Hand-in Exercise 2 - O-minimal Structures

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Problem 1.
Let $F$ denote an ordered field and let $R$ be a nontrivial ordered $F$-linear space as defined in (7.2). Construe $R$ as a model-theoretic structure for the language $L_F = \{<, 0, -, +\} \cup \{\cdot : \lambda \in F\}$ of ordered abelian groups augmented by a unary function symbol $\cdot$ for each $\lambda \in F$, to be interpreted as multiplication by the scalar $\lambda$. Prove:

1. The subsets of $R^m$ definable in the $L_F$-structure $\mathcal{R}$ using constants are exactly the semilinear sets in $R^m$.

2. The maps $R \to R$ that are additive and definable using constants are exactly the scalar multiplications by elements of $F$. A map $f$ is additive iff
   \[ \forall r_1, r_2 \in R : f(r_1 + r_2) = f(r_1) + f(r_2). \]

Solution

1. (2 points)
   This little exercise is a good example of the concepts defined in paragraph 5, model-theoretic structures. We have to prove that the subsets of $R^m$ definable in the structure $\text{Def}(\mathcal{R}_R)$ are exactly the semilinear sets in $R^m$. Since every affine function is definable using constants from $R$, we conclude that all basic semilinear sets in $R^m$ are definable using constants. This means that the basic semilinear sets are definable using constants in every structure on $R$ that contains the relations and the functions of the $L_F$-structure. Since every structure has to be a boolean algebra on every level of the structure, we conclude that every structure, containing the basic semilinear sets, defines the semilinear sets. Hence the semilinear sets are defined in $\text{Def}(\mathcal{R}_R)$. Furthermore, Corollary (7.6) shows that $(S_m)_{m \in \mathbb{N}}$, with $S_m$ the boolean algebra of semilinear subsets of $R^m$, is actually a structure. We conclude that $\text{Def}(\mathcal{R}_R)$ is said structure and that the subsets of $R^m$ definable in $\text{Def}(\mathcal{R}_R)$ are exactly the semilinear sets in $R^m$. \( \square \)

2. (8 points)
   Notice that the scalar multiplications are indeed definable and additive. (Additivity is a property of scalar multiplication in a vector space).
   Let $f : R \to R$ be an additive map, definable in the $L_F$-structure $\mathcal{R}$ using constants. Following definition (7.2), we see that $R$ is an ordered additive group and using proposition (4.2), we conclude that $R$ is abelian, divisible and torsion-free. Writing the identity element of $R$ as $0$, we see that $f(0) = f(0 + 0) = f(0) + f(0)$. Since $R$ is torsion-free, $f(0)$ has
to be the identity element, so \( f(0) = 0 \). Furthermore, writing the additive inverse of an element \( r \in R \) as \(-r\), we see that \( 0 = f(0) = f(r + (-r)) = f(r) + f(-r) \), which means that \(-f(r) = f(-r)\).

In point 1 of the exercise, we saw that \( \text{Def}(R) \) is the structure defined in corollary (7.6), so we can apply the same corollary to see that there is a partition of \( R \) into basic semilinear sets \( A_i \), \((1 \leq i \leq k)\), such that \( f|_{A_i} \) is the restriction to \( A_i \) of an affine function on \( R \), for each \( i \in \{1, \ldots, k\} \). Using this we can write \( f(x) = \lambda x + a_i \) for all \( x \in A_i \), with \( \lambda \in F \), \( a_i \in R \), \( i \in \{1, \ldots, k\} \). Since \( R \) is infinite (for example because it is torsion-free) and our partition finite of definable subsets, there is at least one \( A_i \) such that \( A_i \) contains an interval. Take WLOG \( A_1 \) as such an element in our partition and let \( y, z \in R \) s.t. \((y, z) \subset A_1 \). Let \( x \in (y, z) \) and \( r \in R \) s.t. \( x + r \in (y, z) \). Then we have for all \( r' \in (0, r) \), (so \( x + r' \in (y, z) \subset A_1 \)), the following:

\[
\begin{align*}
  f(r') &= f(x + r' - x) = f(x + r') + f(-x) = f(x + r') - f(x) = \lambda_1(x + r') + a_1 - (\lambda_1 x + a_1) \\
  &= \lambda_1 x + \lambda_1 r' + a_1 - \lambda_1 x - a_1 = \lambda_1 r'.
\end{align*}
\]

Here we used the usual properties of scalar multiplication in a vector space. Write this \( \lambda_1 \) as \( \lambda \). We’ll now first prove that for every \( A_i \) containing an interval, \( \lambda_i = \lambda \). Next we’ll prove that for all \( x \in R \), \( x \in A_j \), that \( f(x) = \lambda_j x + a_j = \lambda x \), concluding our prove that every additive and definable map: \( R \to R \) is a scalar multiplication by elements of \( F \).

Let \( A_1 \) be an element in our partition containing an interval. Then there are \( x \in A_i \), \( r' \in (0, r) \) s.t. \( x + r' \in A_i \). Now we have that \( \lambda_1 x + \lambda_1 r' + a_i = \lambda_i (x + r') + a_i = f(x + r') = f(x) + f(r') = \lambda x + a_i + \lambda r' \). This means that \( \lambda r' = \lambda_j r' \), which implies that \( \lambda = \lambda_j \).

Suppose not and assume WLOG that \( \lambda > \lambda_j \), because we have a linear order on \( F \). Then \( (\lambda - \lambda_j) r' > 0 \), but \( \lambda - \lambda_j > 0 \) and \( r' > 0 \). This is in direct contradiction with definition 7.2 of an ordered \( F \)-linear space. We conclude that for every \( A_i \) containing an interval \( \lambda_i = \lambda \).

Next let \( A_j \) be any element of our partition and let \( x \in A_j \). Notice that we have a finite number of sets in our partition, each being a finite union of intervals and points. Since \( R \) is torsion-free and since \( F \) has an infinite number of elements, we conclude that there exist two different \( n_1, n_2 \in F \) s.t. \( n_1 x = (1 + \cdots + 1)x = x + \cdots + x \in A_k \) and \( n_2 x \in A_k \), where \( A_k \) is an element in our partition containing an interval. Hence we have that \( n_1(\lambda_j x + a_j) = n_1 f(x) = f(x) + \cdots + f(x) = f(x + \cdots + x) = f(n_1 x) = \lambda n_1 x + a_k \) and \( n_2(\lambda_j x + a_j) = n_2 f(x) = f(x) + \cdots + f(x) = f(x + \cdots + x) = f(n_2 x) = \lambda n_2 x + a_k \).

If we subtract these expressions from each other, we find that \( (n_1 - n_2)(\lambda_j x + a_j) = n_1(\lambda_j x + a_j) - n_2(\lambda_j x + a_j) = \lambda n_1 x + a_k - (\lambda n_2 x + a_k) = (n_1 - n_2)\lambda x \). Again we have used the usual properties of scalar multiplication in a vector space. Furthermore we used that \( F \) has commutative multiplication, since it is by definition an ordered field. Now multiplying with the multiplicative inverse of \((n_1 - n_2)\), which exists since \( n_1 \neq n_2 \), we find that \( f(x) = \lambda_j x + a_j = \lambda x \). This holds for every \( A_j \) in our partition and every \( x \in A_j \), so it holds for every \( x \in R \). We conclude that the maps \( R \to R \) that are additive and definable using constants are exactly the scalar multiplications by elements of \( F \).