in \mathcal{P}(X \times \Gamma)$ say, let $\chi \in \mathcal{P}(\Gamma \times PX)$ be

$$\chi(i,s) \stackrel{\text{def}}{\Leftrightarrow} (\forall x : X \,.\, In_X(x,s) \Leftrightarrow R(i,x)) \mathbin{\&} G(i,i)$$

Routine calculation in the internal logic of $(\mathbf{C}, \mathcal{P})$ shows that $\chi$ is a morphism from $(\Gamma, G)$ to $(PX, Eq_X)$, that the subobject (16) is the pullback of (15) along $id \times \chi$, and that $\chi$ is the unique morphism in $\mathbf{C}[\mathcal{P}]$ with this property. So $(PX, Eq_X)$ is indeed a powerobject for $(X, E)$. Thus when $(\mathbf{C}, \mathcal{P})$ satisfies (CA), $\mathbf{C}[\mathcal{P}]$ has finite limits (Theorem 3.4) and powerobjects and hence is a topos. □

In Axiom 4.1, one way to satisfy (14) is to insist that its 'Skolemized' version holds, i.e. that there is a $\mathbf{C}$-morphism $f : \Gamma \longrightarrow PX$ satisfying

$$\forall i : \Gamma \,.\, \forall x : X \,.\, In_X(x, f(i)) \Leftrightarrow R(x,i)$$

i.e. such that $R = \mathcal{P}(id_X \times f)(In_X)$ in $\mathcal{P}(X \times \Gamma)$. (Of course, such an $f$ is not necessarily unique up to equality of $\mathbf{C}$-morphisms.) This leads to the definition of tripos.

**Definition 4.3 (Triposes).** Let $\mathbf{C}$ be a category with finite products. A $\mathbf{C}$-*tripos* is a first order hyperdoctrine $\mathcal{P}$ over $\mathbf{C}$ equipped with the following extra structure. For each $\mathbf{C}$-object $X$ there is a $\mathbf{C}$-object $PX$ and a $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$ such that given any $\Gamma$ and $R \in \mathcal{P}(X \times \Gamma)$, there is a $\mathbf{C}$-morphism $\{R\} : \Gamma \longrightarrow PX$ with $R = \mathcal{P}(id_X \times \{R\})(In_X)$. Since this implies that $\mathcal{P}$ satisfies (CA) we know from the above theorem that $\mathbf{C}[\mathcal{P}]$ is a topos—the *topos generated by the* $\mathbf{C}$*-tripos* $\mathcal{P}$.

If the base category $\mathbf{C}$ happens to be cartesian closed, one can further simplify this Skolemized version of (CA).

**Theorem 4.4 (Generic predicates).** Let $\mathbf{C}$ be a category with finite products and $\mathcal{P}$ a first order hyperdoctrine over $\mathbf{C}$

(i) If $\mathcal{P}$ is a tripos, then it possesses a *generic predicate* (Hyland, Johnstone, and Pitts 1980, Definition 1.2(iii)). By definition this means that there is some $\mathbf{C}$-object *Prop* and $\mathcal{P}$-predicate $Prf \in \mathcal{P}(Prop)$ such that for any $\Gamma$ and $A \in \mathcal{P}(\Gamma)$ there is a $\mathbf{C}$-morphism $\ulcorner A \urcorner : \Gamma \longrightarrow Prop$ with $A = \mathcal{P}(\ulcorner A \urcorner)(Prf)$.
(ii) Conversely, assuming $\mathbf{C}$ is cartesian closed, if $\mathcal{P}$ has a generic predicate, then it is a tripos.

*Proof.* For part (i), using the tripos structure of $\mathcal{P}$ we can take $Prop = P1$ and $Prf = \mathcal{P}(\langle !, id_{P1}\rangle)(In_1)$, using the isomorphism $\langle !, id_{P1}\rangle : P1 \cong 1 \times P1$. For any $A \in \mathcal{P}(\Gamma)$, we get $\mathcal{P}(\pi_2)(A) \in \mathcal{P}(1 \times \Gamma)$ and can define $\ulcorner A \urcorner$ to be $\{\mathcal{P}(\pi_2)(A)\} : \Gamma \longrightarrow P1$. A simple calculation shows that $\mathcal{P}(\ulcorner A \urcorner)(Prf) = A$. Hence we do have a generic predicate.

