For these questions, we assume an o-minimal structure \((\mathbb{R},<,S)\) extending an ordered field \((\mathbb{R},<,0,1,+,-,\cdot)\).

\section{\(C^k\) functions}

Let \(f = (f_1, f_2, ..., f_n) : U \to \mathbb{R}^n\) be a definable map on an open set \(U \subset \mathbb{R}^m\). We give an inductive definition for \(f\) to be \(C^k\), where \(k\) is a positive integer:

\(f\) is \(C^1\) if it satisfies the original definition.

\(f\) is \(C^{k+1}\) if \(f\) is \(C^1\) and \(df : U \to \mathbb{R}^{nm}\) is \(C^k\).

a) Show that for an open \(U \subset \mathbb{R}^m\), the inclusion map \(U \to \mathbb{R}^m\) is \(C^k\) for all \(k > 0\).

b) Assume \(f : U \to \mathbb{R}^m\) (\(U \subset \mathbb{R}^m\) open and \(f\) is \(C^k\)) and \(g : V \to \mathbb{R}^p\) (\(V \subset \mathbb{R}^n\) open and \(g\) is \(C^k\)). Proof that \(g \circ f : W \to \mathbb{R}^p\) is \(C^k\) for any set \(W \subset f^{-1}(V)\) open in \(\mathbb{R}^m\).

c) For \(f : A \to \mathbb{R}\) with \(A \subset \mathbb{R}\) and \(k > 0\). Proof that there is decomposition of \(\mathbb{R}\) partitioning \(A\) such that on all open cells \(C\) in the partition, \(f|_{C}\) is \(C^k\).

\section{Good linear spaces}

Let \(A \subset \mathbb{R}^m\) definable with \(\dim(A) \leq k < m\). Show that there is a linear space \(L\) in \(\mathbb{R}^m\) of dimension \((m - k)\) such that for all \(v \in \mathbb{R}^m\) we have that with \(L_v := \{v + x : x \in L\}\), the intersection \(L_v \cap A\) is finite.

Hint: use the good directions lemma (7.4.2)

\section{Open faces of complexes}

Let \(K\) be a complex of \(\mathbb{R}^m\), in the ordered field \((\mathbb{R},<,0,1,+,-,\cdot)\).

a) Let \(\sigma \in K\) with \(\dim(\sigma) = \dim(|K|)\). Show that \(\sigma\) is open in \(|K|\).

Note: You may assume that the dimension of \(|K|\) is the maximal dimension of its elements.

b) Give an example of a complex \(K\) with an element \(\sigma \in K\) for which \(\sigma\) is not open in \(K\).