Seminar O-minimal Structures

hand-in exercise 8, due: 2014/12/19

We fix an o-minimal structure \((R, <, S)\).

exercise 1. In this exercise you must try to find definable sets for which \(f_C\) grows “at least as fast” as a given function.

a. In this exercise you show that the polynomial bounds \(p_d(n)\) which we will obtain from the main theorem are “tight”. Show that for every \(d > 0\) there exists a definable relation \(S \subseteq R^{d-1} \times R\) such that, if we put \(C = \{ S_x \mid x \in R^{d-1} \}\), then \(f_C(n) = p_d(n)\) for every \(n\).

b. Assume for this part that \((R, <, S)\) expands an ordered abelian group \((R, <, +, 0)\). Show that for every natural number \(c\) there exists a definable set \(S \subseteq R \times R\) and a natural number \(N\) such that, putting \(C = \{ S_x \mid x \in R \}\), we have \(f_C(n) > cn\) for every \(n \geq N\).

exercise 2. Let \(S \subseteq R \times R^q\). Again, put \(C = \{ S_x \mid x \in R \}\), and put \(G = \{ S^y \mid y \in R^q \} \subseteq \mathcal{P}(R)\). We have seen that there exists an \(e \in \mathbb{N}\) such that \(f^G(n) \leq p_3(en)\). Therefore, \(f_C\) is of at most quadratic growth. This exercise asks you to improve on this.

a. Show that for any decomposition \(D = \{ E_1, \ldots, E_k \}\) of \(R\), the atoms of the boolean algebra \(B(E_1, \ldots, E_k)\) are precisely \(E_1, \ldots, E_k\).

b. Assume \(S_1, \ldots, S_k \subseteq R\) are definably connected. Show that there is a decomposition of \(R\) into at most \(4k + 1\) cells, partitioning each of the \(S_i\).

c. Show that there exist natural numbers \(c\) and \(N\) such that if \(F \subseteq R^q\) is any finite set with at least \(n \geq N\) elements, then \(|C \cap F| \leq cn\).