For part (ii) suppose that $\mathbf{C}$ is cartesian closed and that the hyperdoctrine $\mathcal{P}$ has a generic predicate $Prf \in \mathcal{P}(Prop)$. For each $\mathbf{C}$-object $X$ define $PX$ to be the exponential $Prop^X$ and the membership predicate $In_X \in \mathcal{P}(X \times Prop^X)$ to be $\mathcal{P}(ev_X)(Prf)$, where $ev_X : X \times Prop^X \longrightarrow Prop$ is evaluation (counit of the exponential adjunction $X \times (-) \dashv (-)^X$ at $Prop$). For any $R \in \mathcal{P}(X \times \Gamma)$ we have $\ulcorner R \urcorner : X \times \Gamma \longrightarrow Prop$ and can take its transpose across the exponential adjunction to get a morphism $\{R\} : \Gamma \longrightarrow Prop^X$. It is straightforward to see that this has the property required in Definition 4.3. $\square$

**Example 4.5.** The hyperdoctrines in Examples 2.2 and 2.3 both possess generic predicates and hence by Theorem 4.4 are $\mathbf{Set}$-triposes (since $\mathbf{Set}$ is cartesian closed).

In the first case we can take $Prop$ to be (the underlying set of) $H$ and $Prf \in \mathcal{P}(H) = H^H$ to be the identity function; for any $A \in \mathcal{P}(X)$, $\ulcorner A \urcorner$ is just $A$ itself. The topos generated by this tripos is precisely Higg's *category of $H$-valued sets*, equivalent to the category of sheaves on the complete Heyting algebra $H$: see (Fourman and Scott 1979).

In the second example we can take $Prop$ to be the powerset $P(\mathbb{A})$ of the PCA $\mathbb{A}$ and $Prf \in \mathcal{P}(P(\mathbb{A}))$ to be equivalence class of the identity function; for any $A \in \mathcal{P}(\Gamma)$, choosing a representative $\Phi \in P(\mathbb{A})^\Gamma$ for it, we can take $\ulcorner A \urcorner = \Phi$ since $\mathcal{P}(\Phi)(Prf) = \mathcal{P}(\Phi)([id_{P(\mathbb{A})}]) = [id_{P(\mathbb{A})} \circ \Phi] = [\Phi] = A$. The topos generated by this tripos is the so-called *realizability topos* of the partial combinatory algebra $\mathbb{A}$: see (Hyland, Johnstone, and Pitts 1980; Pitts 1981; van Oosten 1991; Longley 1995).

There are two minor differences between Definition 4.3 and the definition of tripos given in (Hyland, Johnstone, and Pitts 1980) or (Pitts 1981). The first has to do with generalised quantifiers; the second has to do with the use of preorders rather than partial orders. These differences are discussed in the next two remarks.

**Remark 4.6 (Generalised quantification).** The original definition of tripos assumes that $\mathbf{C}$ has all finite limits, rather than just finite products, and that there are adjoints $(\exists_f, \forall_f)$ for all the monotone functions $\mathcal{P}(f)$, rather than just for the case when $f$ is a product projection or diagonal; furthermore these adjoints are required to be stable in the sense that 'Beck-Chevalley' conditions hold:

$$\begin{array}{ccc}
\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "k"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} & \text{implies} & \begin{tikzcd} \mathcal{P}(W) \arrow[r, "\forall_h"] & \mathcal{P}(Z) \\ \mathcal{P}(Y) \arrow[u, "\mathcal{P}(k)"] \arrow[r, "\forall_f"'] & \mathcal{P}(X) \arrow[u, "\mathcal{P}(g)"'] \end{tikzcd} \\
\text{pullback in } \mathbf{C} & & \text{commutes in } \mathbf{Poset}.
\end{array}$$

(If this holds for all pullbacks, then a similar condition holds for the left adjoints $\exists_f$ as well.) From the work of Lawvere (1969) we know that in a first order hyperdoctrine as defined in

Section 2 such *generalised quantifiers* are definable from the usual ones:

$$(\forall_f A)(x) \overset{\text{def}}{\Leftrightarrow} \forall y : Y.\ x =_X f(y) \ \Rightarrow\ A(y)$$
$$(\exists_f A)(x) \overset{\text{def}}{\Leftrightarrow} \exists y : Y.\ x =_X f(y) \ \&\ A(y).$$

These formulas do define adjoints to $\mathcal{P}(f)$ and these adjoints satisfy the Beck-Chevalley condition for certain pullback squares—the ones that exist by dint of the finite product structure in $\mathbf{C}$. However, there is no reason why the Beck-Chevalley condition should hold for all the pullback squares that happen to exist in $\mathbf{C}$. In this sense the definitions in (Hyland, Johnstone, and Pitts 1980) and (Pitts 1981) assume a bit more than is strictly necessary.

