solution to exercise 1

For part (1), let $A \subseteq R$ definable. Then $A$ is a finite union of intervals and points: $A = \bigcup_{i=1}^{n}(p_i, q_i) \cup \{r_1, \ldots, r_l\}$, with $-\infty \leq p_i < q_i \leq +\infty$ for every $i$, and all $r_j \in R$. If for some $i$, $q_i = +\infty$, then for every $x > p_i$, $x \notin R \setminus A$. So $p_i$ is an upper bound for $R \setminus A$. Otherwise, let $s = \max\{q_1, \ldots, q_b, r_1, \ldots, r_l\}$. Then clearly $s$ is an upper bound for $A$. For the statement about lower bounds the argument is entirely similar.

For part (2), first observe that every definable infinite subset $A \subseteq R$ must contain an interval, for otherwise it would be a finite union of just points. Second, if $X$ is dense in $R$, that is: for every $p, q$, if $p < q$ then there exists $x$ with $p < x < q$, then $X$ is cofinite. For, if $R \setminus X$ were infinite, then we would have an interval $(p, q) \subseteq R \setminus X$, contradicting $X$ being dense in $R$. Now, if $X$ is dense in $Y$, then $X \cup (R \setminus Y)$ is dense in $R$. For suppose $p < q$ are such that $(p, q) \cap (X \cup (R \setminus Y)) = \emptyset$, then $(p, q) \subseteq Y \setminus X$. But then $(p, q)$ is open in $Y$ and does not intersect $X$, contradicting $X$ being dense in $Y$. So $X \cup (R \setminus Y)$ is cofinite. Its complement, $Y \setminus X$, is therefore finite. Since finite sets are closed in $Y$, we see that $X$ is open in $Y$.

solution to exercise 2

First we prove: for every definable $A \subseteq R$, $bd(A)$ is finite. Suppose not, then by assumption (3) there exist $p, q$ such that $p < q$ and $(p, q) \subseteq bd(A)$. In particular, $(p, q) \subseteq cl(A)$, so $A \cap (p, q)$ is dense in $(p, q)$ (since every non-empty interval $(p', q') \subseteq (p, q)$ is the neighborhood of a point in $cl(A)$ and therefore intersects $A$). Then by assumption (4), $A \cap (p, q)$ is open in $(p, q)$. Since $A \cap (p, q)$ is non-empty, we can find a non-empty interval $(p', q') \subseteq A \cap (p, q)$. But then $(p, q) \cap int(A) \neq \emptyset$, contradicting the assumption that $(p, q) \subseteq bd(A)$.

Now let $A$ be an arbitrary definable subset of $R$. We claim that for every interval $(a, b)$ such that $(a, b) \cap bd(A) = \emptyset$, either $(a, b) \subseteq A$ or $(a, b) \subseteq R \setminus A$. We distinguish the case where one or both of the endpoints is $\pm \infty$ from the case where the endpoints are both in $R$.

Start with the case where at least one of the endpoints is $\pm \infty$. We can assume without loss of generality that $b = +\infty$. Note that the special case where $(a, b) = (-\infty, +\infty)$, that is: $bd(A) = \emptyset$, also falls under this assumption. If $A = \emptyset$ or $A = R$ we are done. Otherwise, either $A$ or $R \setminus A$ has an upper bound in $R$. Without loss of generality assume $R \setminus A$ has an upper bound and put $c = \sup(R \setminus A)$ by assumption (2). Observe that for every $p < c$ there exists $x \in R \setminus A$ such that $p < x < c$, since $c$ is the least upper bound for $R \setminus A$. And for every $x$ such that $c < x$, $x \in A$. So for every $p, q$ such that $p < c < q$, both $(p, q) \cap (R \setminus A) \neq \emptyset$ and $(p, q) \cap A \neq \emptyset$. So $c \in bd(A)$. Hence by assumption, $c \notin (a, +\infty)$. If $a = -\infty$, this constitutes a contradiction and we can conclude that $A = \emptyset$ or $A = R$ and we are done. Otherwise we must have $c \leq a$, hence for every $x > a$, $x \in A$. That is, $(a, +\infty) \subseteq A$, as required.

Now consider $(a, b)$ where $a, b \in R$. If $int(A) \cap (a, b) = \emptyset$, then $(R \setminus A) \cap (a, b)$ is dense in $(a, b)$, hence open in $(a, b)$. In that case, if $a < x < b$ and $x \in A$, then certainly $x \in cl(A)$ and by assumption $x \notin int(A)$. But then $x \in bd(A)$ contradicting our assumption that $(a, b) \cap bd(A) = \emptyset$. So such $x$ cannot exist, and therefore $(a, b) \subseteq R \setminus A$. Similarly, if $int(R \setminus A) \cap (a, b) = \emptyset$, then $(a, b) \subseteq A$. We are left with the case that there exist $p, q$ such that $a < p < q < b$ and...
(p, q) \subseteq A, and r, s such that (r, s) \subseteq R \setminus A. We will derive a contradiction. Clearly, either q < r or s < p. Without loss of generality, assume q < r. Define

\[ D = \{ x \in R \mid \forall y \in R, p < y < x \rightarrow y \in A \} \]

Note that D is definable and q \in D, so we may define c = \text{sup}(D). Note that q \leq c \leq r so c \in (a, b). Now, for every p' < c there exists x such that \text{max}(p, p') < x < c and therefore x \in A. So, clearly c \in \text{cl}(A). And c \notin \text{int}(A), for otherwise we could find x \in D with x > c. But then c \in \text{bd}(A), and c \notin (a, b) contradicting our assumption.

Finally, using that the boundary \text{bd}(A) = \text{bd}(R \setminus A) is finite, let it be enumerated in order by b_1 < \ldots < b_k, and in addition put b_0 = -\infty and b_{k+1} = +\infty. Since for every i \leq k, either (b_i, b_{i+1}) \subseteq A or (b_i, b_{i+1}) \subseteq R \setminus A, we can indeed write A as a finite union of intervals and points.