O-minimal Structures - Assignment 3

Exercise 1 (4 points)

In this exercise, let $X$ and $Y$ be nonempty topological spaces and assume that $Y$ is compact. We equip $X \times Y$ with the product topology.

(a) (2 points) Let $\pi : X \times Y \to X$ denote the projection map from $X \times Y$ onto $X$. Prove that the image of a closed set under $\pi$ is again closed.

(b) (2 points) Suppose that the graph $\Gamma(f)$ of the function $f : X \to Y$ is closed. Prove that $f$ is continuous.

Exercise 2 (5 points)

In this exercise, let $F$ be an ordered field.

(a) (3 points) Let $f = f(X_1, \ldots, X_m) \in F[X_1, \ldots, X_m]$, and let $d_1, \ldots, d_m \in \mathbb{N}$ be such that $\deg_{X_i}(f) \leq d_i$ for $1 \leq i \leq m$. We put $\deg_{X_i}(0) = -\infty$ by convention. Furthermore, let $A_1 \times \cdots \times A_m \subset F^m$, with $|A_1| > d_1, \ldots, |A_m| > d_m$. Prove that if the restriction of $f$ to $A_1 \times \cdots \times A_m$ is identically zero, then $f = 0$.

(b) (2 points) Let $f = f(X_1, \ldots, X_m) \in F[X_1, \ldots, X_m]$, with $f \neq 0$. Prove that the zero set $Z(f) = \{ a \in F^m : f(a) = 0 \}$ is a closed subset of $F^m$ with empty interior.

Grade = total points + 1