**Remark 4.7 (Canonically presented hyperdoctrines).** The original definition of tripos was phrased in terms of indexed *preordered* sets $\mathbf{C}^{op} \longrightarrow \mathbf{Preord}$, rather than indexed posets $\mathbf{C}^{op} \longrightarrow \mathbf{Poset}$. Each setting has its conveniences and it is easy to pass between the two. However, one advantage of using preorders is that one can often identify predicates on $X$ with functions from $X$ to some fixed object. For example if we present the realizability triposes of Example 2.3 using indexed preordered sets, then we can take $\mathcal{P}(X)$ to be $P(\mathbb{A})^X$ rather than a quotient of it. Triposes in which predicates are functions are called *canonically presented* in (Hyland, Johnstone, and Pitts 1980; Pitts 1981).

In the partially ordered setting we are using here, we can say that a first order hyperdoctrine $(\mathbf{C}, \mathcal{P})$ as in Definition 2.1 is *canonically presented by a* $\mathbf{C}$*-object Prop* if for each $\mathbf{C}$-object $\Gamma$ there is a surjective function $e_\Gamma : \mathbf{C}(\Gamma, \mathit{Prop}) \twoheadrightarrow \mathcal{P}(\Gamma)$, natural in $\Gamma$. For then we can make $\mathbf{C}(-, \mathit{Prop})$ into a $\mathbf{C}$-indexed preordered set equivalent to $\mathcal{P}$ by declaring $f \leq f'$ in $\mathbf{C}(\Gamma, \mathit{Prop})$ to mean that $e_\Gamma(f) \leq e_\Gamma(f')$ holds in $\mathcal{P}(\Gamma)$. Note that from Theorem 4.4(i) we have that if $\mathcal{P}$ is a $\mathbf{C}$-tripos, then $\mathcal{P} : \mathbf{C}^{op} \longrightarrow \mathbf{Poset}$ can be canonically presented by $P1$. However, if $\mathcal{P}$ merely satisfies (CA), then it is not necessarily canonically presentable. The following example shows this. It already occurs in (Pitts 1981, Section 2.9). However, I was not aware at that time of the general result (Theorem 4.2) of which it is an instance. (If I had been, doubtless the definition of tripos would have been different.)

**Example 4.8 (A non-tripos satisfying CA).** Let $\mathbf{Fin}$ denote the category of finite sets and functions. From any infinite Boolean algebra $B$ we can define a hyperdoctrine $\mathcal{P}_B$ over $\mathbf{Fin}$ that satisfies (CA), but which is not a tripos, as follows.

In fact $\mathcal{P}_B$ is just like Example 2.2 except that we restrict the base category to be finite sets so that the quantifiers use only the finite meets and joins assumed to exist in $B$. Thus for each finite set $X$, define $\mathcal{P}_B(X)$ to be the the $X$-fold product $B^X$ of the Boolean algebra $B$; and for each $f : X \longrightarrow Y$ in $\mathbf{Fin}$, define $\mathcal{P}_B(f) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ to be $B^f$. Equality predicates $=_X$ in $\mathcal{P}_B(X \times X)$ are given by the functions

$$=_X(x, x') \overset{\text{def}}{=} \begin{cases} \top & \text{if } x = x' \\ \bot & \text{if } x \neq x' \end{cases}$$

and quantification is given by

$$(\exists X)_\Gamma(A) \overset{\text{def}}{=} \lambda i \in \Gamma\,.\, \textstyle\bigvee_{x \in X} A(i, x) \qquad (\forall X)_\Gamma(A) \overset{\text{def}}{=} \lambda i \in \Gamma\,.\, \textstyle\bigwedge_{x \in X} A(i, x).$$

The fact that $\mathcal{P}_B$ is a first order hyperdoctrine of course only depends upon the Heyting algebra structure of $B$. However, to see that it also satisfies the Comprehension Axiom 4.1 we make essential use of the fact that $B$ has complements ($\neg$) rather than just relative pseudocomplements ($\rightarrow$).

For each finite set $X$, we take $PX$ to be the set $\{\perp, \top\}^X$ of functions from $X$ to the two-element Boolean subalgebra $\{\perp, \top\}$ of $B$. The membership $\mathcal{P}$-predicate $In_X \in \mathcal{P}(X \times PX)$ is given by function application. Then for any $R \in \mathcal{P}_B(X \times \Gamma)$ we have

$$\begin{aligned}\top &= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \top \\ &= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} R(x,i) \vee \neg R(x,i) \\ &= \textstyle\bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \bigvee_{b\in\{\perp,\top\}} b \Leftrightarrow R(x,i) \\ &= \textstyle\bigwedge_{i\in\Gamma} \bigvee_{s\in\{\perp,\top\}^X} \bigwedge_{x\in X} s(x) \Leftrightarrow R(x,i)\end{aligned}$$

the last step using the fact that $B$ is a distributive lattice. Thus by definition of $PX$ and $In_X$, the formula

$$\forall i : \Gamma \,.\, \exists s : PX \,.\, \forall x : X \,.\, In_X(x,s) \Leftrightarrow R(x,i)$$

of the internal language of $\mathcal{P}$ is satisfied. Hence (CA) holds and $\mathbf{Fin}[\mathcal{P}_B]$ is a topos for any Boolean algebra $B$, whether or not it is infinite.

