Tame Topology and O-minimal Structures-Dimensions,
Homework Set

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In the following exercises we fix an O-minimal structure \((\mathbb{R}, <, S)\):

**Exercise 1:** (3 points) (Dimensions of Sets from Definable Families)
Let \(A\) and \(B\) be definable subsets of \(\mathbb{R}^{m+n}\), with \(A\) non-empty. Assume that, for every \(a \in \mathbb{R}^m\), \(\dim(B_a) < \dim(A_a)\). Prove that \(\dim B < \dim A\). (For definition of \(A_a\) and \(B_a\), check p.59 (3.1).)

**Exercise 2:** (Local Dimension, p.69 (1.17) Exercise 2, 3, 4.)

1. (2 points) Let \(A \subseteq \mathbb{R}^m\) be definable and \(a \in \mathbb{R}^m\). Show there is a number \(d \in \{-\infty, 0, \cdots, \dim A\}\) such that there is an open box \(U \subseteq \mathbb{R}^m\) with \(a \in U\), and for all open box \(V \subseteq \mathbb{R}^m\), if \(a \in V\) and \(V \subseteq U\), then \(\dim(V \cap A) = d\).

   **Remark:** The number \(d\) defined by this property is called the local dimension of \(A\) at \(a\), notation \(\dim_a(A)\). Note that \(\dim_a(A) = -\infty\) iff \(a \notin \text{cl}(A)\).

2. (2 points) Show that if \(A \subseteq \mathbb{R}^m\) is a \(d\)-dimensional cell, then \(\dim_a(A) = d\) for all \(a \in \text{cl}(A)\). (**Hint:** use the homeomorphism \(p_A\) defined in p.51 (2.7).)

3. (3 points) Let \(A \subseteq \mathbb{R}^m\) be a definable set and \(d \in \{0, \cdots, \dim A\}\). Show that the set \(\{a \in \mathbb{R}^m : \dim_a(A) \geq d\}\) is a definable closed subset of \(\text{cl}(A)\). (**Hint:** apply cell decomposition theorem to \(\text{cl}(A)\), then show the set \(\{a \in \mathbb{R}^m : \dim_a(A) \geq d\}\) is the closure of a finite union of cells.)

   Show also that if \(A \neq \emptyset\), then \(\dim(\{a \in \text{cl}(A) : \dim_a(A) < d\}) < d\).