However, if $B$ is infinite then $\mathcal{P}_B$ cannot be made into a tripos for any choice of $X \mapsto (PX, In_X)$. For if it could, then by Theorem 4.4 it would possess a generic predicate and hence be canonically presented by some object *Prop* in $\mathbf{Fin}$. So in particular there would be a surjection from the finite set $\mathit{Prop} \cong \mathbf{Fin}(1, \mathit{Prop})$ onto $\mathcal{P}_B(1) \cong B$, which is impossible.

**Remark 4.9 (An open problem in topos theory).** If $\mathbf{C}$ is a category with finite products and $\mathcal{P}$ a first order hyperdoctrine over $\mathbf{C}$, then Lemmas 3.2 and 3.3 imply that the Heyting algebra $Sub_{\mathbf{C}[\mathcal{P}]}(1)$ of subobjects of the terminal object 1 in the logos $\mathbf{C}[\mathcal{P}]$ is isomorphic to $\mathcal{P}(1)$. In Example 4.8, $\mathcal{P}_B(1)$ is the Boolean algebra $B$. Thus by Theorem 4.2, $\mathbf{Fin}[\mathcal{P}_B]$ is a topos with $Sub_{\mathbf{Fin}[\mathcal{P}_B]}(1)$ isomorphic to $B$. In general the subobjects of 1 in a topos $\mathbf{E}$ (i.e. its 'truth-values') form a Heyting algebra. We have just seen that every Boolean algebra can arise as $Sub_{\mathbf{E}}(1)$ for some topos $\mathbf{E}$. However, it is not known whether every Heyting algebra can arise in this way. (Probably the free Heyting algebra on countably many generators cannot be the Heyting algebra of truth-values of a topos; see Pitts 1992, Section 1 for more on this topic.)

## 5. Conclusion

The notion of 'tripos' was motivated by the desire to explain in what sense Higg's description of sheaf toposes as $H$-valued sets and Hyland's realizability toposes are instances of the same construction. The construction itself involves building a category of partial equivalence relations and can be seen as the universal way of realizing the predicates of a first order hyperdoctrine as subobjects in a logos having effective equivalence relations (Theorem 3.6). This yields a topos if and only if the hyperdoctrine satisfies a certain comprehension property (Theorem 4.2). Triposes satisfy this property, but there are examples of non-triposes satisfying this form of comprehension (Example 4.8).

So should the definition of tripos in (Pitts 1981) have used this more general form? The main use for triposes seems to occur when one has some non-standard notion of predicate and one wishes to see that it can be used to generate a topos. For examples see (van Oosten 1991; Hofmann 1999; Awody, Birkedal, and Scott 1999). In this respect the condition (CA) seems useful, because it is more permissive that its Skolemized form. However, triposes often arise by applying various constructions to other triposes. In particular, (Pitts 1981) establishes quite a rich theory of triposes akin to that for sheaf theory, involving notions of geometric morphism, Lawvere-Tierney topologies, etc. I do not know how far this theory extends to the case of hyperdoctrines satisfying the (CA) axiom, but I guess it is not very far. For example, one of the most useful results in (Pitts 1981) concerns the question of *iteration*: if $\mathcal{P}$ is a tripos over $\mathbf{C}$ and $\mathcal{R}$ a tripos over $\mathbf{C}[\mathcal{P}]$, when is $\Delta_{\mathcal{R}} \circ \Delta_{\mathcal{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{P}] \longrightarrow \mathbf{C}[\mathcal{P}][\mathcal{R}]$ the topos of partial equivalence relations of a $\mathbf{C}$-tripos? Theorem 6.2 of (Pitts 1981) provides a practically useful answer to this question—namely that $\mathcal{R}'(-) = \mathcal{R}(\Delta_{\mathcal{P}}(-))$ is a $\mathbf{C}$-tripos with $\mathbf{C}[\mathcal{R}']$ equivalent to $\mathbf{C}[\mathcal{P}][\mathcal{R}]$, provided $\mathcal{R}$ has 'fibrewise quantification'. Fibrewise quantification is a concept that applies to triposes based on toposes and occurs frequently (e.g. Examples 2.2 and 2.3 have fibrewise quantification). It means that the quantifiers in $\mathcal{R}$ are induced by morphisms $\bigwedge_{\mathcal{R}}, \bigvee_{\mathcal{R}} : \Omega^{Prop} \longrightarrow Prop$ in a certain obvious fashion (cf. Hyland, Johnstone, and Pitts 1980, Proposition 1.12), where $\Omega$ is the subobject classifier of the topos and *Prop* the carrier of the generic predicate of the tripos (Theorem 4.4). We saw in Example 4.8 that hyperdoctrines satisfying (CA) do not necessarily have generic predicates, so the notion of fibrewise quantification and its consequences for iteration are not so useful in that setting.

## References


Awody, S., L. Birkedal, and D. S. Scott (1999). Local realizability toposes and a modal logic for computability. In L. Birkedal and G. Rosolini (Eds.), *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, Volume 23 of *Electronic Notes in Theoretical Computer Science*. Elsevier.

Birkedal, L. and G. Rosolini (Eds.) (1999). *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, Volume 23 of *Electronic Notes in Theoretical Computer Science*. Elsevier.

Fourman, M. P. and D. S. Scott (1979). Sheaves and logic. In M. P. Fourman, C. J. Mulvey, and D. S. Scott (Eds.), *Applications of Sheaves, Proceedings, Durham 1977*, Volume 753 of *Lecture Notes in Mathematics*, pp. 302–401. Springer-Verlag, Berlin.

Hofmann, M. (1999). Semantical analysis of higher-order abstract syntax. In *14th Annual Symposium on Logic in Computer Science*, pp. 204–213. IEEE Computer Society Press, Washington.

Hyland, J. M. E. (1982). The effective topos. In A. S. Troelstra and D. van Dalen (Eds.), *The L. E. J. Brouwer Centenary Symposium*, pp. 165–216. North-Holland, Amsterdam.

Hyland, J. M. E. (1988). A small complete category. *Annals of Pure and Applied Logic 40*, 135–165.

Hyland, J. M. E., P. T. Johnstone, and A. M. Pitts (1980). Tripos theory. *Math. Proc. Cambridge Philos. Soc. 88*, 205–232.

Johnstone, P. T. (1977). *Topos Theory*. Number 10 in LMS Mathematical Monographs. Academic Press, London.

Johnstone, P. T. and R. Paré (Eds.) (1978). *Indexed Categories and Their Applications*, Volume 661 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Kleene, S. C. (1945). On the interpretation of intuitionistic number theory. *Journal of Symbolic Logic 10*, 109–124.

Lawvere, F. W. (1969). Adjointness in foundations. *Dialectica 23*, 281–296.

Lawvere, F. W. (1970). Equality in hyperdoctrines and the comprehension schema as an adjoint functor. In A. Heller (Ed.), *Applications of Categorical Algebra*, pp. 1–14. American Mathematical Society, Providence RI.

Longley, J. (1995). *Realizability Toposes and Language Semantics*. Ph. D. thesis, Edinburgh University.

Makkai, M. (1993). The fibrational formulation of intuitionistic predicate logic. I: Completeness according to Gödel, Kripke, and Läuchli. *Notre Dame Journal of Formal Logic 34*, 334–377.

Makkai, M. and G. E. Reyes (1977). *First Order Categorical Logic*, Volume 611 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Phoa, W. (1990). *Domain Theory in Realizability Toposes*. Ph. D. thesis, Cambridge University.

Pitts, A. M. (1981). *The Theory of Triposes*. Ph. D. thesis, Cambridge University.

Pitts, A. M. (1989). Conceptual completeness for first order intuitionistic logic: an application of categorical logic. *Annals Pure Applied Logic 41*, 33–81.

Pitts, A. M. (1992). On an interpretation of second order quantification in first order intuitionistic propositional logic. *Jour. Symbolic Logic 57*, 33–52.

Pitts, A. M. (2000). Categorical logic. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum (Eds.), *Handbook of Logic in Computer Science, Volume 5. Algebraic and Logical Structures*, Chapter 2. Oxford University Press.

Reus, B. and T. Streicher (1999). General synthetic domain theory—a logical approach. *Mathematical Structures in Computer Science 9*, 177–223.

Rosolini, G. (1990). About modest sets. *International Journal of Foundations of Computer Science 1*, 341–353.

van Oosten, J. (1991). *Exercises in Realizability*. Ph. D. thesis, Universiteit van Amsterdam.